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SELF-ADJOINT OPERATORS GENERATED FROM NON-LAGRANGIAN SYMMETRIC DIFFERENTIAL EQUATIONS HAVING ORTHOGONAL POLYNOMIAL EIGENFUNCTIONS

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ABSTRACT. We discuss the self-adjoint spectral theory associated with a certain fourth-order *non-Lagrangian symmetrizable* ordinary differential equation $l_4[y] = \lambda y$ that has a sequence of orthogonal polynomial solutions. This example was first discovered by Jung, Kwon, and Lee. In their paper, they derive the remarkable formula for these polynomials $\{Q_n(x)\}_{n=0}^{\infty}$:

$$Q_n(x) = n \int_1^x PL_{n-1}(t)dt, \quad n \in \mathbf{N},$$

where $\{PL_n(x)\}_{n=0}^{\infty}$ are the left Legendre type polynomials. The left Legendre type polynomials and the spectral analysis of the associated *symmetric* fourth-order differential equation that they satisfy have been extensively studied previously by Krall, Loveland, Everitt, and Littlejohn.

Despite the non-symmetrizability of the expression $l_4[\cdot]$, we show that there exists a self-adjoint operator S in a certain Hilbert space H generated by $l_4[\cdot]$ that has the "polynomial" sequence of ordered pairs $\{\langle Q_n(x), Q_n'(-1)\rangle\}_{n=0}^{\infty}$ as a complete set of eigenfunctions in H. This operator S is related to the derivative of the self-adjoint operator T which has the left Legendre type polynomials $\{PL_n(x)\}_{n=0}^{\infty}$ as eigenfunctions. We also develop a left-definite theory for $l_4[\cdot]$. This unexpected example casts further difficulties in the efforts to extend and generalize certain classification results in orthogonal polynomials and differential equations.

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1. Introduction. The theory of self-adjoint extensions of the minimal operator $\mathcal{L}_0$ generated from a real, ordinary differential expression $l[\cdot]$ is well known and extensively described in the classic texts of Naimark [**22**] and Akhiezer and Glazman [**1**]. These extensions are studied in a Hilbert space $L^2_w(I)$ of functions that are Lebesgue measurable and square integrable with respect to some positive (almost everywhere), locally integrable function $w(x)$ on some interval $I \subset \mathbf{R}$. Moreover, and this is the key for the existence of self-adjoint extensions of $\mathcal{L}_0$ in $L^2_w(I)$, the expression $l[\cdot]$ is symmetrizable (see [**20**]) in the sense that $w(x)l[\cdot]$ is formally Lagrangian symmetric.

In this paper, we study a certain fourth-order differential expression $l_4[\cdot]$, discovered by Jung, Kwon and Lee (see [**7**]), that is *not* symmetrizable in the Lagrangian sense and, yet, there is a *self-adjoint* realization of this expression in some Hilbert-Sobolev space H. Moreover, the expression $l_4[\cdot]$ is important from the viewpoint of the theory of orthogonal polynomials; indeed, a sequence $\{Q_n(x)\}_{n=0}^{\infty}$ of polynomial eigenfunctions of $l_4[\cdot]$ exists that generates the complete orthogonal set $\{\langle Q_n(x), Q_n'(-1)\rangle\}_{n=0}^{\infty}$ in H.

Besides being an unusual example from the theory of self-adjoint differential operators, this is an unexpected example from the viewpoint of orthogonal polynomials. Indeed, until this example was found, *all* known differential equations having a sequence of orthogonal polynomial solutions are symmetrizable in the sense of Lagrange. In fact, Kwon and Yoon (see [**18**]) show that if a differential equation $l[y] = \lambda y$ has a sequence of orthogonal polynomial solutions that are orthogonal with respect to a bilinear form of the type

$$\int_{\mathbf{R}} f(x)\bar{g}(x)\, d\mu, \tag{1.1}$$

where μ is a (possibly signed) Borel measure, then $l[\cdot]$ is Lagrangian symmetrizable. Although, the orthogonality of the above sequence $\{Q_n(x)\}_{n=0}^{\infty}$ is not with respect to a form of the type (1.1), it was generally believed that the Lagrangian symmetrizability of $l[\cdot]$ would follow from any orthogonalizing bilinear form; see [**5**] for a general discussion of orthogonal polynomial solutions to differential equations. Furthermore, as we show later in this paper, this example is not a singular, isolated one; in fact, we produce more examples with this phenomenon later in this paper.

It is precisely structural theorems of the type proven by Kwon and Yoon in [**18**] that are needed to understand and solve some important, and open, classification theorems in the area of orthogonal polynomials and differential equations. One of these is the so-called $BKS(N, M)$ problem (see [**19**]), named after Bochner and Krall for their early investigations into orthogonal polynomial solutions to differential equations (see [**2**] and [**13**]):

Problem 1.1. The $BKS(N, M)$ problem. *Let $N \in \mathbf{N}$ and $M \in \mathbf{N}_0$. Classify, up to both a real and a complex linear change of variables, all ordinary differential expressions of order N of the form*

$$L_N[y](x) := a_N(x)y^{(N)}(x) + a_{N-1}(x)y^{(N-1)}(x) + \cdots + a_0(x)y(x) \tag{1.2}$$

for which

(i) *there exists a polynomial $p_n(x)$ of degree n (for each $n \in \mathbf{N}_0$) satisfying*

$$L_N[p_n](x) = \lambda_n p_n(x), \tag{1.3}$$

for some complex number λ_n, and

(ii) *there exist $M + 1$ moment functionals $\sigma_0, \sigma_1, \dots, \sigma_M$, with $\sigma_M \neq 0$, (equivalently, $M+1$ signed Borel measures $\mu_0, \mu_1, \dots \mu_M$ with $\int_{\mathbf{R}} |p|^2 \, d\mu_M \neq 0$ for some polynomial $p \neq 0$) such that the polynomials $\{p_n(x\}_{n=0}^{\infty}$ are orthogonal with respect to the Sobolev bilinear form*

$$(p, q)_M := \sum_{k=0}^{M} \langle \sigma_k, p^{(k)}\overline{q}^{(k)} \rangle \tag{1.4}$$
$$\left(\textit{equivalently, } (p, q)_M = \sum_{k=0}^{M} \int_{\mathbf{R}} p^{(k)}\overline{q}^{(k)} \, d\mu_k \right);$$

that is to say, there exists constants $K_n \neq 0$, $n \in \mathbf{N}_0$ such that

$$(p_n, p_m)_M = K_n \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta function. If $\{p_n(x)\}_{n=0}^{\infty}$ satisfies (1.3) and is orthogonal with respect to a bilinear form of the type (1.4), we write $\{p_n\} \in BKS(N, M)$.

At the time of this writing, only the $BKS(2,0)$, $BKS(2,1)$, $BKS(2,2)$ and $BKS(4,0)$ problems have been solved (see [**19**] for specific references). Indeed, Bochner [**2**] and Krall [**14**] solved the $BKS(2,0)$ problem under a complex linear change of variable, Kwon and Littlejohn [**15**] solved the $BKS(2,0)$ problem under a real linear change of variable, Krall [**13**] solved the $BKS(4,0)$ problem (under a complex linear change of variable), Kwon and Littlejohn [**16**] solved the $BKS(2,1)$ problem, and Kwon, Littlejohn and Lee [**17**] determined the contents of the $BKS(2,2)$ set. Furthermore, examples are known for many of the other BKS classes. All the equations in these known classes are symmetrizable and have a self-adjoint realization in some Hilbert or Krein space.

The structure of the $BKS(N,0)$ class, thanks in part to results as in [**18**], seems to be fairly well understood although no global solution to this problem is currently known. In fact, the exact contents of $BKS(6,0)$ are not explicitly known. Several conjectures about the content of this class are made in [**5**]; in particular, Conjecture 5.3 in [**5**] states that the only orthogonal polynomials (up to a complex linear change of variable) in the $BKS(N,0)$ class are the Hermite, Laguerre, Laguerre type, Jacobi, Jacobi type, and Bessel polynomials for certain restricted values of the polynomial parameters. Significant progress has also been made on determining the minimal order N of the corresponding differential equations; for example, see [**8**] and [**10**].

While it is possible to make *reasonable* conjectures for $BKS(N,0)$, it is not possible, at the moment, to make educated conjectures about $BKS(N,M)$ for $M \geq 1$ and $N > 2$. Many results for these classes are surprising and seemingly roguish in nature. Indeed, the example produced by Jung, Kwon, and Lee has dashed all hopes for a pattern of symmetrizability that seemed to be emerging in this general classification problem.

The contents of this paper are as follows. In Section 2 we discuss the main result in [**7**] that led these authors to the discovery of the differential expression $l_4[\cdot]$. Also in this section, we will study the properties of the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ and their relationship with the left Legendre type polynomials $\{PL_n(x)\}_{n=0}^{\infty}$ (see [**6**] and [**21**]). As we will see, the derivatives of the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ *are* the left Legendre type polynomials, which satisfy a fourth-order *formally symmetric* equation. It is this reason that there is a self-adjoint realization of the nonsymmetrizable expression $l_4[\cdot]$. Section 3 will deal with the spectral analysis of the fourth-order left Legendre type differential expression $m_4[y]$; we only state the main results of this analysis since this work was previously established in [**6**], [**12**] and [**21**]. In Section 4 we introduce the appropriate Sobolev-Hilbert space H where both the self-adjoint realization of $l_4[\cdot]$ and the orthogonality of the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ (to be precise, the sequence of ordered pairs $\{\langle Q_n, Q_n'(-1)\rangle\}_{n=0}^{\infty}$) live. Section 5 will deal with the self-adjoint operator S in H and its properties, generated by $l_4[\cdot]$. Section 6 will be concerned with the left-definite theory generated from the expression $l_4[\cdot]$ and a new type of orthogonality for $\{Q_n(x)\}_{n=0}^{\infty}$. Lastly, in Section 7, we discuss further examples of nonsymmetrizable differential equations having orthogonal polynomial solutions in the $BKS(6,1)$ and $BKS(8,1)$ classes as well as a conjecture concerning the $BKS(N,1)$ class. Also, in this section, we give a generalization of the main Jung, Kwon, Lee theorem (see Theorem 2.1 below). From this generalization, we produce a new example of a nonsymmetrizable fourth-order differential equation having a sequence of orthogonal polynomial solutions in the $BKS(4,2)$ class.

2. The result of Jung, Kwon and Lee. Recall (see [**3**]) that a moment functional $\tau : \mathcal{P} \to \mathbf{R}$ is a real-valued linear function on the set of one-variable polynomials $\mathcal{P}$. By Boas's theorem (see [**3**, p. 74]) τ has a representation as

$$\langle \tau, p\rangle = \int_{\mathbf{R}} p(x)\, d\mu, \quad p \in \mathcal{P},$$

where μ is some finite, (possibly) signed Borel measure. The moments of τ are the numbers

$$\tau_n := \langle \tau, x^n\rangle, \quad n \in \mathbf{N}_0.$$

We say that τ is quasi-definite, respectively, positive-definite, if, for each $n \in \mathbf{N}_0$,

$$\det(\tau_{i+j})_{i,j=0}^{n} \neq 0 \quad \text{respectively, } \det(\tau_{i+j})_{i,j=0}^{n} > 0.$$

A sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$, where $\deg(p_n) = n$ for each $n \in \mathbf{N}_0$, is said to be an orthogonal polynomial sequence with respect to τ if there exist nonzero numbers K_n, $n \in \mathbf{N}_0$, such that $\langle \tau, p_n p_m \rangle = K_n \delta_{n,m}$, $n, m \in \mathbf{N}_0$, where $\delta_{n,m}$ is the Kronecker delta function. It is well known (see [**3**, Chapter I, Section 3]) that τ is quasi-definite if and only if there exists an orthogonal polynomial sequence with respect to τ.

