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# SEMI-DISCRETE GALERKIN APPROXIMATIONS FOR THE SINGLE-LAYER EQUATION ON LIPSCHITZ CURVES

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ABSTRACT. We study a semi-discrete Galerkin method for solving the single-layer equation $\mathcal{V}u = f$ with an approximating subspace of piecewise constant functions. Error bounds in Sobolev norms $\|\cdot\|_s$ with $-1 \le s < 1/2$ are proven and are of the same order as for the original Galerkin method. The distinctive features of the present work are that we handle irregular meshes and do not rely on Fourier methods. The main assumptions are that the quadrature rule used to approximate the inner product is a composite rule and that the underlying quadrature rule that is mapped to each subinterval has a sufficiently small *Peano constant*.

**1. Introduction.** The single-layer equation

$$\mathcal{V}u = f \tag{1}$$

is an important boundary integral equation. It arises, for example, in the solution of the Laplace equation on interior or exterior domains. If $\Omega$ is a two-dimensional domain with Lipschitz boundary $\Gamma$, as we shall assume in this paper, then the single-layer operator $\mathcal{V}$ takes the form

$$\mathcal{V}u(t) := -\frac{1}{\pi}\int_\Gamma \log|t-s|u(s)\,ds = f(t), \quad t \in \Gamma, \tag{2}$$

where $|t-s|$ denotes the Euclidean distance between $t$ and $s$, and $ds$ is the element of arc length. The curve $\Gamma$ could, for example, be a polygon or 'curved polygon,' without cusps.

In this paper we study a semi-discrete Galerkin method, or 'qualocation method,' with an approximating subspace of piecewise constant functions. Let $S_h$ be the space of piecewise constant functions on a partition

$$\Gamma = \cup\Gamma_k,$$

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with $h = \max h_k$ and $h_k$ the length of $\Gamma_k$. Then the Galerkin approximation to $u$ is: find $u_h^G \in S_h$ such that

$$(3) \qquad (\mathcal{V}u_h^G, \chi_h) = (f, \chi_h), \quad \forall \chi_h \in S_h,$$

with

$$(g, w) \equiv \int_\Gamma g(s)\overline{w(s)}\, ds.$$

The Galerkin approximation has been well studied, for example, see [**8, 22**], and it possesses excellent stability and convergence properties. It is, however, difficult and expensive to implement exactly, because of the two levels of integration involved in (3). Indeed, it is normally impossible to implement it exactly, in that one is usually forced to resort to approximate integration for at least the outer integral, i.e., the inner product integral, in (3).

The semi-discrete approximations to be studied in this paper have the form: find $u_h \in S_h$ where $u_h$ satisfies

$$(4) \qquad (\mathcal{V}u_h, \chi_h)_h = (f, \chi_h)_h, \quad \forall \chi_h \in S_h.$$

In this equation $(\cdot,\cdot)_h$ is a quadrature approximation of the exact integral $(\cdot,\cdot)$, obtained by using an appropriately scaled version of a basic quadrature rule, denoted by $q$, on each subinterval of the mesh. In essence, we shall show that the method is stable for small enough $h$ and has satisfactory convergence properties, provided the rule $q$ is sufficiently 'rich,' where richness means, roughly, having enough quadrature points. We shall make these ideas precise later. A distinctive feature of the present work is that we allow the mesh to be irregular (but it must be quasi-uniform).

An important special case of a qualocation approximation is the collocation method, obtained by choosing $q$ to be the 1-point quadrature rule based on evaluation at, for example, the midpoint. In this case it is easy to see that (4) is equivalent to

$$\mathcal{V}u_h(\tau_k) = f(\tau_k), \quad k = 1, \dots, N_h,$$

where $\tau_k$ is the midpoint in the $k$th subinterval $\Gamma_k$, and $N_h$ is the number of subintervals.

The challenge for the collocation method, as for all semi-discrete or fully discrete Galerkin methods for the single-layer equation, is to prove stability. For the piecewise constant collocation method on smooth curves, there exists a very satisfactory theory, cf. [**2, 17**], if the mesh is uniform (or is uniform with respect to smooth parametrization). This theory, which is based on Fourier analysis, has been extended to the collocation method on a torus by Costabel and McLean [**9**].

