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GREEN'S FUNCTION AND MAXIMUM PRINCIPLE FOR HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH IMPULSES

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ABSTRACT. We find Green's function representation of the solutions of a boundary value problem for an impulsive linear differential equation of nth order. Then we derive a maximum principle valid to develop monotone iterative techniques for some nonlinear differential equations with impulses.

1. Introduction. The theory of impulsive differential equations is experiencing a rapid development in past years. The reason is that it is richer than the corresponding theory of classical differential equations and it is more adequate to represent some processes arising in various disciplines, see monograph [**10**] and references therein.

In recent years many papers devoted to the study of boundary value problems for nonlinear differential equations with impulses have appeared. See, for instance, [**4**], [**6**], [**7**], [**8**], [**9**], [**10**], [**11**], [**12**].

One of the techniques employed in these papers is the method of upper and lower solutions coupled with iterative methods. In general, the applicability of this kind of technique depends strongly on the sign of Green's function representations for the solutions of certain linear problems associated to the considered boundary value problem.

For differential equations with impulses, the study of these integral representations needs further development in order to apply the monotone iterative techniques in a systematic way. For first order problems, some results can be found in recent papers due to Eloe and Henderson [**6**] and Nieto [**12**].

In Section 2 of the present paper, we find an integral representation for the solutions of the following nth order linear differential equation with constant impulses and nonhomogeneous periodic boundary conditions

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(1.1)
$$\left.\begin{aligned}&u^{(n)}(t)+\sum_{i=0}^{n-1}a_iu^{(i)}(t)=\sigma(t)\quad\text{for a.e. } t\in I,\ t\neq t_k, k=1,\dots,p\\&u^{(i)}(t_k^+)-u^{(i)}(t_k^-)=\mu_{ik},\quad k=1,\dots,p;\ i=0,\dots,n-1\\&u^{(i)}(0)-u^{(i)}(T)=\lambda_i,\quad i=0,\dots,n-1,\end{aligned}\right\}$$

where $I=[0,T]$, $\sigma\in L^1(I)$, $a_i,\lambda_i\in\mathbf{R}$, $i=0,\dots,n-1$, $a_0\neq 0$ and $\mu_{ik}\in\mathbf{R}$ for each $k=1,\dots,p$ and $i=0,\dots,n-1$.

Here $u^{(i)}$ denotes the ith derivative of the function $u, u^{(0)}=u$.

In Section 3 we briefly explain how the methods of Section 2 can be used to derive an existence result, using monotone iterative techniques, for a class of nonlinear boundary value problems for impulsive differential equations, following the ideas of [**4**].

2. Green's function and maximum principle. In this section we shall deal with problem (1.1).

We shall prove that problem (1.1) is uniquely solvable if the following boundary value problem without impulses has a unique solution

$$\left.\begin{aligned}&z^{(n)}(t)+\sum_{i=0}^{n-1}a_iz^{(i)}(t)=0\quad\text{for a.e. } t\in I\\&z^{(i)}(0)-z^{(i)}(T)=0,\quad i=0,\dots,n-2\\&z^{(n-1)}(0)-z^{(n-1)}(T)=1.\end{aligned}\right\}\tag{2.1}$$

First we recall some results for the nth order periodic boundary value problem, see [**2**, Lemma 2.1].

Lemma 2.1. *Assume that the problem* (2.1) *has a unique solution. Then the boundary value problem*

$$\left.\begin{aligned}&v^{(n)}(t)+\sum_{i=0}^{n-1}a_iv^{(i)}(t)=\sigma(t)\quad\textit{for a.e. } t\in I\\&v^{(i)}(0)-v^{(i)}(T)=\lambda_i,\quad i=0,\dots,n-1\end{aligned}\right\}\tag{2.2}$$

has a unique solution $v \in W^{n,1}(I)$ *for each* $\sigma \in L^1(I)$.