For what follows in this section, the notation $\langle \tau, pq \rangle$ denotes the action of the moment functional τ on the polynomial $p(x)q(x)$. Beginning in Section 4, the notation $\langle f, c \rangle$ will denote an ordered pair in a certain inner product space, with the first component being a function f and the second component of this pair being a complex number c; the contexts of these two identical notations should be clear and should not cause any confusion.

In [**7**], the authors prove the following theorem:

Theorem 2.1. *Consider the Sobolev bilinear form*

$$\phi(p, q) = \lambda p(c)\overline{q}(c) + \langle \tau, p'\overline{q}' \rangle \tag{2.1}$$

where τ is a quasi-definite moment functional and λ, c are real constants. Then $\phi(\cdot, \cdot)$ is quasi-definite if and only if $\lambda \neq 0$. Moreover, let $\{P_n(x)\}_{n=0}^\infty$ be the monic orthogonal polynomial sequence with respect to τ and, for fixed $\lambda \neq 0$, let $\{Q_n(x)\}_{n=0}^\infty$ denote the monic orthogonal polynomial sequence associated with $\phi(\cdot, \cdot)$. Then

$$Q_0(x) = 1; \qquad Q_n(x) = n \int_c^x P_{n-1}(t)\, dt, \quad n \in \mathbf{N}.$$

In particular, $Q_n(c) = 0$ for each $n \in \mathbf{N}$.

(i) *Suppose further that $\{P_n(x)\}_{n=0}^\infty \in BKS(N, 0)$; specifically, suppose that $y = P_n(x)$ satisfies the Nth-order differential equation*

$$m_N[y](x) = \mu_n y(x), \quad n \in \mathbf{N}_0,$$

where

$$m_N[y](x) = a_N(x)y^{(N)}(x) + a_{N-1}(x)y^{(N-1)}(x) + \cdots + a_1(x)y'(x). \tag{2.2}$$

If

$$\sum_{j=0}^{N-k} (-1)^j a_{k+j}^{(j)}(c) = 0, \quad 1 \leq k \leq N, \tag{2.3}$$

then $\{Q_n(x)\}_{n=0}^{\infty} \in BKS(N,1)$ *with* $y = Q_n(x)$ *satisfying the* N*th-order differential equation*

$$l_N[y](x) = \lambda_n y(x), \quad n \in \mathbf{N}_0,$$

where

$$l_N[y](x) = b_N(x)y^{(N)}(x) + b_{N-1}(x)y^{(N-1)}(x) + \cdots + b_1(x)y'(x), \tag{2.4}$$

with the coefficients of $l_N[\cdot]$ *being given by*

$$b_k(x) = \sum_{j=0}^{N-k} (-1)^j a_{k+j}^{(j)}(x), \quad 1 \le k \le N, \tag{2.5}$$

and

$$\lambda_n = \mu_{n-1} + b_1'(x), \quad \mu_{-1} = 0;\ n \in \mathbf{N}_0.$$

Moreover,

$$b_k(c) = 0, \quad 1 \le k \le N, \tag{2.6}$$

and, formally, $l_N'[y] = m_N[y']$.

(ii) *Conversely, suppose that* $\{Q_n(x)\}_{n=0}^{\infty} \in BKS(N,1)$ *with* $y = Q_n(x)$ *satisfying the* N*th-order differential equation*

$$l_N[y](x) = \lambda_n y(x),$$

where $l_N[\cdot]$ *is given in* (2.4). *Then the coefficients of* $l_N[\cdot]$ *satisfy* (2.6). *Furthermore* $\{P_n(x)\}_{n=0}^{\infty} \in BKS(N,0)$ *with* $y = P_n(x)$ *satisfying*

$$m_N[y](x) = \mu_n y(x), \quad n \in \mathbf{N}_0,$$

where $m_N[\cdot]$ *is given in* (2.2) *with the coefficients satisfying*

$$a_k(x) = \begin{cases} b_N(x) & \text{for } k = N \\ b_{k+1}'(x) + b_k(x) & \text{for } 1 \le k \le N-1, \end{cases}$$

and

$$\mu_n = \lambda_{n+1} - b_1'(x), \quad n \in \mathbf{N}_0.$$

Moreover, (2.3) *is satisfied and, formally,* $l'_N[y] = m_N[y']$.

Jung, Kwon and Lee applied this theorem to obtain a new orthogonal polynomial sequence $\{Q_n(x)\}_{n=0}^{\infty}$ in the $BKS(4,1)$ class. Indeed, from Theorem 2.1, the associated polynomial set $\{P_n(x)\}_{n=0}^{\infty}$ belongs to the $BKS(4,0)$ class and, hence, are necessarily one of the polynomial sets in the $BKS(4,0)$ class. The *only* fourth-order equation from this class that fits the conditions of Theorem 2.1 is

$$\begin{aligned} m_4[y](x) &:= (x^2-1)^2 y^{(4)} + 8x(x^2-1)y''' \\ &\quad + (x+1)((4A+14)x - 4A - 10)y'' \\ &\quad + ((8A+4)x+4)y' + ky \\ &= \left((1-x^2)^2 y''\right)'' \\ &\quad - 2\left[(1-x)\left((2A+1)x + 2A + 3\right)y'\right]' + ky, \end{aligned} \tag{2.7}$$

where $A \neq 0$ and k are fixed constants. The expression $m_4[\cdot]$ is called the *left Legendre type* differential expression. For each $n \in \mathbf{N}_0$, the equation

$$m_4[y](x) = \mu_n y(x),$$

where

$$\mu_n = n(n+1)(n^2+n+4A) + k \tag{2.8}$$

has a polynomial solution $PL_n(x)$ of degree n. These polynomials are called the *left Legendre type polynomials*; the nth degree monic left Legendre type polynomial $y = PL_n(x)$ is explicitly given by

$$PL_n(x) := \sum_{j=0}^{n} (-1)^j \frac{2^{n-j}(n!)^2(n+j)!(n^2+n+2A-j)}{(2n)!(j!)^2(n-j)!(n^2+2A)}(1-x)^j, \quad n \in \mathbf{N}_0. \tag{2.9}$$

In terms of the general Jacobi type polynomials studied by Koornwinder [**11**], we have

$$PL_n(x) = P_n^{(0,0,(1/A),0)}(x), \quad n \in \mathbf{N}_0. \tag{2.10}$$

Furthermore, when $A > 0$, the left Legendre type polynomials form a complete orthogonal set in the Hilbert space $L^2_\mu[-1,1)$ defined by

(2.11)
$$L^2_\mu[-1,1) := \{f : [-1,1) \to \mathbf{C} \mid f \text{ is Lebesgue msble and } (f,f)_\mu < \infty\},$$

where the positive-definite inner product $(\cdot,\cdot)_\mu$ is defined by

$$\begin{aligned}(p,q)_\mu := \int_{[-1,1)} p(x)\overline{q}(x)d\mu = \frac{p(-1)\bar{q}(-1)}{A} \\ + \int_{-1}^{1} p(x)\overline{q}(x)\,dx, \quad f,g \in L^2_\mu[-1,1).\end{aligned} \tag{2.12}$$

In fact, they satisfy the following orthogonality relation:

$$\begin{aligned}(PL_n, PL_m)_\mu = \frac{(n!)^4 2^{2n+1}(n^2+2n+2A+1)}{(2n+1)!(2n)!(n^2+2A)}\ \delta_{n,m}, \\ n,m \in \mathbf{N}_0.\end{aligned} \tag{2.13}$$

It follows from the conditions of the above theorem that:

(i) $c = 1$,

(ii) the monic polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ are given by

$$Q_0(x) = 1; \qquad Q_n(x) = n\int_1^x PL_{n-1}(t)\,dt, \quad n \in \mathbf{N}, \tag{2.14}$$

and, from (2.5),

(iii) the fourth-order differential expression $l_4[\cdot]$ is given explicitly by

$$\begin{aligned}l_4[y](x) := (x^2-1)^2 y^{(4)} + 4x(x^2-1)y''' \\ + 2(x-1)[(1+2A)x + 2A + 3]y'' + ky,\end{aligned} \tag{2.15}$$

and satisfies

$$l_4'[y] = m_4[y']. \tag{2.16}$$

Observe that $l_4[\cdot]$ is not Lagrangian symmetrizable; indeed, this follows since there is no first-order derivative term in the expression. On the other hand, from (2.7), $m_4[\cdot]$ is formally Lagrangian symmetric. For each $n \in \mathbf{N}_0$, $y = Q_n(x)$ satisfies

$$l_4[Q_n](x) = \lambda_n Q_n(x), \tag{2.17}$$

where

$$\lambda_n = n(n-1)(n^2 - n + 4A) + k, \quad n \in \mathbf{N}_0. \tag{2.18}$$

Furthermore, from (2.1) and (2.2), these polynomials are orthogonal with respect to the Sobolev inner product

$$(f, g)_1 := f(1)\bar{g}(1) + \frac{1}{A} f'(-1)\bar{g}'(-1) + \int_{-1}^{1} f'(x)\bar{g}'(x)\, dx. \tag{2.19}$$

In fact, $(Q_0, Q_0)_1 = 1$ and

(2.20)

$$(Q_n, Q_m)_1 = \frac{(n!)^2((n-1)!)^2 2^{2n-1}(n^2 + 2A)}{(2n-1)!(2n-2)!(n^2 - 2n + 2A + 1)}\, \delta_{n,m}, \quad n, m \in \mathbf{N}_0.$$

3. Spectral analysis of $m_4[\cdot]$. In this section, we review the work found in [**6**], [**12**] and [**21**] on the spectral analysis of the self-adjoint operator generated by $m_4[\cdot]$ having the left Legendre type polynomials (2.9) as eigenfunctions. For a further reference, a complete and comprehensive treatment of the spectral study of the Legendre type polynomials, which satisfy the fourth-order differential equation

$$(x^2-1)^2 y^{(4)} + 8x(x^2-1)y^{(3)} + (4A+12)(x^2-1)y'' + 8Axy' + ky = \lambda y,$$

can be found in [**4**]; many of the arguments used in [**4**] carry over with some modifications to the spectral study of the left Legendre type differential equation (2.7).

Until further notice, we will henceforth assume that A and k are fixed, positive constants (in Section 7 we will, at one point, set $A = -1$).

From the orthogonality of $\{PL_n(x)\}_{n=0}^{\infty}$, the study of any self-adjoint operator generated from $m_4[\cdot]$, having these polynomials as eigenfunctions, is necessarily in the Hilbert space $L^2_\mu[-1, 1)$. Even though the classical Glazman-Naimark-Krein theory [**22**], developed for the classical Lebesgue-Hilbert space $L^2(-1,1)$, cannot be directly applied to the spectral study of $m_4[\cdot]$ in $L^2_\mu[-1,1)$, this theory is essential in our ultimate construction of the self-adjoint operator T in $L^2_\mu[-1,1)$, having the left Legendre type polynomials as eigenfunctions.