In a different direction, the Fourier-based theory has been extended to qualocation methods with more sophisticated quadrature rules, cf. [**7, 18, 20**]. In these methods the basic quadrature rule $q$ on each element is taken to have two or more points, with the parameters in the rule chosen to enhance the order of convergence in appropriate negative norms. But always the Fourier nature of the analysis requires that the mesh be uniform.

For nonuniform meshes the situation is much less satisfactory, although Chandler [**5, 6**] has made some progress for piecewise constant collocation on circles in the function space setting of functions whose average value is zero. Still less is known about piecewise constant collocation on surfaces.

It should be mentioned that, for smoothest splines of odd degree (such as the continuous piecewise linear functions), a very satisfactory stability and convergence theory of the break point collocation method has been developed by Arnold and Wendland [**1**]. However, this theory is not applicable for splines of even degree.

Hsiao, Kopp and Wendland [**12, 13**] have studied a different kind of approximation to the Galerkin method for the single-layer equation on curves, the *Galerkin-collocation approximation*. In this method the principal convolutional part of the operator is treated exactly, and quadrature is used only for the term that represents the departure of the curve from a circle. Here the approach is quite different, in that we do not treat the principle part exactly.

There exists one study of a fully discrete approximation to the Galerkin method, namely a study by Penzel [**15**] for the three-dimensional case in which the surface is the three-dimensional equivalent of (2), on a square plate. In this work Penzel uses a uniform mesh in both the $x$ and $y$ directions, and he exploits the convolution structure of the kernel to obtain concrete expressions for the error in the Galerkin matrix elements with respect to the standard basis. Penzel's results for this special case have some of the flavor of ours, in that he proves stability and convergence if the underlying quadrature rule on each element (in his case a product Simpson's rule) is sufficiently refined. It also has some relation to the method of Hsiao, Kopp and Wendland mentioned above, in that the diagonal elements are treated in a special way.

In this paper we do not directly consider the effects of approximations to the boundary $\Gamma$. However, Nedelec [**14**] has considered the effects of such approximations for the case of the Galerkin method, and his results apply with equal force to the semi-discrete methods considered here.

In Sections 2 and 3 we look more closely at the quadrature approximation. In Section 4 we state the main stability and convergence theorem, and we give its proof in Section 5. The paper concludes with some additional remarks in Section 6.

**2. Approximation scheme and quadrature errors.** We shall assume that the meshes $\Gamma = \cup\Gamma_k$ are quasi-uniform, so that if the length of $\Gamma_k$ is $h_k$, then there exists $\rho > 0$, independent of $h$, such that

$$\frac{h}{\min h_i} \leq \rho, \tag{5}$$

where $h = \max h_k$.

The approximate inner product $(\cdot,\cdot)_h$ in (4) is then defined by

$$(v,w)_h = \sum_k (v,w)_{h,\Gamma_k},$$

where the quantity $(v,w)_{h,\Gamma_k}$ is an approximation to

$$(v,w)_{\Gamma_k} \equiv \int_{\Gamma_k} v(s)\overline{w(s)}\,ds, \tag{6}$$

obtained by applying a scaled version of a fixed quadrature rule $q$,

$$q(g) = \sum_{j=1}^{m} w_j g(x_j) \approx \int_0^1 g(x)\,dx, \tag{7}$$

with

$$\sum_{j=1}^{m} w_j = 1.$$

Generally we will use the simpler notation $qg \equiv q(g)$. The quantity $(\cdot,\cdot)_h$ is also often called a *discrete inner product.*

**3. The Peano constant.** The stability and convergence theorem in the next section does not require that the quadrature rule $q$ integrate exactly any polynomials other than the constant functions. It does, however, require that the rule be 'rich' enough, in the sense of having a small enough *Peano constant.* In the present context the required Peano kernel theory is trivial and is included only for completeness.

If $g$ and its derivative $g'$ belong to $L_2(0,1)$, then we can conclude that

$$\begin{aligned} g(x) &= g(0) + \int_0^x g'(y)\,dy \\ &= g(0) + \int_0^1 H(x-y)g'(y)\,dy, \end{aligned} \tag{8}$$

where $H$ is the Heaviside step function. Since the quadrature rule $q$ is exact for the first term, we can write the quadrature error as

$$Eg := qg - \int_0^1 g(y)\,dy = \int_0^1 K(y)g'(y)\,dy, \tag{9}$$

where

$$K(y) := E_x H(x-y), \tag{10}$$

and the notation $E_x$ means the error functional $E$ is to be applied to the function $H(x-y)$ with respect to the variable $x$. This yields

$$|Eg| \le \kappa \|g'\|_{L_2(0,1)}, \tag{11}$$

where

$$\kappa := \|K\|_{L_2(0,1)}. \tag{12}$$

Of course $K$ is just the first Peano kernel, cf. [**10**]; and $\kappa$, which is called here the *Peano constant*, is just the $L_2$ norm of the first Peano kernel.