Moreover, v *is given by the following expression*:

$$v(t) = \int_0^T G(t,s)\sigma(s)\,ds + \sum_{i=0}^{n-1} r_i(t)\lambda_i, \tag{2.3}$$

where $r_i \in C^\infty(I)$, $i = 0, 1, \dots, n-1$ *and*

$$G(t,s) = \begin{cases} r_{n-1}(t-s) & 0 \le s \le t \le T \\ r_{n-1}(T+t-s) & 0 \le t < s \le T. \end{cases} \tag{2.4}$$

Function r_{n-1} *is the unique solution of* (2.1) *and functions* r_i *can be obtained as*:

$$r_i(t) = r_{n-1}^{(n-1-i)}(t) + \sum_{j=i+1}^{n-1} a_j r_{n-1}^{(j-i-1)}(t),$$
$$t \in I, \; i = 0, \dots, n-2.$$

Remark 2.1. From Lemma 2.1 it follows that the uniqueness of solution of problem (2.2) is independent of the values of $\lambda_i \in \mathbf{R}$, $i = 0, \dots, n-1$. Furthermore, it is equivalent to the uniqueness of solution of problem (2.1), which is easy to study since it reduces to check if a certain linear algebraic system $Ax = b$, with A an $n \times n$ matrix and $b \in \mathbf{R}^n$, has a unique solution.

In order to define precisely the concept of solution for the nth order impulsive problems considered in this paper, we introduce the following sets of functions:

$$C_p^m = \{u : I \to \mathbf{R} : u^{(l)} \text{ is continuous for } t \ne t_k, \text{ and there exist } u^{(l)}(t_k^+), u^{(l)}(t_k^-), \; l = 0, \dots, m, \; k = 1, \dots, p\}$$

and

$$W_p^n = \{u \in C_p^{n-1} : u_{|(t_k, t_{k+1})} \in W^{n,1}(t_k, t_{k+1}), \; k = 0, \dots, p\}.$$

We recall that C_p^m and W_p^n are Banach spaces with the norms

$$\|u\|_{C_p^m} = \sum_{k=0}^{p} \|u|_{[t_k,t_{k+1}]}\|_{C^m[t_k,t_{k+1}]}$$

and

$$\|u\|_{W_p^n} = \sum_{k=0}^{p} \|u|_{(t_k,t_{k+1})}\|_{W^{n,1}(t_k,t_{k+1})}.$$

By using the results of Lemma 2.1, we shall find an integral representation for the solutions of problem (1.1).

Lemma 2.2. *Let $a_0,\dots,a_{n-1} \in \mathbf{R}$ be fixed. Then problem* (2.1) *has a unique solution if and only if problem* (1.1) *has a unique solution $u \in W_p^n$.*

In such a case the solution of problem (1.1) *is given by the expression*

$$u = v + w, \tag{2.5}$$

where v is defined by (2.3) *and*

$$w(t) = \sum_{k=0}^{n-1} w_k(t), \tag{2.6}$$

with

$$w_k(t) = \sum_{j=1}^{p} \frac{\partial^k G}{\partial t^k}(t,t_j)\mu_{(n-1-k)j}, \quad t \neq t_j,\ j=1,\dots,p$$

being G the Green function given by (2.4).

Proof. Suppose that problem (2.1) has a unique solution v defined by (2.3).

For all $i \in \{0,\dots,n-1\}$, each function w_k satisfies

$$\begin{aligned} w_k^{(i)}(t_m^+) - w_k^{(i)}(t_m^-) &= \sum_{j=1}^{p}\left[\frac{\partial^{i+k}G}{\partial t^{i+k}}(t_m^+,t_j) - \frac{\partial^{i+k}G}{\partial t^{i+k}}(t_m^-,t_j)\right]\mu_{(n-1-k)j} \\ &= [r_{n-1}^{(i+k)}(0) - r_{n-1}^{(i+k)}(T)]\mu_{(n-1-k)m} \\ &= \begin{cases} 0 & i+k \neq n-1 \\ \mu_{im} & i+k = n-1, \end{cases} \end{aligned}$$

for each $m = 1, \dots, p$, and

$$\begin{aligned} w_k^{(i)}(0) - w_k^{(i)}(T) \\ &= \sum_{j=1}^{p} [r_{n-1}^{(i+k)}(T - t_j) - r_{n-1}^{(i+k)}(T - t_j)]\mu_{(n-1-k)j} = 0 \end{aligned}$$

in view of the definition of function G and the fact that r_{n-1} is the solution of the problem (2.1).