To begin, the *maximal domain* $\Delta \subset L^2(-1,1)$ associated with $m_4[\cdot]$ is defined to be

$$\begin{aligned}\Delta = \{f : (-1,1) \longrightarrow \mathbf{C} \mid f^{(j)} \in AC_{\mathrm{loc}}(-1,1),\\ j = 0,1,2,3;\ f, m_4[f] \in L^2(-1,1)\}.\end{aligned} \tag{3.1}$$

For any $f, g \in \Delta$, we have *Green's formula*

$$\int_{-1}^{+1} [m_4[f](x)\bar{g}(x) - f(x)\overline{m_4[g]}(x)]\, dx = [f,g](x) \mid_{-1}^{1}, \tag{3.2}$$

where $[\cdot,\cdot]$ is the sesquilinear concomitant defined by

(3.3)
$$\begin{aligned}&[f,g](x)\\ &:= \left\{[(1-x^2)^2 f''(x)]' - 2(1-x)[(2A+1)x + 2A + 3]f'(x)\right\}\bar{g}(x)\\ &\quad - \left\{[(1-x^2)^2 \bar{g}''(x)]' - 2(1-x)[(2A+1)x + 2A + 3]\bar{g}'(x)\right\} f(x)\\ &\quad - (1-x^2)^2 \left\{f''(x)\bar{g}'(x) - \bar{g}''(x)f'(x)\right\}, \quad x \in (-1,1).\end{aligned}$$

Furthermore, for any compact interval $[\alpha,\beta] \subset (-1,1)$, we have *Dirichlet's formula*

(3.4)
$$\begin{aligned}&\int_\alpha^\beta m_4[f](x)\bar{g}(x)\, dx\\ &\quad = \Big[\left\{[(1-x^2)^2 f''(x)]' - 2(1-x)[(2A+1)x + 2A + 3]f'(x)\right\}\\ &\qquad\qquad \cdot\bar{g}(x) - (1-x^2)^2 f''(x)\bar{g}'(x)\Big]_\alpha^\beta\\ &\qquad + \int_\alpha^\beta \big\{(1-x^2)^2 f''(x)\bar{g}''(x) + 2(1-x)[(2A+1)x + 2A + 3]\\ &\qquad\qquad \cdot f'(x)\bar{g}'(x) + kf(x)\bar{g}(x)\big\}\, dx.\end{aligned}$$

Note that, by definition of Δ, the limits $\lim_{x\to\pm 1}[f,g](x) := [f,g](\pm 1)$ exist and are finite for all $f, g \in \Delta$. In [**6**] and [**21**], the following theorem is established concerning smoothness properties of functions in Δ.

Theorem 3.1. *Let $f, g \in \Delta$. Then*

(i) $f'' \in L^2(-1, 0]$ *so that* $f, f' \in AC[-1, 0]$;

(ii) $\lim_{x\to -1}[(1-x^2)^2 f''(x)]' = 0$;

(iii) $\lim_{x\to -1}(1-x^2)^2 f''(x)\bar{g}'(x) = 0$;

(iv) $\lim_{x\to -1}[f, g](x) = -8\,[f'(-1)\bar{g}(-1) - \bar{g}'(-1)f(-1)]$.

In order to define the self-adjoint operator T in $L^2_\mu[-1, 1)$, generated from $m_4[\cdot]$, having the Legendre type polynomials $\{PL_n(x)\}_{n=0}^\infty$ as eigenfunctions, it is necessary to define the following two "Glazman boundary" functions and to consider a certain subspace of Δ. Indeed, construct $h_1, h_2 \in C^4[-1, 1] \subset \Delta$ such that

$$h_1(x) = \begin{cases} 0 & \text{for } x \text{ near } -1 \\ 1 & \text{for } x \text{ near } 1, \end{cases} \qquad h_2(x) = \begin{cases} 0 & \text{for } x \text{ near } -1 \\ 1-x & \text{for } x \text{ near } 1. \end{cases} \tag{3.5}$$

Define

$$\delta := \{f \in \Delta \mid [f, h_j](1) = 0,\ j = 1, 2\}. \tag{3.6}$$

An elementary calculation shows that $\{PL_n(x)\}_{n=0}^\infty \subset \delta$; consequently, since the space of polynomials $\mathcal{P}$ is dense in $L^2_\mu[-1, 1]$, we see that δ is a dense subspace of $L^2_\mu[-1, 1]$. The following theorem (see [**6**] and [**21**]) shows that functions in δ enjoy rather surprising smoothness conditions on the closed interval $[-1, 1]$.

Theorem 3.2. *Suppose $f, g \in \delta$. Then*

(i) $f'' \in L^2(-1, 1)$ *so that* $f, f' \in AC[-1, 1]$;

(ii) $\lim_{x\to \pm 1}[(1-x^2)^2 f''(x)]' = 0$;

(iii) $\lim_{x\to \pm 1}(1-x^2)^2 f''(x)\bar{g}'(x) = 0$;

(iv) $\lim_{x\to +1}[f, g](x) = 0$.

From Theorems 3.1 and 3.2, it follows that Green's formula, restricted to functions $f, g \in \delta$, reduces to

$$\int_{-1}^{+1} \left[m_4[f](x)\bar{g}(x) - f(x)\overline{m_4[g]}(x)\right] dx = 8\,[f'(-1)\bar{g}(-1) - f(-1)\bar{g}'(-1)]. \tag{3.7}$$

Moreover, from Theorem 3.2 (i), (ii), (iii) and (iv), we deduce from (3.4) that, for $f, g \in \delta$,

$$\begin{aligned}\int_{-1}^{+1} m_4[f](x)\bar{g}(x)\,dx &= \int_{-1}^{+1} \big\{(1-x^2)^2 f''(x)\bar{g}''(x) + 2(1-x)[(2A+1)x + 2A + 3] \\ &\qquad \cdot f'(x)\bar{g}'(x) + kf(x)\bar{g}(x)\big\}\,dx \\ &\quad + 8f'(-1)g(-1).\end{aligned} \tag{3.8}$$

We are now in a position to define the operator T in $L^2_\mu[-1,1)$, generated from $m_4[\cdot]$, having the Legendre type polynomials as eigenfunctions.

Definition 3.1. Let $T : \mathcal{D}(T) \subset L^2_\mu[-1,1) \to L^2_\mu[-1,1)$ be the operator defined by

$$T[f](x) = \begin{cases} -8Af'(-1) + kf(-1) & \text{if } x = -1 \\ m_4[f](x) & \text{for a.e. } x \in (-1,1) \end{cases} \qquad f \in \mathcal{D}(T) := \delta, \tag{3.9}$$

where δ is defined in (3.6).

An elementary calculation shows, for each $n \in \mathbf{N}_0$, that $T[PL_n](x) = \mu_n PL_n(x)$ where μ_n is defined in (2.8). Notice, from the definition of T and (3.8), we have

$$\begin{aligned}(T[PL_n], PL_m)_\mu &= k(PL_n, PL_m)_\mu \\ &\quad + \int_{-1}^{1} \big\{(1-x^2)^2 PL_n''(x)PL_m''(x) + 2(1-x)((2A+1)x + 2A + 3) \\ &\qquad \cdot PL_n'(x)PL_m'(x)\big\}\,dx,\end{aligned}$$

and, in particular, that

(3.10)
$$
\begin{aligned}
k(PL_n, PL_m)_\mu + \int_{-1}^{1} &\big\{(1-x^2)^2 PL_n''(x) PL_m''(x) \\
&+ 2(1-x)((2A+1)x + 2A + 3) PL_n'(x) PL_m'(x)\big\}\, dx \\
= \frac{(n!)^4 2^{2n+1}(n^2+2n+2A+1)(n(n+1)(n^2+n+4A)+k)}{(2n+1)!(2n)!(n^2+2A)}\, \delta_{n,m}&, \\
n, m \in \mathbf{N}_0&.
\end{aligned}
$$

This is the *left-definite* orthogonality relationship which will be discussed in further detail in Section 6 below.

Using (3.7), it is routine to check that T is symmetric in $L^2_\mu[-1,1)$. In fact, T is self-adjoint as claimed in the next theorem (see [**21**] and [**4**]). This fact, together with several key results needed to establish this theorem below, holds the key for the spectral analysis of the expression $l_4[\cdot]$ in the Hilbert space H defined below in Section 4.

Theorem 3.3. *For $A, k > 0$, the operator T, defined in* (3.9)*, is a self-adjoint operator in $L^2_\mu[-1,1)$.*

Moreover, the spectrum of T is simple, discrete, and given by $\sigma(T) = \{\mu_n \mid n \in \mathbf{N}_0\}$, where μ_n is defined in (2.8)*. The corresponding eigenfunctions are the left Legendre type polynomials $\{PL_n(x)\}_{n=0}^\infty$.*

Remark 3.1. The proof, given in [**21**, pp. 121–128] (see also [**4**]), of the self-adjointness of the operator T involves several steps, including a key application of the classical Glazman-Krein-Naimark (GKN) theory (see [**22**, pp. 74–76]). Indeed, a self-adjoint operator N in the classical Lebesgue space $L^2(-1,1)$ is constructed having GKN boundary conditions. Since this operator N is important for later use in this paper, we now describe this operator and some of its properties. Since $x = \pm 1$ are both regular singular points of $m_4[y] = 0$, we can apply the method of Frobenius to show that $m_4[\cdot]$ is in the limit-4 case at $x = 1$ and in the limit-3 case at $x = -1$. Hence, from the GKN theory, two appropriate separated boundary conditions at $x = 1$ and one appropriate separated boundary condition at $x = -1$ are needed in order to obtain a self-adjoint operator in $L^2(-1,1)$ generated from $m_4[\cdot]$. In particular, the operator $N : L^2(-1,1) \to L^2(-1,1)$ defined by

$$N[f](x) = m_4[f](x), \quad \text{a.e. } x \in (-1,1),$$
$$f \in \mathcal{D}(N) := \{f \in \Delta \mid [f,h_1](1) = [f,h_2](1) = [f,h_3](-1) = 0\}$$

is self-adjoint, where $h_3 \in C^4[-1,1]$ is constructed so that

$$h_3(x) = \begin{cases} 1 & x \text{ near } -1 \\ 0 & x \text{ near } +1. \end{cases}$$

Moreover, N is bounded below in $L^2(-1,1)$ by kI, where I is the identity operator. Since $k > 0$, we can therefore conclude that $0 \in \rho(N)$, the resolvent set of N. From the method of Frobenius, the indicial equation about $x = -1$ is $(\rho-2)(\rho-1)\rho(\rho+1) = 0$ and four linearly independent solutions of $m_4[y] = 0$, expanded about the point $x = -1$, have the following optimal forms:

$$\begin{aligned}
\psi_1(x) &= (x+1)^2 \sum_{n=0}^{\infty} a_n(x+1)^n, \quad (a_0 = 1),\\
\psi_2(x) &= (x+1)\sum_{n=0}^{\infty} b_n(x+1)^n + (x+1)^2\log(1+x)\sum_{n=0}^{\infty} c_n(x+1)^n,\\
&\quad \left(b_0 = \frac{1}{60},\ c_0 = \frac{A}{180}\right),\\
\psi_3(x) &= \sum_{n=0}^{\infty} d_n(x+1)^n + (1+x)^2\log(1+x)\sum_{n=0}^{\infty} e_n(x+1)^n,\\
&\quad \left(d_0 = -\frac{1}{1800},\ e_0 = \frac{-16A+2k}{43200}\right),\\
\psi_4(x) &= \sum_{n=0}^{\infty} f_n(x+1)^{n-1} + \log(1+x)\sum_{n=0}^{\infty} g_n(x+1)^n,\\
&\quad \left(f_n = -\frac{1}{7200},\ g_0 = -\frac{3A}{7200}\right).
\end{aligned}$$

It is easy to see that $\psi_i \in \Delta$ for $i = 1,2,3$. Moreover, we can choose constants $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{C}$ such that

$$\psi(x) := \alpha_1\psi_1(x) + \alpha_2\psi_2(x) + \alpha_3\psi_3(x) \tag{3.11}$$

satisfies

$$[\psi, h_1](1) = [\psi, h_2](1) = 0,$$

where $[\cdot,\cdot]$ is the bilinear form defined in (3.3); that is to say, $\psi \in \delta$. It follows then that $\psi \notin \mathcal{D}(N)$. For, if $\psi \in \mathcal{D}(N)$, then ψ is an eigenfunction of N corresponding to the eigenvalue $\lambda = 0$, contradicting the fact that $0 \in \rho(N)$. By scalar multiplication and Theorem 3.1 (iv), we can therefore assume that $\psi'(-1) = 1$.