The Peano constant $\kappa$ can be made as small as desired by a suitable choice of the rule $q$. For example, if $q$ is the composite $m$-point midpoint rule

$$qg := \frac{1}{m}\sum_{k=1}^{m} g\left(\frac{2k-1}{2m}\right),$$

then it is easily seen that

$$K(y) = \frac{k}{m} - y, \quad \text{for } \frac{2k-1}{2m} < y < \frac{2k+1}{2m},$$

for $k = 1, 2, \dots, m$. Then it is straightforward to calculate

$$\kappa = \frac{1}{2\sqrt{3}m}, \quad m \geq 1.$$

For the case of an $m$-point Gauss rule, shifted to the interval $[0, 1]$, Petras [**16**] has shown that

$$\kappa \leq \frac{\pi}{6m}.$$

More generally, if $q$ is an $m$-point interpolatory rule with positive weights (such as the Clenshaw-Curtis rule), then the bound

$$\kappa \leq \frac{\pi}{2m}$$

is an immediate consequence of a uniform bound on the first Peano kernel given by Brass [**4**, Theorem 2].

**4. The main result.** The following theorem, which is the main result of this paper, expresses the fact that the semi-discrete Galerkin method is stable, provided the Peano constant $\kappa$ is sufficiently small; and, moreover, the method has the same order of convergence as the Galerkin method in all Sobolev norms down to the $\|\cdot\|_{-1}$ norm.

The Sobolev spaces $H^s \equiv H^s(\Gamma)$ for $|s| \le 1$ are as defined with respect to the arc length parameterization of $\Gamma$; see [**8**]. For $-1 \le s < 0$, the norms are defined by duality:

$$\|v\|_s = \sup_{w \in H^{-s}} \frac{|(v,w)|}{\|w\|_{-s}}, \quad s < 0.$$

Note that $H^0 = L_2$ and that $H^1$ is the subspace of $H^0$ of absolutely continuous functions whose first tangential derivatives are in $L_2$.

The assumption in the theorem that $\mathcal{V}$ is one-to-one is satisfied provided the transfinite diameter of $\Gamma$ is not equal to 1, see, for example, [**8, 21, 23**]. It is also known, cf. [**21**], that

$$\mathcal{V} : H^0 \underset{\text{onto}}{\overset{1-1}{\longrightarrow}} H^1 \tag{13}$$

and

$$\mathcal{V}^{-1} : H^1 \underset{\text{onto}}{\overset{1-1}{\longrightarrow}} H^0 \quad \text{are bounded}$$

for arbitrary Lipschitz curves. Thus the assumption $f \in H^1$ in the theorem ensures that equation (1) has a solution $u \in H^0$. The convergence result allows for the possibility that $u$ is actually more regular than this, as can happen, for example, if $\Gamma$ has additional smoothness.

**Theorem 1.** *Assume that $\mathcal{V}$ is one-to-one and that $f \in H^1$. Assume also that the meshes for the piecewise constant approximating space $S_h$ are quasi-uniform. Let $h_0 > 0$ be such that $u_h^G$ exists and is unique and stable for $h < h_0$. Then there exists $\kappa_0 > 0$ such that if the Peano constant $\kappa < \kappa_0$ and if $h < h_0$, then $u_h$ exists and is unique. If $u \in H^t$ with $0 \le t \le 1$, then for $s$ satisfying*

$$-1 \le s \le t \quad \text{and} \quad s < 1/2, \tag{14}$$

*there exists $C$ such that*

$$\|u - u_h\|_s \le C h^{t-s} \|u\|_t. \tag{15}$$

The theorem is proven in the next section. Note that the 'best' convergence result given in the theorem is

$$\|u - u_h\|_{-1} \leq Ch^{t+1}\|u\|_t, \quad 0 \leq t \leq 1. \tag{16}$$

As is well known, the convergence result for the Galerkin method is better if $\Gamma$ is sufficiently smooth, in that it allows $s$ in (14) to go down to $-2$; and in that norm, one obtains one more power of $h$ than in the error bound (16). We expect that the convergence theorem for the semi-discrete method can only be extended in this way for special choices of the rule $q$. (The reader might think it is interesting that so far we have assumed of $q$ *only* that it integrates constants exactly and has a small enough Peano constant.)