Moreover, for all $t \in I$, $t \neq t_j$, $j = 1, \dots, p$, we have

$$\begin{aligned} w_k^{(n)}(t) &= \sum_{j=1}^{p} \left[\frac{\partial^{k+n} G}{\partial t^{k+n}}(t, t_j)\mu_{(n-1-k)j} \right] \\ &= -\sum_{i=0}^{n-1} a_i \left[\sum_{j=1}^{p} \frac{\partial^{k+i} G}{\partial t^{k+i}}(t, t_j)\mu_{(n-1-k)j} \right] \\ &= -\sum_{i=0}^{n-1} a_i w_k^{(i)}(t). \end{aligned}$$

Consequently, w satisfies

$$\begin{aligned} w^{(n)}(t) + \sum_{i=0}^{n-1} a_i w^{(i)}(t) = 0, \quad & t \in I,\ t \neq t_j,\ j = 1, \dots, p \\ w^{(i)}(0) - w^{(i)}(T) = 0; \quad & i = 0, \dots, n-1 \\ w^{(i)}(t_j^+) - w^{(i)}(t_j^-) = \mu_{ij}; \quad & j = 1, \dots, p;\ i = 0, \dots, n-1. \end{aligned}$$

Finally, using that v is the solution of (2.2), we have that u given by (2.5) is the solution of the problem (1.1).

Since the difference of two solutions of (1.1) is a solution of (2.2) for $\sigma \equiv 0$ and $\lambda_i = 0$, $i = 0, \dots, n-1$, and this problem has only the trivial solution, the uniqueness of solution for problem (1.1) follows.

Since problem (2.1) is a particular case of problem (1.1), the reversed implication is proved. $\square$

Now, to derive results from this expression of the Green function, we define the set $F_n = \{u \in W_p^n, u \in C^{n-2}(I), u^{(i)}(0) = u^{(i)}(T), i =$

$0, \dots, n-2, u^{(n-1)}(0) - u^{(n-1)}(T) \geq 0, u^{(n-1)}(t_k^+) - u^{(n-1)}(t_k^-) \geq 0, k = 1, \dots, p\}$.

We say that the operator

$$[T_n(u)](t) = u^{(n)}(t) + \sum_{i=0}^{n-1} a_i u^{(i)}(t) \tag{2.7}$$

is inverse positive, respectively inverse negative, on F_n if and only if problem (1.1) has a unique solution for all $\sigma \in L^1(I)$, and λ_i, $\mu_{ij} \in \mathbf{R}$, $i = 0, \dots, n-1$, $j = 1, \dots, p$ and for all $u \in F_n$ such that $T_n(u) \geq 0$ on I we have that $u \geq 0$ on I, respectively $T_n(u) \geq 0 \Rightarrow u \leq 0$.

Lemma 2.2 permits us to prove the following maximum principle for the problem (1.1).

Lemma 2.3. *Let $a_1, \dots, a_{n-1} \in \mathbf{R}$ be fixed. Then the operator T_n is inverse positive on F_n if and only if a_0 is a real number such that* (2.1) *has a unique solution $r_{n-1} \geq 0$ on I.*

Proof. By using expressions (2.5) and (2.6) we have that if $u \in F_n$ there exist $\sigma \geq 0$ on I, $\lambda_{n-1} \geq 0$ and $\mu_{(n-1)j} \geq 0$, $j = 1, \dots, p$, such that

$$u(t) = \int_0^T G(t,s)\sigma(s)\, ds + r_{n-1}(t)\lambda_{n-1} + \sum_{j=1}^{p} G(t, t_j)\mu_{(n-1)j}.$$

Since $r_{n-1} \geq 0$ on I and $\sigma \geq 0$ on I, it is obvious that $u \geq 0$ on I.