Remark 3.2. Another key result in establishing the self-adjointness of the operator T is the fact that the related symmetric operator $T_1 : L^2_\mu[-1,1) \to L^2_\mu[-1,1)$ defined by

$$T_1[f](x) = \begin{cases} -8Af'(-1) & \text{if } x = -1 \\ m_4[f](x) & \text{a.e. } x \in (-1,1) \end{cases}$$
$$\mathcal{D}(T_1) = \delta$$

is onto $L^2_\mu[-1,1)$ and therefore is self-adjoint (see [**1**, Section 41]). In particular, if $g \in L^2(-1,1)$, there exists $h \in \delta$ such that $m_4[h](x) = g(x)$ for $x \in (-1,1)$. This fact will be important in subsequent discussions below.

4. The space H. The maximal vector space of functions for which the bilinear form $(\cdot,\cdot)_1$, given in (2.19), is well-defined is

$$Y := \{f : [-1,1] \longrightarrow \mathbf{C} \mid f'(-1) \text{ exists and is finite}; \\ f' \text{ exists a.e. } x \in (-1,1];\ f' \in L^2(-1,1)\}.$$

However, $(Y, (\cdot,\cdot)_1)$ is not a Hilbert space; in fact, $(\cdot,\cdot)_1$ is only a pseudo inner product on $Y \times Y$. In this section we find the completion of this space. To begin, define

$$H_0 := \{f : [-1,1] \longrightarrow \mathbf{C} \mid f \in AC[-1,1];\ f' \in L^2(-1,1)\}, \tag{4.1}$$

and endow it with the inner product

$$(f,g)_{H_0} := f(1)\bar{g}(1) + \int_{-1}^{1} f'(x)\bar{g}'(x)\,dx \quad f, g \in H_0.$$

Lemma 4.1. *The space* $(H_0, (\cdot,\cdot)_{H_0})$ *is a Hilbert space.*

Proof. Suppose $\{f_n\}_{n=1}^{\infty} \subset H_0$ is Cauchy. In particular, $\{f_n(1)\}_{n=1}^{\infty} \subset \mathbf{C}$ is Cauchy and $\{f_n'\}_{n=1}^{\infty}$ is Cauchy in $L^2(-1,1)$. Hence, there exists $\alpha \in \mathbf{C}$ and $f \in L^2(-1,1)$ such that

$$|f_n(1) - \alpha| \longrightarrow 0, \quad n \to \infty,$$

and

$$\int_{-1}^{1} |f_n'(x) - f(x)|^2 \, dx \longrightarrow 0, \quad n \to \infty.$$

Define $g : [-1,1] \to \mathbf{C}$ by

$$g(x) = \alpha - \int_x^1 f(x)\, dx.$$

Then $g \in AC[-1,1]$; moreover, $g(1) = \alpha$ and, for almost every $x \in [-1,1]$, $g'(x) = f(x) \in L^2(-1,1)$. Hence, $g \in H_0$. Finally

$$\begin{aligned} \|f_n - g\|_{H_0}^2 &= |f_n(1) - g(1)|^2 + \int_{-1}^{1} |f_n'(x) - g'(x)|^2 \, dx \\ &= |f_n(1) - \alpha|^2 + \int_{-1}^{1} |f_n'(x) - f(x)|^2 \, dx \\ &\longrightarrow 0, \quad n \to \infty. \qquad \square \end{aligned}$$

Let $H := H_0 \times \mathbf{C}$ and, for ordered pairs $\langle f, \alpha \rangle$ and $\langle g, \beta \rangle$ in H, define the inner product $(\cdot,\cdot)_H$ on H as

$$(\langle f, \alpha \rangle, \, \langle g, \beta \rangle)_H := (f,g)_{H_0} + \frac{\alpha\overline{\beta}}{A}.$$

Observe that

$$(\langle f, f'(-1) \rangle, \, \langle g, g'(-1) \rangle) = (f,g)_1, \quad f, g \in Y. \tag{4.2}$$

We now show that $(H, (\cdot,\cdot)_H)$ is a complete Hilbert space and that completion of Y in the norm generated from (2.19) is $(H, (\cdot,\cdot)_H)$.

Theorem 4.1. *The space* $(H, (\cdot,\cdot)_H)$ *is complete.*

Proof. Suppose $\{\langle f_n, \alpha_n\rangle\}_{n=0}^{\infty}$ is Cauchy in H. It follows that $\{f_n'\}_{n=0}^{\infty}$ is Cauchy in $L^2(-1,1)$, $\{\alpha_n\}_{n=0}^{\infty}$ is Cauchy in $\mathbf{C}$, and $\{f_n(1)\}_{n=0}^{\infty}$ is Cauchy in $\mathbf{C}$. Hence there exist $g \in L^2(-1,1)$, $\alpha, \beta \in \mathbf{C}$, such that

$$\| f_n' - g \|_2 \longrightarrow 0, \qquad |\alpha_n - \alpha| \longrightarrow 0, \qquad |f_n(1) - \beta| \longrightarrow 0, \quad n \to \infty.$$

Define $h : [-1,1] \to \mathbf{C}$ by

$$h(x) = \beta - \int_x^1 g(t)\, dt, \quad x \in [-1,1].$$

Then $h \in AC[-1,1]$ and $h' = g \in L^2(-1,1)$ so $h \in H_0$; consequently, $\langle h, \alpha\rangle \in H$. Moreover, noting that $h(1) = \beta$, we see that

$$\begin{aligned}\| \langle h,\alpha\rangle - \langle f_n, \alpha_n\rangle \|_H^2 &= \frac{|\alpha - \alpha_n|^2}{A} + |h(1) - f_n(1)|^2 + \| h' - f_n' \|_2^2 \\ &\longrightarrow 0 \quad \text{as } n \to \infty.\end{aligned}$$

In order to see the relationship between Y and H, we define the operator $P : Y \to H$ by

$$P(f) = \left\langle f(1) - \int_x^1 f'(t)\, dt,\ f'(-1)\right\rangle.$$

Notice, from (4.2), that

$$(f,g)_1 = (P(f), P(g))_H, \quad f, g \in Y; \tag{4.3}$$

furthermore, if $f \in Y$ and $f \in AC[-1,1]$, then $P(f) = \langle f, f'(-1)\rangle$.

Theorem 4.2. *The mapping* P *is a linear isometry onto* H. *Hence,* Y *can be represented as the Hilbert space* $(H, (\cdot,\cdot)_H)$ *through the mapping* P. *More specifically,* $Y/\ker(P)$ *is isometrically isomorphic to* H.

Proof. Clearly, P is linear and (4.3) shows that P is a pseudo-distance preserving map between Y and $P(Y)$. It remains to show that $P(Y) = H$. Since the kernel of P is

$$\ker(P) = \{f \in Y \mid \| f \|_H = 0\},$$

it will follow that $Y/\ker(P)$ is isometrically isomorphic to H and hence that H can be identified as the completion of Y under the isometry P. Let $(f, c) \in H$. Since $f \in AC[-1,1]$, f is uniformly continuous on $[-1,1]$. Consequently, for each $n \in \mathbf{N}$, there exists $m_n \in \mathbf{N}$ such that if $x, y \in [-1,1]$ and $|x-y| \leq 1/(n(n+1)m_n)$, then

$$|f(x) - f(y)| \leq \frac{1}{n(n+1)}.$$

For each $n \in \mathbf{N}$, let $t_n = -1 + (1/n)$. For each $n \in \mathbf{N}$ and each $i \in \{0, 1, \ldots, m_n\}$, let $u_{n,i} = ((m_n - i)/m_n)t_n + (i/m_n)t_{n+1}$. Observe that

$$\begin{aligned} 0 = t_1 = u_{1,0} > u_{1,1} > \cdots > u_{1,m_1} = t_2 = u_{2,0} \\ > \cdots > u_{2,m_2} = t_3 = u_{3,0} > \cdots \end{aligned}$$

with the limit of elements in this ordering being -1. Define a function $h : [-1,1] \to \mathbf{C}$ by

$$h(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x) - f(u_{n,i}) + c(1 + u_{n,i}) & \text{if } x \in [u_{n,i}, u_{n,i-1}) \\ & \text{for some } n \in \mathbf{N} \text{ and } 0 \leq i \leq m_n \\ f(x) & \text{if } x \in [0,1]. \end{cases}$$

Note that $h(-1) = 0$, $h'(x) = f'(x)$ almost everywhere $x \in [-1,1]$ and $h(1) = f(1)$. In particular, since $f \in AC[-1,1]$, we have $h(1) - \int_{-1}^{x} h'(t)\, dt = f(1) - \int_{-1}^{x} f'(t)\, dt = f$. Also, for $x \in [u_{n,i}, u_{n,i-1})$,

$$\frac{h(x) - h(-1)}{x+1} = c + c\,\frac{u_{n,i} - x}{x+1} + \frac{f(x) - f(u_{n,i})}{x+1}.$$

Now $|f(x) - f(u_{n,i})| \leq 1/(n(n+1))$ since $|x - u_{n,i}| \leq |u_{n,i-1} - u_{n,i}| = 1/(n(n+1)m_n)$. Thus

$$\left|\frac{f(x) - f(u_{n,i})}{x+1}\right| \leq \frac{1}{n(n+1)}\,\frac{1}{t_{n+1}+1} = \frac{1}{n} \to 0 \quad \text{as } x \to -1^+,$$

and

$$\left|\frac{u_{n,i} - x}{x+1}\right| \leq \frac{1}{n(n+1)m_n}\,\frac{1}{t_{n+1}+1} \leq \frac{1}{n} \to 0 \quad \text{as } x \to -1^+.$$

Thus $h'(-1)$ exists and equals c. Hence, $h \in Y$ and $P(h) = \langle f, c\rangle$ so that P is onto H. □

Recall that $\mathcal{P}$ denotes the set of all (complex-valued) polynomials in the real variable x. We now set out to show that the set

$$\mathcal{P}_H := P(\mathcal{P}) = \{\langle p, p'(-1)\rangle \mid p \in \mathcal{P}\} \tag{4.4}$$

is dense in H.