**5. Proof of the theorem.** For convenience we first collect some properties of the Galerkin method. It is well known, see, for example, [**8**], that the single layer operator is strongly elliptic. In consequence, there is $h_0 > 0$ such that for $h \leq h_0$, the solution $u_h^G$ of the Galerkin equation (4) exists and is unique; and this solution $u_h^G$ is optimal in the sense that

$$\|u_h^G - u\|_{-1/2} \leq C \inf_{v_h \in S_h} \|v_h - u\|_{-1/2} \leq Ch^{1/2}\|u\|_0. \tag{17}$$

We also note that the Galerkin method can be expressed as: find $u_h^G \in S_h$ such that

$$\mathcal{P}_h \mathcal{V} u_h^G = \mathcal{P}_h f, \tag{18}$$

where $\mathcal{P}_h$ is the $L_2$-orthogonal projection onto $S_h$ defined by

$$\mathcal{P}_h g \in S_h, \qquad (\mathcal{P}_h g, \chi_h) = (g, \chi_h), \quad \forall\, \chi_h \in S_h. \tag{19}$$

Let $\mathcal{V}_h : S_h \to S_h$ be the self-adjoint operator defined by

$$\mathcal{V}_h = \mathcal{P}_h \mathcal{V}|_{S_h}. \tag{20}$$

Then for $h < h_0$ we know already that $\mathcal{V}_h$ is bijective, so that the Galerkin equation and its solution can be written as

$$\mathcal{V}_h u_h^G = \mathcal{P}_h f, \qquad u_h^G = \mathcal{V}_h^{-1} \mathcal{P}_h f. \tag{21}$$

We need two simple consequences of these results for the Galerkin method.

**Lemma 2.** *There exists* $\gamma > 0$ *and* $h_0 > 0$ *such that for* $v \in H^1$ *and* $h < h_0$,

$$\|\mathcal{V}_h^{-1}\mathcal{P}_h v\|_0 \le \gamma \|v\|_1. \tag{22}$$

*Proof.* We note first that $U_h^G := \mathcal{V}_h^{-1}\mathcal{P}_h v$ is the Galerkin solution of (1) if $f = v$, so that the exact solution is $U := \mathcal{V}^{-1}v$. Let $\tilde{U}_h \in S_h$ be an optimal approximation to $U$ in both the $H^0$ and $H^{-1/2}$ norms, so that, in particular,

$$\|U - \tilde{U}_h\|_0 \le C\|U\|_0, \qquad \|U - \tilde{U}_h\|_{-1/2} \le Ch^{1/2}\|U\|_0.$$

(The existence of such simultaneous approximations is well known; for example, see [**3**, p. 95].) Then by use of the triangle inequality and a standard *inverse estimate* for members of $S_h$ (which follows from the quasi-uniformity assumption),

$$\begin{aligned}\|U_h^G\|_0 &\le \|U_h^G - \tilde{U}_h\|_0 + \|\tilde{U}_h - U\|_0 + \|U\|_0\\ &\le Ch^{-1/2}\|U_h^G - \tilde{U}_h\|_{-1/2} + C\|U\|_0\\ &\le Ch^{-1/2}(\|U_h^G - U\|_{-1/2} + \|\tilde{U}_h - U\|_{-1/2}) + C\|U\|_0\\ &\le \gamma\|U\|_0.\end{aligned}$$

**Lemma 3.** *Let* $\gamma$ *and* $h_0$ *be as in Lemma* 2. *For* $h < h_0$ *and* $w_h \in S_h$,

$$\|w_h\|_{-1} \le \gamma\|\mathcal{P}_h\mathcal{V}w_h\|_0. \tag{23}$$

*Proof.* For $v \in H^1$, we have

$$\begin{aligned}(w_h, v) &= (w_h, \mathcal{P}_h v)\\ &= (\mathcal{V}_h^{-1}\mathcal{V}_h w_h, \mathcal{P}_h v)\\ &= (\mathcal{V}_h w_h, \mathcal{V}_h^{-1}\mathcal{P}_h v).\end{aligned}$$