Considering $\sigma \equiv 0$, $\lambda_{n-1} = 1$ and $\mu_{(n-1)j} = 0$, $j = 1, \dots, p$, we prove the necessary condition. $\square$

Analogously, one can prove the following anti-maximum principle.

Lemma 2.4. *Let $a_1, \dots, a_{n-1} \in \mathbf{R}$ be fixed. Then the operator T_n is inverse negative on F_n if and only if a_0 is a real number such that* (2.1) *has a unique solution $r_{n-1} \leq 0$ on I.*

Following the ideas of [**5**, Corollary 3.1] it is not difficult to prove the following result.

Lemma 2.5. *Let $a_1, \dots, a_{n-1} \in \mathbf{R}$ be fixed. Then there exists $M \in (0, \infty]$ such that the following properties are verified*:

1. *The operator T_n is inverse positive on F_n if and only if $a_0 \in (0, M)$ or $a_0 \in (0, M]$.*

2. *The operator T_n is inverse negative on F_n if and only if $a_0 \in (-M, 0)$ or $a_0 \in [-M, 0)$.*

Clearly, in order to apply our previous maximum and anti-maximum principles, one has to know the values of $a_0 > 0$ for which r_{n-1} is a nonnegative (nonpositive) function on I.

It is well known [**13**] that for first order equations $M = \infty$ is obtained both for T_1 inverse positive or inverse negative on F_1. The same property remains valid for T_2 inverse negative on F_2.

The study of the nth order case is now in progress, and some results were obtained recently. In this direction see the papers by Omari and Trombetta [**13**] and Cabada [**2**], in which the optimal estimations are obtained for second and third order problems, respectively. It is important to note that obtaining the expression of such estimations is a difficult and tedious problem. For instance, when $n = 3$ and $a_2 = a_1 = 0$, the best estimate of $M > 0$ for which T_3 is inverse positive or inverse negative on F_3 is given by $M = m^3$, where m is the unique solution of the equation

$$\arctan\left(\frac{\sin\sqrt{3}m\pi}{\cos\sqrt{3}m\pi - e^{m\pi}}\right) + \pi = \frac{\sqrt{3}}{3}\log\left(\frac{e^{3m\pi} - e^{m\pi}}{\sqrt{1 + e^{2m\pi} - 2e^{m\pi}\cos\sqrt{3}m\pi}}\right)$$

with $\arctan\theta \in [-(\pi/2), (\pi/2)]$.

When $n = 4$ this question is solved in [**1**] and [**5**] when the coefficients a_i are zero for $i = 1, 2, 3$.

For $n \in \mathbf{N}$ and $a_i = 0$, $i = 1, \dots, n-1$, some estimations can be found in [**3**]. The solution of this question for the general case remains as a difficult open problem.

3. Monotone iterative method. In this section we consider the following boundary value problem for a nonlinear differential equation with impulses:

$$\text{(3.1)} \quad \left.\begin{aligned}
&u^{(n)}(t)+\sum_{i=1}^{n-1}a_i u^{(i)}(t)=f(t,u(t)) \quad \text{for a.e. } t\in I,\ t\neq t_k,\ k=1,\dots,p\\
&u^{(i)}(t_k^+)=u^{(i)}(t_k^-)+\mu_{ik}, \quad k=1,\dots,p;\ i=0,\dots,n-2\\
&u^{(n-1)}(t_k^+)=u^{(n-1)}(t_k^-)+I_k(u(t_k^-)), \quad k=1,\dots,p\\
&u^{(i)}(0)-u^{(i)}(T)-\lambda_i, \quad i=0,\dots,n-1,
\end{aligned}\right\}$$