Theorem 4.3. *The set* $\{\langle Q_n, Q_n'(-1)\rangle\}_{n=0}^{\infty}$ *is a complete orthogonal set in* $(H, (\cdot,\cdot)_H)$*; here* $Q_n(x)$ *is the polynomial defined in* (2.14)*. Consequently,* $\mathcal{P}_H$ *is dense in* $(H, (\cdot,\cdot)_H)$.

Proof. First, for any $n, m \in \mathbf{N}_0$, we have from (2.20) that

$$\begin{aligned}
&(\langle Q_n, Q_n'(-1)\rangle,\, \langle Q_m, Q_m'(-1)\rangle)_H \\
&\qquad = Q_n(1)Q_m(1) + \frac{1}{A}\,Q_n'(-1)Q_m'(-1) + \int_{-1}^{1} Q_n'(t)Q_m'(t)\,dt \\
&\qquad = (Q_n, Q_m)_1 \\
&\qquad = \frac{(n!)^2((n-1)!)^2 2^{2n-1}(n^2+2A)}{(2n-1)!(2n-2)!(n^2-2n+2A+1)}\,\delta_{n,m}
\end{aligned}$$

so $\{\langle Q_n, Q_n'(-1)\rangle\}_{n=0}^{\infty}$ is an orthogonal set in $(H, (\cdot,\cdot)_H)$. Suppose $\langle f, \alpha\rangle \in H$ satisfies

$$(\langle f, \alpha\rangle,\, \langle Q_n, Q_n'(-1)\rangle)_H = 0, \quad n \in \mathbf{N}_0.$$

That is to say,

$$f(1)Q_n(1) + \frac{1}{A}\,Q_n'(-1)\alpha + \int_{-1}^{1} f'(x)Q_n'(x)\,dx = 0, \quad n \in \mathbf{N}_0. \tag{4.5}$$

In particular, for $n = 0$, we see that $f(1) = 0$. Furthermore, from (2.14), we have $Q_n'(x) = nPL_{n-1}(x)$ so (4.5) can be rewritten as

$$\frac{1}{A} PL_{n-1}(-1)\alpha + \int_{-1}^{1} f'(x) PL_{n-1}(x)\, dx = 0, \quad n \in \mathbf{N}. \tag{4.6}$$

If we let $n = 1$, then (4.6) implies that

$$\frac{\alpha}{A} + \int_{-1}^{1} f'(x)\, dx = 0. \tag{4.7}$$

It follows, since $f(1) = 0$ and f is absolutely continuous on $[-1, 1]$, that

$$f(-1) = \frac{\alpha}{A}.$$

Hence we can rewrite (4.6) as

$$f(-1)PL_{n-1}(-1) + \int_{-1}^{1} f'(x) PL_{n-1}(x)\, dx = 0, \quad n \in \mathbf{N}. \tag{4.8}$$

Integrating by parts, we see that (4.8) simplifies to

$$\int_{-1}^{1} f(x) PL_{n-1}'(x)\, dx = 0, \quad n \in \mathbf{N}.$$

Since $\{PL_n'(x)\}_{n=0}^{\infty}$ is a basis for the space of polynomials we have, by Weierstrass' approximation theorem, that $f(x) \equiv 0$ on $[-1, 1]$. Hence, since $A > 0$, we see from (4.7) that $\alpha = 0$, i.e., $\langle f, \alpha \rangle = \langle \mathbf{0}, 0 \rangle$ showing that $\{\langle Q_n, Q_n'(-1) \rangle\}_{n=0}^{\infty}$ is a complete orthogonal set in $(H, (\cdot, \cdot)_H)$. The density of $\mathcal{P}_H$ now follows, completing the proof of the theorem. $\square$

5. The self-adjoint operator S**.** From the fundamental relation (2.16), it is natural to define the following two sets:

$$\Delta' := \{f : (-1, 1) \longrightarrow \mathbf{C} \mid f' \in \Delta\} \tag{5.1}$$

and

$$\delta' := \{f : (-1, 1) \longrightarrow \mathbf{C} \mid f' \in \delta\}, \tag{5.2}$$

where Δ and δ are defined in, respectively, (3.1) and (3.6). Since

$$\begin{aligned}\Delta' = \{f : (-1,1) \longrightarrow \mathbf{C} \mid f^{(j)} \in AC_{\mathrm{loc}}(-1,1),\\ j = 0,1,2,3,4;\ f', l_4'[f] \in L^2(-1,1)\},\end{aligned} \tag{5.3}$$

we can view Δ' as the maximal domain of $l_4[\cdot]$ in $L^2(-1,1)$ endowed with inner product

$$(f,g)' := \int_{-1}^{1} f'(t)\overline{g}'(t)\,dt.$$

Observe that

$$\mathcal{P} \subset \delta',$$

where $\mathcal{P}$ is the space of all polynomials. The space δ', as we will shortly see, will prove crucial in developing the spectral theory of $l_4[\cdot]$ in the Hilbert space H, defined in the previous section. The next result follows immediately from Theorem 3.1 and Theorem 3.2.

Theorem 5.1. *Suppose $f, g \in \delta'$. Then*

(i) $f''' \in L^2(-1,1)$ so that $f, f', f'' \in AC[-1,1]$;

(ii) $\lim_{x\to\pm 1}[(1-x^2)^2 f'''(x)]' = 0$;

(iii) $\lim_{x\to\pm 1}(1-x^2)^2 f'''(x)\overline{g}''(x) = 0$;

(iv) $\lim_{x\to +1}[f', g'](x) = 0$;

(v) $\lim_{x\to -1}[f', g'](x) = -8[f''(-1)\overline{g}'(-1) - \overline{g}''(-1)f'(-1)]$.

Notice, from the definition of Δ' and Theorem 3.2, that if $f \in \Delta'$, then $f'(-1)$ exists and both f and $l_4[f]$ belong to H_0, where H_0 is defined in (4.1). Consequently, $\langle f, f'(-1)\rangle \in H$ and $\langle l_4[f], c\rangle \in H$ for all $f \in \Delta'$ and all $c \in \mathbf{C}$. Moreover, from Theorem 5.1 and the fact that we can write

$$\begin{aligned}l_4[f](x) = ((x^2-1)^2 f'''(x))' + 2(x-1)[(1+2A)x + 2A + 3]\\ \cdot f''(x) + kf(x), \quad f \in \delta';\ x \in (-1,1),\end{aligned}$$

we have

$$\lim_{x\to 1} l_4[f](x) = kf(1), \quad f \in \delta'. \tag{5.4}$$

Define

$$\mathcal{D} := \{\langle f, f'(-1)\rangle \mid f \in \delta'\}. \tag{5.5}$$

Since $\mathcal{P}_H \subset \mathcal{D} \subset H$, we see, from Theorem 4.4, that $\mathcal{D}$ is a dense subspace of H.

Definition 5.1. The operator $S : \mathcal{D}(S) \subset H \to H$ is given by

$$\begin{gathered} S[\langle f, f'(-1)\rangle] = \langle l_4[f], -8Af''(-1) + kf'(-1)\rangle \\ \langle f, f'(-1)\rangle \in \mathcal{D}(S) := \mathcal{D}. \end{gathered} \tag{5.6}$$

In this section we show that S is self-adjoint with discrete spectrum

$$\sigma(S) = \{\lambda_n \mid n \in \mathbf{N}_0\},$$

where λ_n is defined in (2.18). We begin by showing:

Theorem 5.2. *The operator S is symmetric and bounded below by kI in H, where I is the identity operator in H. In particular, $0 \in \rho(S)$, the resolvent set of S. Furthermore, for each $n \in \mathbf{N}_0$, $\langle Q_n, Q_n'(-1)\rangle$ is an eigenfunction of S corresponding to the eigenvalue λ_n.*

Proof. Let $\langle f, f'(-1)\rangle$, $\langle g, g'(-1)\rangle \in \mathcal{D}$. Then, from (5.4) and Theorem 5.1, we have

$$\begin{aligned} &\left(S\left[\langle f, f'(-1)\rangle\right], \langle g, g'(-1)\rangle\right)_H \\ &\qquad = (\langle l_4[f], -8Af''(-1) + kf'(-1)\rangle, \langle g, g'(-1)\rangle)_H \\ &\qquad = l_4[f](1)\bar{g}(1) + \frac{1}{A}\,(-8Af''(-1) + kf'(-1))\bar{g}'(-1) \\ &\qquad\quad + \int_{-1}^{1} l_4'[f](x)\bar{g}'(x)\,dx \\ &\qquad = kf(1)\bar{g}(1) + \frac{1}{A}\,(-8Af''(-1) + kf'(-1))\bar{g}'(-1) \\ &\qquad\quad + \int_{-1}^{1} m_4[f'](x)\bar{g}'(x)\,dx \end{aligned}$$

$$\begin{aligned}
&= kf(1)\bar{g}(1) + \frac{1}{A}\left(-8Af''(-1) + kf'(-1)\right)\bar{g}'(-1) \\
&\quad + \int_{-1}^{1} f'(x)\bar{m}_4[g'](x)\,dx \\
&\quad + \lim_{x\to 1}[f', g'](x) - \lim_{x\to -1}[f', g'](x) \\
&= kf(1)\bar{g}(1) + \frac{1}{A}\left(-8Af''(-1) + kf'(-1)\right)\bar{g}'(-1) \\
&\quad + \int_{-1}^{1} f'(x)\overline{m_4[g']}(x)\,dx \\
&\quad + 8f''(-1)\bar{g}'(-1) - 8f'(-1)\bar{g}''(-1) \\
&= kf(1)\bar{g}(1) - 8f'(-1)\bar{g}''(-1) + \frac{k}{A}\,f'(-1)\bar{g}'(-1) \\
&\quad + \int_{-1}^{1} f'(x)\overline{l_4'[g]}(x)\,dx \\
&= f(1)\overline{l_4[g]}(1) + \frac{1}{A}\left(-8A\bar{g}''(-1) + k\bar{g}'(-1)\right)f'(-1) \\
&\quad + \int_{-1}^{1} f'(x)\overline{\ell_4'[g]}(x)\,dx \\
&= \big(\langle f, f'(-1)\rangle,\ S\left[\langle g, g'(-1)\rangle\right]\big)_H,
\end{aligned}$$

showing that S is symmetric. Moreover, from (3.8), we see that for all $\langle f, f'(-1)\rangle \in \mathcal{D}$,