Then, from Lemma 2,

$$\begin{aligned}|(w_h, v)| &\le \|\mathcal{V}_h w_h\|_0 \|\mathcal{V}_h^{-1}\mathcal{P}_h v\|_0 \\ &\le \gamma \|\mathcal{V}_h w_h\|_0 \|v\|_1,\end{aligned}$$

from which (23) follows immediately. □

Now we turn to the semi-discrete Galerkin approximation defined by (4). We see that this approximation can be written in terms of the discrete orthogonal projection $\mathcal{B}_h : H^1 \to S_h$ defined by

$$\mathcal{B}_h g \in S_h, \qquad (\mathcal{B}_h g, \chi_h)_h = (g, \chi_h)_h, \quad \forall\, \chi_h \in S_h. \tag{24}$$

The operator $\mathcal{B}_h$ is well-defined, since the matrix of the system (24) with respect to the standard basis functions $\{\phi_i\}$ (for which support $(\phi_i) = \bar{\Gamma}_i$) is the diagonal matrix

$$[(\phi_i, \phi_j)_h] = [(\phi_i, \phi_j)],$$

in which all diagonal elements are nonzero.

In terms of this operator $\mathcal{B}_h$, the semi-discrete Galerkin method (4) becomes: find $u_h \in S_h$ such that

$$\mathcal{B}_h \mathcal{V} u_h = \mathcal{B}_h f. \tag{25}$$

The key to the analysis is the following estimate involving the Peano constant $\kappa$.

**Proposition 4.** *There exists $C > 0$ such that, for $v \in H^0$,*

$$\|\mathcal{B}_h \mathcal{V} v - \mathcal{P}_h \mathcal{V} v\|_0 \le C\kappa h \|v\|_0, \tag{26}$$

*where $C$ is independent of $v$, $h$ and the rule $q$.*

*Proof.* Because $\mathcal{B}_h \mathcal{V} v - \mathcal{P}_h \mathcal{V} v$ is constant on each element $\Gamma_h$ of the boundary,

$$\begin{aligned}\|\mathcal{B}_h \mathcal{V} v - \mathcal{P}_h \mathcal{V} v\|_0^2 &= \sum_k \int_{\Gamma_k} (\mathcal{B}_h \mathcal{V} v - \mathcal{P}_h \mathcal{V} v)^2\, ds \\ &= \sum_k h_k ((\mathcal{B}_h \mathcal{V} v - \mathcal{P}_h \mathcal{V} v)|_{\Gamma_k})^2.\end{aligned} \tag{27}$$

Now in this piecewise constant setting,

$$(\mathcal{P}_h w)|_{\Gamma_k} = \frac{1}{h_k}\int_{\Gamma_k} w(s)\,ds = \int_0^1 \hat{w}(\hat{s})\,d\hat{s},$$

where in the last step the map $s \mapsto \hat{s}$ is the translation and magnification map that carries the arc length parameter on $\Gamma_k$ onto the unit interval, and $\hat{w}(\hat{s}) \equiv w(s)$. In a similar way,

$$(\mathcal{B}_h w)|_{\Gamma_k} = q\hat{w}.$$

Thus

$$(\mathcal{B}_h w - \mathcal{P}_h w)|_{\Gamma_k} = E\hat{w},$$

and hence, from (11),

$$((\mathcal{B}_h w - \mathcal{P}_h w)|_{\Gamma_k})^2 \le \kappa^2 \|\hat{w}'\|^2_{L^2(0,1)} = \kappa^2 h_k \int_{\Gamma_k} |w'(s)|^2\,ds.$$

Finally, (27) gives

$$\|\mathcal{B}_h \mathcal{V} v - \mathcal{P}_h \mathcal{V} v\|_0^2 \le \kappa^2 h^2 \sum_k \int_{\Gamma_k} |(\mathcal{V}v)'|^2\,ds = \kappa^2 h^2 \|(\mathcal{V}v)'\|_0^2.$$

With the aid of (13),

$$\|\mathcal{B}_h \mathcal{V} v - \mathcal{P}_h \mathcal{V} v\|_0 \le \kappa h \|(\mathcal{V}v)'\|_0 \le C\kappa h \|v\|_0. \qquad \square$$

**Corollary 5.** *If the meshes are quasi-uniform, then there exists* $C_0 > 0$ *such that*

$$\|\mathcal{B}_h \mathcal{V} w_h - \mathcal{P}_h \mathcal{V} w_h\|_0 \le C_0 \kappa \|w_h\|_{-1}, \quad \forall\, w_h \in S_h,$$

*where* $C_0$ *is independent of* $w_h$, $h$ *and the rule* $q$.