where $f : I\times \mathbf{R}\to \mathbf{R}$, $I_k : \mathbf{R}\to\mathbf{R}$ for $k=1,\dots,p$, $a_i\in\mathbf{R}$, $i=1,\dots,n-1$, $\lambda_i\in\mathbf{R}$, $i=0,\dots,n-1$ and $\mu_{ik}\in\mathbf{R}$ for each $k=1,\dots,p$ and $i=0,\dots,n-2$.

By defining appropriate concepts of upper and lower solutions, we are able to obtain an existence result for (3.1) using the maximum principle achieved in Section 2.

For it, define

$$\begin{aligned}
D_L=\{u\in W_p^n : u^{(i)}(t_k^+)-u^{(i)}(t_k^-)=\mu_{ik},\ k{=}1,\dots,p,\ i{=}0,\dots,n{-}2\\
\text{and}\quad u^{(i)}(0)-u^{(i)}(T)=\lambda_i,\ i{=}0,\dots,n{-}2\}.
\end{aligned}$$

Let us observe that $u\in W_p^n$ is a solution of (3.1) if and only if $u\in D_L$ and $L(u)=N(u)$, where $L : W_p^n\to L^1(0,T)\times\mathbf{R}^{p+1}$ and $N : C_p^0\to L^1(0,T)\times\mathbf{R}^{p+1}$ are two operators defined by

$$L(u)=\Big(T_n(u),\{u^{(n-1)}(t_k^+)-u^{(n-1)}(t_k^-)\}_{k=1}^p,u^{(n-1)}(0)-u^{(n-1)}(T)\Big),$$
$$N(u)=\big(f(\cdot,u)+a_0u,\{I_k(u(t_k^-))\}_{k=1}^p,\lambda_{n-1}\big),$$

where T_n is the operator defined in (2.7).

In $L^1(0,T)\times\mathbf{R}^{p+1}$ we are considering the norm given by

$$\|(g,x_1,\dots,x_{p+1})\|=\|g\|_1+\|(x_1,\dots,x_{p+1})\|_\infty.$$

Next we introduce the concepts of lower and upper solutions for (3.1).

Definition 3.1. We shall say that α is a lower solution of (3.1) if $\alpha \in D_L$ and $L(\alpha) \le N(\alpha)$.

Analogously, an upper solution of (3.1) is a function $\beta \in D_L$ satisfying $L(\beta) \ge N(\beta)$.

Here we are considering in $L^1(0,T) \times \mathbf{R}^{p+1}$ the partial order defined by

$$(g, x_1, \dots, x_{p+1}) \le (h, y_1, \dots, y_{p+1}) \quad \text{if } g(t) \le h(t) \quad \text{for a.e. } t \in I \\ \text{and } x_i \le y_i \quad \text{for all} \quad i = 1, \dots, p+1.$$

With previous notations, we have the following direct consequence of Lemma 2.3.

Proposition 3.1. *Let $a_1, \dots, a_{n-1} \in \mathbf{R}$ be fixed. Suppose $a_0 > 0$ such that T_n is inverse positive on F_n, and let $v_1, v_2 \in D_L$ such that $Lv_1 \le Lv_2$. Then $v_1 \le v_2$.*

This proposition reads that L is inverse positive on D_L. Now, following standard arguments, see the proof of [**4**, Theorem 3.1], it is not difficult to prove the following result for problem (3.1).

Theorem 3.1. *Let α and β be respectively a lower and an upper solution of* (3.1) *with $\alpha \le \beta$. Assume that the following conditions are satisfied*:

(H_1) f is a Carathéodory function.

(H_2) $f(t,u) - f(t,v) \ge -a_0(u - v)$ for almost every $t \in I$ and $\alpha(t) \le v \le u \le \beta(t)$ with $a_0 > 0$ such that T_n is inverse positive on F_n.