$$\begin{aligned}
&\big(S\big[\langle f, f'(-1)\rangle\big],\ \langle f, f'(-1)\rangle\big)_H \\
&\quad = k\,|f(1)|^2 + \frac{1}{A}\left(-8Af''(-1) + kf'(-1)\right)\bar{f}'(-1) \\
&\qquad + \int_{-1}^{1} m_4[f'](x)\overline{f}'(x)\,dx \\
&\quad = k\,|f(1)|^2 + \frac{1}{A}\left(-8Af''(-1) + kf'(-1)\right)\bar{f}'(-1) + 8f''(-1)\bar{f}'(-1) \\
&\qquad + \int_{-1}^{+1} \big\{(1-x^2)^2 f'''(x)\bar{f}'''(x) \\
&\qquad\qquad + 2(1-x)[(2A+1)x + 2A + 3]f''(x)\bar{f}''(x)\big\}\,dx \\
&\qquad + k\int_{-1}^{1} f'(x)\bar{f}'(x)\,dx
\end{aligned}$$

$$\begin{aligned}
&= k\big(\langle f, f'(-1)\rangle, \langle f, f'(-1)\rangle\big)_H \\
&\quad + \int_{-1}^{+1} \big\{(1-x^2)^2 f'''(x)\bar{f}'''(x) \\
&\qquad\qquad + 2(1-x)[(2A+1)x + 2A + 3] f''(x)\bar{f}''(x)\big\}\, dx \\
&\geq k\big(\langle f, f'(-1)\rangle, \langle f, f'(-1)\rangle\big)_H,
\end{aligned}$$

since the functions $(1-x^2)^2$ and $2(1-x)[(2A+1)x+2A+3]$ are nonnegative on $[-1,1]$. This shows that S is bounded below by kI in H. On account of (2.17), in order to show that the ordered pair $\langle Q_n, Q_n'(-1)\rangle$ is an eigenfunction of S for each $n \in \mathbf{N}_0$, it suffices to show that

$$-8AQ_n''(-1) + kQ_n'(-1) = \lambda_n Q_n(-1).$$

Since $y = PL_{n-1}(x)$ satisfies $m_4[PL_{n-1}](x) = (n(n-1)(n^2-n+4A)+k)PL_{n-1}(x)$, we see that in particular

$$\begin{aligned}
(n(n-1)&(n^2-n+4A)+k)PL_{n-1}(-1) \\
&= m_4[PL_{n-1}](-1) \\
&= -8APL_{n-1}'(-1) + kPL_{n-1}(-1).
\end{aligned} \tag{5.7}$$

Substitution of $Q_n'(-1)/n = PL_{n-1}(-1)$ and $Q_n''(-1)/n = PL_{n-1}'(-1)$ into (5.7) yields the result and completes the proof. $\square$

In order to show that S is self-adjoint in H, we first need to establish the self-adjointness of the related operator $S_1 : H \to H$ defined by

$$\begin{aligned}
S_1(\langle f, f'(-1)\rangle) &= \langle l_4[f], -8Af''(-1)\rangle \\
\mathcal{D}(S_1) &= \mathcal{D}.
\end{aligned}$$

Theorem 5.3. *The operator S_1 is self-adjoint.*

Proof. The proof that S_1 is symmetric is identical to the proof in Theorem 5.2. To show that S_1 is self-adjoint, we show that S_1 is onto H. It will follow from [**1**, Section 41] that S_1 is self-adjoint. Let $\langle g, c\rangle \in H$; notice that $g \in AC[-1,1]$ and $g' \in L^2(-1,1)$. From Remark 3.2, there exists $h \in \delta$ such that

$$m_4[h](x) = g'(x), \quad \text{a.e. } x \in (-1,1).$$

Moreover let $\psi(x)$ be the function defined in (3.11); then $\psi \in \delta$, $\psi'(-1) \neq 0$, and ψ is a solution of $m_4[y] = 0$ on $(-1, 1)$. Define $F : [-1, 1] \to \mathbf{C}$ by

$$\begin{aligned} F(x) = &\int_{-1}^{x} h(t)\,dt - \int_{-1}^{1} h(t)\,dt \\ &+ \frac{g(1)}{k} + \frac{(8Ah'(-1) + c)}{8A\psi'(-1)} \int_{x}^{1} \psi(t)\,dt, \end{aligned} \tag{5.8}$$

so that

$$F(1) = \frac{g(1)}{k}. \tag{5.9}$$

Since

$$F'(x) = h(x) - \frac{(8Ah'(-1) + c)}{8A\psi'(-1)}\ \psi(x) \in \delta,$$

we have that $F \in \delta'$. We show that

$$S_1(\langle F, F'(-1)\rangle) = \langle g, c\rangle.$$

Now

$$\begin{aligned} g'(x) = m_4[h](x) &= m_4\left[F' + \frac{(8Ah'(-1) + c)}{8A\psi'(-1)}\,\psi\right](x) \\ &= m_4[F'](x) = l_4'[F](x), \end{aligned}$$

and hence

$$g(x) = l_4[F](x) + A, \quad x \in [-1, 1],$$

for some constant $A \in \mathbf{C}$. In particular, from (5.4) and (5.9), we see that

$$\begin{aligned} g(1) &= l_4[F](1) + A \\ &= kF(1) + A \\ &= g(1) + A, \end{aligned}$$

so $A = 0$. That is to say,

$$l_4[F](x) = g(x) \quad \text{on } [-1, 1]. \tag{5.10}$$

From Theorem 5.1, we have that $F'' \in AC[-1, 1]$ and, from (5.8), we have

$$F''(x) = h'(x) - \frac{(8Ah'(-1) + c)}{8A\psi'(-1)}\ \psi'(x)$$

so that

$$(5.11) \qquad -8AF''(-1) = -8Ah'(-1) + \frac{(8Ah'(-1)+c)}{\psi'(-1)}\,\psi'(-1) = c.$$

From (5.10) and (5.11), we see that $S_1(\langle F, F'(-1)\rangle) = \langle g, c\rangle$ showing that S_1 is onto. $\square$

We now define $S_2 : H \to H$ by $S_2(\langle f, c\rangle) = \langle \mathbf{0}, kc\rangle$ for all $\langle f, c\rangle \in H$. It is easy to see that S_2 is symmetric (and bounded) in H and hence self-adjoint.

Theorem 5.4. *The operator S, defined in* (5.6)*, is self-adjoint and has, as a complete orthogonal set of eigenfunctions, the polynomial set $\{\langle Q_n, Q_n'(-1)\rangle\}_{n=0}^{\infty}$. The spectrum is discrete and given by*

$$\sigma(S) = \{\lambda_n \mid n \in \mathbf{N}_0\},$$

where λ_n is defined in (2.18).

Proof. The orthogonality and completeness of $\{\langle Q_n, Q_n'(-1)\rangle\}_{n=0}^{\infty}$ is established in Theorem 4.4 while Theorem 5.2 contains a proof that the ordered pairs $\langle Q_n, Q_n'(-1)\rangle$, $n \in \mathbf{N}_0$, are eigenfunctions. Notice that $S = S_1 + S_2$. Since both S_1 and S_2 are self-adjoint, they are both closed operators. In fact, since S_2 is also bounded, S is a closed symmetric operator in H. Moreover, since S_1 is self-adjoint, its deficiency indices are both zero. Hence (from [**22**, Section 14.7]), since S_1 is a closed symmetric operator in H and S_2 is a bounded, symmetric operator in H, the operators S_1 and $S = S_1 + S_2$ have the same deficiency indices. That is to say, the deficiency indices of the closed symmetric operator S are both zero. Consequently (from [**22**, Section 14.4]), S is self-adjoint. Since $\lambda_n \to \infty$ as $n \to \infty$, it follows that the spectrum of S is discrete and consists only of the eigenvalues $\{\lambda_n \mid n \in \mathbf{N}_0\}$. $\square$

6. Left-definite analysis of $l_4[\cdot]$. In order to develop the left-definite theory of the expression $l_4[\cdot]$, we first establish the following lemma.

Lemma 6.1. $\Delta' \subset \Delta$ *and* $\delta' \subset \delta$.

Proof. From (5.3), it is clear that $\delta' \subset \Delta' \subset \Delta$. Let $f \in \delta'$. Then, from Theorem 3.1, $\lim_{x\to 1}[f, h_1](x)$ exists and equals

$$\lim_{x\to 1}[f, h_1](x) := \lim_{x\to 1}((1-x^2)^2 f''(x))',$$

where $[\cdot,\cdot]$ is defined in (3.3) and $h_1(x)$ is defined in (3.5). Suppose that $\lim_{x\to 1}[f, h_1](x) = c \neq 0$. Without loss of generality, suppose that $c > 0$. Then there exists $x_0 > 0$ such that

$$((1-x^2)^2 f''(x))' > \frac{c}{2} := C, \quad x_0 \leq x < 1.$$

Integrating this inequality on $[x_0, x]$ yields

$$(1-x^2)^2 f''(x) \geq Cx + D$$

for some constant D. Dividing, we see that

$$f''(x) \geq \frac{Cx + D}{(1-x^2)^2}, \quad x_0 \leq x < 1.$$

However, this contradicts the fact that $f'' \in AC[-1, 1]$. Hence

$$\lim_{x\to 1}[f, h_1](x) = 0.$$

Using this fact, it follows immediately that

$$\lim_{x\to 1}[f, h_2](x) = 0,$$

and we therefore have $f \in \delta$. $\square$

Recall, from Theorem 5.2, that for $\langle f, f'(-1)\rangle$, $\langle g, g'(-1)\rangle \in \mathcal{D}$,

(6.1)

$$\begin{aligned}(S[\langle f, f'(-1)\rangle],\ &\langle g, g'(-1)\rangle)_H \\ &= k(\langle f, f'(-1)\rangle, \langle g, g'(-1)\rangle)_H \\ &\quad + \int_{-1}^{+1} \big\{(1-x^2)^2 f'''(x)\bar{g}'''(x) + 2(1-x)[(2A+1)x + 2A + 3] \\ &\qquad\qquad f''(x)\bar{g}''(x)\big\}\, dx.\end{aligned}$$

It is this identity that prompts the following definition.

Definition 6.1. The set $\tilde{H}$ is defined as:

$$\tilde{H} := \{f : [-1,1] \to \mathbf{C} \mid f \in AC[-1,1];\ f' \in AC_{\mathrm{loc}}[-1,1) \cap L^2(-1,1);\\ f'' \in AC_{\mathrm{loc}}(-1,1);\ (1-x)^{1/2} f'',\ (1-x^2) f''' \in L^2(-1,1)\},$$

and the set H_{LD} is defined as

$$H_{LD} = \{\langle f, f'(-1)\rangle \mid f \in \widetilde{H}\}.$$

For $\langle f, f'(-1)\rangle$, $\langle g, g'(-1)\rangle \in H_{LD}$, define the inner product $(\cdot,\cdot)_{H_{LD}}$ by

$$\begin{aligned}(\langle f, f'(-1)\rangle, \langle g, g'(-1)\rangle)_{H_{LD}}&\\ := k(\langle f, f'(-1)\rangle,\ &\langle g, g'(-1)\rangle)_H\\ + \int_{-1}^{+1} \big\{&(1-x^2)^2 f'''(x)\bar{g}'''(x)\\ &+ 2(1-x)[(2A+1)x + 2A+3] f''(x)\overline{g}''(x)\big\}\, dx.\end{aligned}$$

We call the inner product space $(H_{LD}, (\cdot,\cdot)_{H_{LD}})$ the left-definite space associated with the expression $l_4[\cdot]$.