*Proof.* Let $v = w_h$ in (26) of Proposition 4. Then apply the inverse assumption on $S_h$,

$$\|w_h\|_0 \le ch^{-1}\|w_h\|_{-1},$$

to complete the proof. □

**Corollary 6.** *Assume the meshes are quasi-uniform. Let* $\gamma$ *and* $h_0$ *be as in Lemma* 2, *and let* $C_0$ *be as in Corollary* 5. *Then, for* $\kappa < (C_0\gamma)^{-1}$ *and* $h < h_0$,

$$\|w_h\|_{-1} \le (1/\gamma - C_0\kappa)^{-1}\|\mathcal{B}_h\mathcal{V}w_h\|_0 \quad \forall\, w_h \in S_h. \tag{28}$$

*Proof.* From the triangle inequality,

$$\begin{aligned}\|\mathcal{B}_h\mathcal{V}w_h\|_0 &\ge \|\mathcal{P}_h\mathcal{V}w_h\|_0 - \|\mathcal{B}_h\mathcal{V}w_h - \mathcal{P}_h\mathcal{V}w_h\|_0\\ &\ge (1/\gamma)\|w_h\|_{-1} - C_0\kappa\|w_h\|_{-1},\end{aligned}$$

where we used Lemma 3 and Corollary 5. □

Now we return to the proof of Theorem 1. If the conditions of Corollary 6 are satisfied, then the operator $\mathcal{B}_h\mathcal{V}$ is a one-to-one operator on the finite dimensional space $S_h$. From this it follows that a solution $u_h \in S_h$ of (25) exists and is unique. Moreover, from (28), (25) and Proposition 4, we have, with

$$\gamma' = \left(\frac{1}{\gamma} - C_0\kappa\right)^{-1},$$

$$\begin{aligned}\|u_h - u_h^G\|_{-1} &\le \gamma'\|\mathcal{B}_h\mathcal{V}(u_h - u_h^G)\|_0\\ &= \gamma'\|\mathcal{B}_h\mathcal{V}(u - u_h^G)\|_0\\ &= \gamma'\|(\mathcal{B}_h - \mathcal{P}_h)\mathcal{V}(u - u_h^G)\|_0\\ &\le \gamma' C\kappa h\|u - u_h^G\|_0\\ &\le Ch^{t+1}\|u\|_t \quad \text{for } 0 \le t \le 1.\end{aligned}$$

In the last step, standard results for the convergence of the Galerkin method are used.

From the inverse inequality we now find, for $s$ satisfying (14), that

$$\|u_h - u_h^G\|_s \le Ch^{-1-s}\|u_h - u_h^G\|_{-1} \le Ch^{t-s}\|u\|_t,$$

so that with the aid of the triangle inequality,

$$\|u_h - u\|_s \le \|u_h - u_h^G\|_s + \|u_h^G - u\|_s \le Ch^{t-s}\|u\|_t.$$

This proves the theorem. □

**6. Conclusion.** The main theorem of this paper asserts that the semi-discrete piecewise constant Galerkin method, for the single layer equation on Lipschitz curves, is stable and has conventional orders of convergence, provided that the Peano constant of the underlying quadrature rule is small enough. The theorem does *not* say exactly how small the Peano constant needs to be. Therefore we are unable to say whether any particular version of the method, for example, collocation at the midpoints, is stable. In this respect, the situation is similar in principle to familiar statements such as that the Galerkin method is stable if $h$ is sufficiently small. The imprecision arises from the same source, namely, the difficulty in practice of keeping track of the constants.

For practitioners, our advice arising from this analysis is: *if the method is unstable with a given quadrature rule $q$, and if reducing $h$ does not help, then replace $q$ by a quadrature rule with a smaller Peano constant and try again.*

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