(H_3) I_k are continuous and nondecreasing functions for each $k = 1, \dots, p$.

Then there exist two monotone sequences $\{\alpha_s\}$ and $\{\beta_s\}$ such that $\alpha_0 = \alpha \le \alpha_s \le \beta_s \le \beta_0 = \beta$ for every $s \in \mathbf{N}$ which converge uniformly to the minimal and the maximal solutions of (3.1) *on $[\alpha, \beta]$, respectively, being $[\alpha, \beta] = \{u \in C_p^0 : \alpha \le u \le \beta\}$.*

As in [**1**], one can prove the following result.

Theorem 3.2. *The assertion proved in Theorem* 3.1 *is optimal in the sense that, for all f a Carathéodory function that does not satisfy condition (H_2), we can find α, β, λ_i, $i = 0, \dots, n-1$ and μ_{ij}, $i = 0, \dots, n-1$, $j = 1, \dots, p$, for which no solution exists lying between α and β.*

Remark 3.1. Note that if $\beta \leq \alpha$ we obtain that the monotone iterative method is valid for the problem (3.1) when the function f satisfies the following condition:

$$f(t,u) - f(t,v) \leq -a_0(u-v) \quad \text{for a.e. } t \in I,$$
$$\beta(t) \leq v \leq u \leq \alpha(t),$$

with $a_0 < 0$ such that T_n is inverse negative on F_n.

This property is optimal in the sense cited in Theorem 3.2.

Remark 3.2. For $\mu_{ik} = 0$, $I_k \equiv 0$, $k = 1, \dots, p$, $i = 0, \dots, n-2$, problem (3.1) is a boundary value problem for classical nth order differential equations without impulses. Thus Theorem 3.1 generalizes the monotone iterative technique for this kind of boundary value problem [**2**, Theorem 2.1].

Remark 3.3. We also note that the nonlinear problem considered in the present section differs from the one studied in [**4**], where the case $u^{(i)}(t_k^+) = I_k(u(t_k))$, $k = 1, \dots, p$, $i = 0, \dots, n-1$, was investigated.

Finally, we present an example in order to illustrate our main results.

Example 3.1. Let us consider the following boundary value problem for a nonlinear second order differential equation with one impulse:

$$\left.\begin{array}{l} u''(t) - 2u'(t) = \sin(u(t)) + h(t), \quad t \in I,\ t \neq \pi \\ u(\pi^+) = u(\pi^-) - \pi/2 \\ u'(\pi^+) = u'(\pi^-) \\ u(0) - u(2\pi) = \pi/2, u'(0) - u'(2\pi) = 0 \end{array}\right\} \tag{3.2}$$

where $I = [0, 2\pi]$ and

$$h(t) = \begin{cases} \sin(t) & t \in [0, \pi], \\ 0 & t \in (\pi, 2\pi]. \end{cases}$$

It is easy to prove that α and β defined by

$$\alpha(t) = \begin{cases} \pi/2 & t \in [0, \pi], \\ 0 & t \in (\pi, 2\pi] \end{cases}$$

and

$$\beta(t) = \begin{cases} 3\pi/2 & t \in [0, \pi], \\ \pi & t \in (\pi, 2\pi], \end{cases}$$

are, respectively, a lower and an upper solution of Problem (3.2) and $\alpha \le \beta$ on I.

Moreover, $T_2(u) = u'' - 2u' + u$ is inverse positive on F_2, see [**13**]. Then, taking $a_0 = 1$ and using the mean value theorem, it is easily seen that $f(t, u) - f(t, v) \ge -a_0(u - v)$ for all $t \in I$ and $v \le u$.

Thus Theorem 3.1 applies and we can approximate the extremal solutions of (3.2) on $[\alpha, \beta]$ by using the monotone iterative technique.

From Remark 3.3 it is obvious that the results of [**4**] are not valid for Problem (3.2).

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