Notice that $\widetilde{H} \subset H_0$ and

$$\mathcal{D} \subset H_{LD} \subset H, \tag{6.3}$$

where $H = H_0 \times \mathbf{C}$ is the Hilbert space defined in Section 4 and $\mathcal{D}$ is the domain of S defined in (5.5). Moreover, from (6.1), we have

$$(\langle f, f'(-1)\rangle, \langle g, g'(-1)\rangle)_{H_{LD}} = (S(\langle f, f'(-1)\rangle), \langle g, g'(-1)\rangle)_H \tag{6.4}$$

for all $\langle f, f'(-1)\rangle$, $\langle g, g'(-1)\rangle \in \mathcal{D}$. In particular, from Theorems 4.4 and 5.2, we have the *left-definite* orthogonality relations

$$(\langle Q_0, Q_0'(-1)\rangle, \langle Q_0, Q_0'(-1)\rangle)_{H_{LD}} = k,$$

and, for $n, m \in \mathbf{N}_0$, $(n,m) \neq (0,0)$,

$$\begin{aligned}&(\langle Q_n, Q_n'(-1)\rangle, \langle Q_m, Q_m'(-1)\rangle)_{H_{LD}}\\ &\quad= \frac{(n(n-1)(n^2-n+4A+k))(n!)^2((n-1)!)^2 2^{2n-1}(n^2+2A)}{(2n-1)!(2n-2)!(n^2-2n+2A+1)}\,\delta_{n,m}.\end{aligned}$$

The following theorem can be established in the same fashion as in [**5**] and [**21**]; we leave the details to the reader.

Theorem 6.1. $(H_{LD}, (\cdot,\cdot)_{H_{LD}})$ is a Hilbert space.

It is not difficult to extend the identity in (6.4). Indeed, it can be shown that

$$\begin{aligned}&(S(\langle f, f'(-1)\rangle), \langle g, g'(-1)\rangle)_H = (\langle f, f'(-1)\rangle, \langle g, g'(-1)\rangle)_{H_{LD}}\\ &\qquad\text{for } \langle f, f'(-1)\rangle \in \mathcal{D},\ \langle g, g'(-1)\rangle \in H_{LD},\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}&(\langle f, f'(-1)\rangle, S(\langle g, g'(-1)\rangle))_H = (\langle f, f'(-1)\rangle, \langle g, g'(-1)\rangle)_{H_{LD}}\\ &\qquad\text{for } \langle f, f'(-1)\rangle \in H_{LD},\ \langle g, g'(-1)\rangle \in \mathcal{D}.\end{aligned} \tag{6.7}$$

We now define the operator $R : H_{LD} \to H_{LD}$ by

$$\begin{gathered}R(\langle f, f'(-1)\rangle) = R_0(S)(\langle f, f'(-1)\rangle)\\ \langle f, f'(-1)\rangle \in \mathcal{D}(R) := H_{LD},\end{gathered}$$

where $R_0(S)$ is the resolvent operator of S corresponding to the regular point $\lambda = 0$ (see Theorem 5.2) that maps H onto $\mathcal{D}$. From (6.3), notice that R is well-defined on H_{LD} and maps into H_{LD}.

Theorem 6.2. *The operator R is self-adjoint and* 1–1. *Consequently, R^{-1} is a self-adjoint operator.*

Proof. Since the domain of R is all of H_{LD}, it suffices to show that R is symmetric. Let $\langle f, f'(-1)\rangle$, $\langle g, g'(-1)\rangle \in H_{LD}$. Since

$R_0(S)(\langle f, f'(-1)\rangle) \in D$, we see from (6.6) that

$$\begin{aligned}&(R(\langle f, f'(-1)\rangle), \langle g, g'(-1)\rangle)_{H_{LD}} \\ &\quad = (R_0(S)(\langle f, f'(-1)\rangle), \langle g, g'(-1)\rangle)_{H_{LD}} \\ &\quad = (S(R_0(S)(\langle f, f'(-1)\rangle), \langle g, g'(-1)\rangle)_H \\ &\quad = (\langle f, f'(-1)\rangle, \langle g, g'(-1)\rangle)_H.\end{aligned}$$

Similarly, using (6.7), it follows that

$$(\langle f, f'(-1)\rangle, R(\langle g, g'(-1)\rangle)_{H_{LD}} = (\langle f, f'(-1)\rangle, \langle g, g'(-1)\rangle)_H,$$

showing that R is symmetric. It is not difficult to see that R is 1–1 and hence R^{-1} is self-adjoint. $\square$

The operator R^{-1} is called the *left-definite operator* associated with the differential expression $l_4[\cdot]$.

Theorem 6.3. *For each $n \in \mathbf{N}_0, \langle Q_n, Q_n'(-1)\rangle$ is an eigenfunction of R^{-1} associated with the eigenvalue λ_n, defined in* (2.18). *Hence R^{-1} is an unbounded operator. Moreover, the sequence $\{\langle Q_n, Q_n'(-1)\rangle\}_{n=0}^{\infty}$ forms a complete orthogonal set in H_{LD}.*

Proof. Since $S(\langle Q_n, Q_n'(-1)\rangle) = \lambda_n \langle Q_n, Q_n'(-1)\rangle$, it follows that

$$\langle Q_n, Q_n'(-1)\rangle = R(S(\langle Q_n, Q_n'(-1)\rangle)) = \lambda_n R(\langle Q_n, Q_n'(-1)\rangle).$$

Applying R^{-1} to both sides yields

$$R^{-1}(\langle Q_n, Q_n'(-1)\rangle) = \lambda_n \langle Q_n, Q_n'(-1)\rangle.$$

The orthogonality of $\{\langle Q_n, Q_n'(-1)\rangle\}_{n=0}^{\infty}$ was established above in (6.5). To show that this polynomial set is complete in H_{LD}, suppose there exists $\langle f, f'(-1)\rangle \in H_{LD}$ such that

$$(\langle Q_n, Q_n'(-1)\rangle, \langle f, f'(-1)\rangle)_{H_{LD}} = 0, \quad n \in \mathbf{N}_0.$$

From (6.6), we see that, for each $n \in \mathbf{N}_0$,

$$\begin{aligned}0 &= (\langle Q_n, Q_n'(-1)\rangle, \langle f, f'(-1)\rangle)_{H_{LD}} \\ &= (S(\langle Q_n, Q_n'(-1)\rangle), \langle f, f'(-1)\rangle)_H \\ &= \lambda_n(\langle Q_n, Q_n'(-1)\rangle, \langle f, f'(-1)\rangle)_H.\end{aligned}$$

Since $\lambda_n > 0$ we find that

$$(\langle Q_n, Q_n'(-1)\rangle, \langle f, f'(-1)\rangle)_H = 0, \quad n \in \mathbf{N}_0.$$

However, from Theorem 4.4, $\{\langle Q_n, Q_n'(-1)\rangle\}_{n=0}^{\infty}$ is complete in H; hence $\langle f, f'(-1)\rangle = \langle \mathbf{0}, 0\rangle$ in H and, hence, in H_{LD} as well. ◻

7. Further results. As far as these authors know, the differential equation (2.15) is the first example of a *nonsymmetrizable* differential equation having orthogonal polynomial eigenfunctions *and* a self-adjoint realization in some Hilbert space. In this section, we offer two more examples of this phenomenon; this will lead us to a conjecture below concerning the $BKS(N,1)$ class. Lastly, in this section, we generalize Theorem 2.1 to connect the $BKS(N,0)$ and $BKS(N,2)$ classes and illustrate this result with another new example.

As this paper shows, the left Legendre type polynomials $\{P_n^{(0,0,(1/A),0)}(x)\}_{n=0}^{\infty}$ (see (2.10)) and the differential equation that they satisfy produced the fourth-order differential expression $l_4[\cdot]$ that is the focus of this paper. We now show that the monic Jacobi type polynomials $\{P_n^{(0,1,(1/A),0)}(x)\}_{n=0}^{\infty}$ and $\{P_n^{(0,2,(1/A),0)}(x)\}_{n=0}^{\infty}$, which satisfy, respectively, sixth-order and eighth-order differential equations, will produce two more examples of nonsymmetrizable equations with self-adjoint realizations having orthogonal polynomial eigenfunctions.

The monic Jacobi type polynomials $\{P_n^{(0,1,(1/A),0)}(x)\}_{n=0}^{\infty}$ (see [**11**]) are in the $BKS(6,0)$ class. They satisfy the symmetrizable (with symmetry factor $f(x) = 1 + x$) sixth-order differential equation

(7.1)
$$\begin{aligned}(x^2-1)^3y^{(6)} &+ 3(1-x^2)^2(7x-1)y^{(5)} + 3(x^2-1)(46x^2-8x-14)y^{(4)}\\ &+ 3(x+1)(110x^2-116x+14)y^{(3)}\\ &+ 6(x+1)((4A+42)x-4A-30)y''\\ &+ ((72A+36)x-24A+36)y' + ky\\ &= \gamma_n y,\end{aligned}$$

where $\gamma_n = n(n+2)(n^4+4n^3+5n^2+2n+24A)+k$ and $A, k > 0$. These polynomials are orthogonal with respect to the inner product

$$(f,g)_6 = \frac{1}{A}\, f(-1)\bar{g}(-1) + \int_{-1}^{1} f(x)\bar{g}(x)(1+x)\,dx.$$

Equation (7.1) fits the hypotheses of Theorem 2.1 with $c = 1$. It follows that the polynomials $\{Q_{n,6}(x)\}_{n=0}^{\infty}$ defined by $Q_{0,6}(x)(x) = 1$ and

$$Q_{n,6}(x)(x) = n \int_1^x P_{n-1}^{(0,1,(1/A),0)}(x)\,dx, \quad n \in \mathbf{N},$$

satisfy the sixth-order *nonsymmetrizable* differential equation

$$\begin{aligned}(x^2-1)^3 y^{(6)} &+ 3(1-x^2)^2(5x-1)y^{(5)} + 3(x^2-1)(21x^2-4x-9)y^{(4)} \\ &+ 6(x-1)(13x^2+16x-5)y^{(3)} \\ &+ 6(x-1)((4A+3)x+4A+9)y'' \\ &+ 24A(x-1)y' + ky \\ &= \tau_n y,\end{aligned}$$

where $\tau_n = n^2(n^4 - 2n^2 + 24A + 1) + k$. Furthermore, the polynomials $\{Q_{n,6}(x)(x)\}_{n=0}^{\infty}$ are orthogonal with respect to the inner product

$$\phi_6(p,q) = p(1)\bar{q}(1) + \frac{1}{A}\,p'(-1)\bar{q}'(-1) + \int_{-1}^{1} p'(x)\overline{q}'(x)(1+x)\,dx.$$

Similarly, the monic Jacobi type polynomials $\{P_n^{(0,2,(1/A),0)}(x)\}_{n=0}^{\infty}$ satisfy the symmetrizable (with symmetry factor $f(x) = (1+x)^2$) eighth-order differential equation

$$\begin{aligned}(x^2-1)^4 y^{(8)} &+ 4(x^2-1)^3(10x-2)y^{(7)} + 96(x^2-1)^2(6x^2-2x-1)y^{(6)} \\ &+ 96(x^2-1)(39x^3-15x^2-21x+5)y^{(5)} \\ &+ 24(x+1)(471x^3-627x^2+93x+71)y^{(4)} \\ &+ 288(x+1)(51x^2-58x+11)y''' \\ &+ 96(x+1)((4A+69)x-4A-51)y'' \\ &+ 192((8A+3)x-4A+3)y' + ky \\ &= \omega_n y\end{aligned}$$

where $\omega_n = n(n+3)(n^6+9n^5+31n^4+51n^3+40n^2+12n+384A)+k$ and $A, k > 0$. These polynomials are orthogonal with respect to the inner product

$$(f,g)_8 = \frac{1}{A}\,f(-1)\overline{g}(-1) + \int_{-1}^{1} f(x)\bar{g}(x)(1+x)^2\,dx.$$

Equation (7.3) also fits the conditions of Theorem 2.1 with $c = 1$, and hence the monic polynomials $\{Q_{n,8}(x)(x)\}_{n=0}^{\infty}$, given by $Q_{0,8}(x) = 1$ and

$$Q_{n,8}(x) = n\int_1^x P_{n-1}^{(0,2,(1/A),0)}(x)\,dx \quad n \in \mathbf{N},$$

satisfy the eighth-order nonsymmetrizable differential equation

$$\begin{aligned}(x^2-1)^4y^{(8)} &+ 4(x^2-1)^3(8x-2)y^{(7)} + 16(x^2-1)^2(22x^2-9x-2)y^{(6)}\\ &+ 4(x^2-1)(408x^3-180x^2-264x+84)y^{(5)}\\ &+ 24(x-1)(131x^3+95x^2-103x-27)y^{(4)}\\ &+ 192(x-1)(11x^2+14x-7)y'''\\ &+ 96(x-1)((4A+3)x+4A+9)y'' + 768A(x-1)y' + ky\\ &= \nu_n y,\end{aligned}$$

where $\upsilon_n = n(n+1)(n^6+3n^5-n^4-7n^3+4n+384A)+k$. Furthermore, these polynomials are orthogonal with respect to the inner product

$$\phi_8(p,q) = p(1)\overline{q}(1) + \frac{1}{A}\,p'(-1)\overline{q}'(-1) + \int_{-1}^1 p'(x)\overline{q}'(x)(1+x)^2\,dx.$$

These two examples lead us to the following conjecture.

Conjecture 7.1. *For each $N \in \mathbf{N}_0$, let $\{P_n^{(0,N,(1/A),0)}(x)\}_{n=0}^{\infty}$ be the monic Jacobi type polynomial sequence which is orthogonal with respect to the inner product*

$$(p,q)_N = \frac{1}{A}\,p(-1)\overline{q}(-1) + \int_{-1}^1 p(x)\overline{q}(x)(1+x)^N\,dx.$$

Define the monic polynomials $\{Q_{n,N}(x)\}_{n=0}^{\infty}$ by

$$\begin{aligned}&Q_{0,N}(x) = 1\\ &Q_{n,N}(x) = n\int_1^x P_{n-1}^{(0,N,(1/A),0)}(t)\,dt, \quad n \in \mathbf{N}.\end{aligned}$$

Then there exists a nonsymmetrizable differential expression $l_{2N+4}[\cdot]$ of order $2N+4$ having the polynomials $\{Q_{n,N}(x)\}_{n=0}^{\infty}$ as eigenfunctions. Furthermore, there exists a self-adjoint operator, generated from

$l_{2N+4}[\cdot]$, *in a Hilbert space generated from the inner product*

$$\phi_{2N+4}(p,q) = p(1)\bar{q}(1) + \frac{1}{A}\,p'(-1)\bar{q}'(-1) + \int_{-1}^{1} p'(x)\bar{q}'(x)(1+x)^N\,dx.$$

In particular, for each $N \in \mathbf{N}_0, \{Q_{n,N}(x)\}_{n=0}^{\infty} \in BKS(2N+4,1)$.

We remark here that Koekoek and Koekoek [**9**] have recently explicitly computed the coefficients of the differential equation for the general Jacobi type polynomials $\{P_n^{(\alpha,\beta,M,N)}(x)\}_{n=0}^{\infty}$ for all parameters $\alpha, \beta > -1$ and $M, N \geq 0$; these polynomials, studied in depth by Koornwinder [**11**], are orthogonal with respect to the inner product

$$\begin{aligned}(p,q)_{\alpha,\beta,M,N} &= Mp(-1)\bar{q}(-1) + Np(1)\bar{q}(1)\\ &\quad + \int_{-1}^{1} p(x)\bar{q}(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx.\end{aligned}$$

In particular, they show that the Jacobi type polynomials $\{P_n^{(0,N,(1/A),0)}(x)\}_{n=0}^{\infty} \in BKS(2N+4,0)$ for each $N \in \mathbf{N}_0$.

We now consider an extension of Theorem 2.1; the proof is similar to that given in [**7**].

Theorem 7.1. *Consider the Sobolev bilinear form*

$$\varphi(p,q) = \lambda p(c)\bar{q}(c) + \mu p'(c)\bar{q}'(c) + \langle \tau, p''\bar{q}''\rangle,$$

where τ *is a quasi-definite moment functional and* λ, μ *are real numbers. Then* $\varphi(\cdot,\cdot)$ *is quasi-definite if and only if both* λ *and* μ *are nonzero. Let* $\{\widetilde{P}_n(x)\}_{n=0}^{\infty}$ *denote the monic orthogonal polynomials with respect to* τ *and, for fixed nonzero real numbers* λ *and* μ*, let* $\{\widetilde{Q}_n(x)\}_{n=0}^{\infty}$ *denote the monic orthogonal polynomials with respect to* $\varphi(\cdot,\cdot)$*. Then*

$$\begin{aligned}&\widetilde{Q}_0(x) = 1; \quad \widetilde{Q}_1(x) = x - c\\ &\widetilde{Q}_n(x) = n(n-1)\int_c^x \int_c^t \widetilde{P}_{n-2}(z)\,dz\,dt, \quad n \geq 2.\end{aligned}$$

In particular, $\widetilde{Q}_1(c) = 0$ *and, for* $n \geq 2$, $\widetilde{Q}_n(c) = \widetilde{Q}_n'(c) = 0$.

(i) *Suppose further that* $\{\widetilde{P}_n(x)\}_{n=0}^{\infty} \in BKS(N,0)$*; specifically, suppose that* $y = \widetilde{P}_n(x)$ *satisfies the* N*th-order differential equation*

$$\tilde{m}_N[y](x) = \mu_n y(x), \quad n \in \mathbf{N}_0,$$

where

$$\tilde{m}_N[y](x) = a_N(x)y^{(N)}(x) + a_{N-1}(x)y^{(N-1)}(x) + \cdots + a_1(x)y'(x). \tag{7.4}$$

If

$$\sum_{j=0}^{N-k} (-1)^j (j+1) a_{k+j}^{(j)}(c) = 0, \quad 1 \le k \le N, \tag{7.5}$$

and

$$\sum_{j=0}^{N-k} (-1)^j (j+1) a_{k+j}^{(j+1)}(c) = 0, \quad 2 \le k \le N, \tag{7.6}$$

then $\{\widetilde{Q}_n(x)\}_{n=0}^{\infty} \in BKS(N,2)$ *with* $y = \widetilde{Q}_n(x)$ *satisfying the* N*th-order differential equation*

$$\tilde{l}_N[y](x) = \lambda_n y(x), \quad n \in \mathbf{N}_0,$$

where

$$\tilde{l}_N[y](x) = b_N(x)y^{(N)}(x) + b_{N-1}(x)y^{(N-1)}(x) + \cdots + b_1(x)y'(x), \tag{7.7}$$

with the coefficients of $\tilde{l}_N[\cdot]$ *being given by*

$$b_k(x) = \sum_{j=0}^{N-k} (-1)^j (j+1) a_{k+j}^{(j)}(x), \quad 1 \le k \le N,$$

and

$$\lambda_n = \begin{cases} nb_1'(x) & \text{for } n = 0, 1 \\ \mu_{n-2} + 2b_1'(x) + b_2''(x) & \text{for } n \ge 2. \end{cases}$$

Moreover,

$$b_k(c) = 0, \quad 1 \le k \le N, \tag{7.8}$$

and

$$b_k'(c) = 0, \quad 2 \le k \le N, \tag{7.9}$$

and, formally, $\tilde{l}_N''[y] = \widetilde{m}_N[y'']$.

(ii) *Conversely, suppose that* $\{\widetilde{Q}_n(x)\}_{n=0}^{\infty} \in BKS(N,2)$ *with* $y = \widetilde{Q}_n(x)$ *satisfying the* N*th-order differential equation*

$$\tilde{l}_N[y](x) = \lambda_n y(x), \quad n \in \mathbf{N}_0),$$

where $\tilde{l}_N[\cdot]$ *is given in* (7.7). *Then the coefficients of* $\tilde{l}_N[\cdot]$ *satisfy* (7.8) *and* (7.9). *Furthermore,* $\{\widetilde{P}_n(x)\}_{n=0}^{\infty} \in BKS(N,0)$ *with* $y = \widetilde{P}_n(x)$ *satisfying*

$$\widetilde{m}_N[y](x) = \mu_n y(x), \quad n \in \mathbf{N}_0,$$

where $\widetilde{m}_N[\cdot]$ *is given in* (7.4) *with the coefficients satisfying*

$$a_k(x) = \begin{cases} b_N(x) & \text{for } k = N \\ b_{N-1}(x) + 2b_N'(x) & \text{for } k = N-1 \\ b_{k+2}''(x) + 2b_{k+1}'(x) + b_k(x) & \text{for } 1 \le k \le N-2, \end{cases}$$

and

$$\mu_n = \lambda_{n+2} - 2b_1'(x) - b_2''(x).$$

Moreover, (7.5) *and* (7.6) *are satisfied and, formally,* $\tilde{l}_N''[y] = \widetilde{m}_N[y'']$.

We apply this result to find a new example in the $BKS(4,2)$ class; indeed, we construct this new equation by again using the left Legendre type expression $m_4[\cdot]$, defined in (2.7). Necessarily, we find that $c = 1$ and the resulting equation is

$$\tilde{l}[y](x) = (x^2-1)^2 y^{(4)} - 2(x-1)^2 y'' + 4(x-1)y' + ky = \lambda y. \tag{7.10}$$

It is clear, since $\tilde{l}[\cdot]$ is missing a third-order derivative, that this equation is not Lagrangian symmetrizable. Moreover, the polynomials $\{\widetilde{Q}_n(x)\}_{n=0}^{\infty}$ defined by

$$\widetilde{Q}_0(x) = 1; \quad \widetilde{Q}_1(x) = x - 1$$
$$\widetilde{Q}_n(x) = n(n-1)\int_1^x \int_1^t PL_{n-2}(z)\,dz\,dt$$

are solutions of

$$\tilde{l}[y](x) = \lambda_n y(x),$$

where $\lambda_n = n^2(n-3)^2 + k$, $n \in \mathbf{N}_0$, and they are orthogonal with respect to the quasi-definite (but not positive-definite) bilinear form

$$\begin{aligned}\varphi(p,q) = \lambda p(1)\bar{q}(1) + \mu p'(1)\bar{q}'(1) - p''(1)\bar{q}''(1) \\ + \int_{-1}^{1} p''(x)\bar{q}''(x)\,dx, \quad \lambda, \mu \neq 0.\end{aligned}$$

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