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AN ALGORITHM FOR THE NUMERICAL RESOLUTION OF A CLASS OF SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. We consider a class of integral equations of Volterra type with constant coefficients containing a logarithmic difference kernel. This equation can be transformed into an equivalent singular equation of Cauchy type which allows us to give the explicit formula for the solution. The numerical method proposed in this paper consists of applying the Lagrange interpolation to the inner Cauchy type singular integral in the latter formula after subtracting the singularity. For the error of this method weighted norm estimates as well as estimates on discrete subsets of knots are given. The paper concludes with some numerical examples.

1. Introduction. In this paper we consider the following integral equation

$$a\int_{-1}^{x} g(t)\,dt + \frac{b}{\pi}\int_{-1}^{1} g(t)\log|x-t|\,dt = f(x), \qquad -1 < x < 1, \tag{1.1}$$

under the hypothesis $a, b \in R$, $a^2 + b^2 = 1$, and $f(x) \in C^{p+\lambda}([-1,1])$, $p \geq 1$, and $0 < \lambda \leq 1$, where $C^{p+\lambda}(A)$ is the class of the functions that have p continuous derivatives in A and the p-th derivative is in the space $\mathrm{Lip}_{\lambda} A$, i.e.,

$$\mathrm{Lip}_{\lambda} A := \left\{ f \in C^0(A) : \sup_{x \neq y \in A} \frac{|f(x) - f(y)|}{|x-y|^{\lambda}} < \infty \right\}.$$

In case $a = 0$ equation (1.1) coincides with Carleman's equation [**1**]. The integral equation (1.1) has also been considered in [**8**] but uses a different approach.

Here, with the aid of the relation

$$h(x) = \int_{-1}^{x} g(t)\,dt - h_0 \frac{1+x}{2} \tag{1.2}$$

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with some constant $h_0 \in c$ and the boundary conditions

$$h(-1) = h(1) = 0, \tag{1.3}$$

we introduce a new unknown function $h(x)$, and the equation (1.1) becomes

$$ah(x) - \frac{b}{\pi}\int_{-1}^{1}\frac{h(t)}{t-x}\,dt = f_0(x), \qquad -1 < x < 1 \tag{1.4}$$

where

$$\begin{aligned} f_0(x) &= f(x) - ah_0\frac{1+x}{2} - \frac{b}{\pi}\left[\left(h(t) + h_0\frac{1+t}{2}\right)\log|x-t|\right]_{-1}^{1} \\ &\quad + \frac{h_0 b}{2\pi}\int_{-1}^{1}\frac{1+t}{t-x}\,dt \\ &= f(x) - ah_0\frac{1+x}{2} + \frac{h_0 b}{\pi} - \frac{h_0 b}{\pi}\log(1-x) \\ &\quad + \frac{h_0 b}{2\pi}(1+x)\log\frac{1-x}{1+x} \\ &= f(x) - h_0\left\{a\frac{1+x}{2} - \frac{b}{\pi} - \frac{b}{2\pi}[(1-x)\log(1-x) \right. \\ &\qquad \left. + (1+x)\log(1+x)]\right\}. \end{aligned}$$

Thus, f_0' is integrable if and only if f' is integrable. Taking into account the boundary conditions (1.3) we look for a solution of (1.4) in the form

$$h(x) = w(x)\bar{h}(x)$$

where the Jacobi weight $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ is defined for index $\chi := (-\alpha+\beta) = -1$. We obtain that (1.4) is solvable for all $f_0 \in L^2_{w^{-1}}$ satisfying

$$\int_{-1}^{1} w^{-1}(x) f_0(x)\,dx = 0. \tag{1.5}$$

Moreover, we have $\bar{h}(x) = (\hat{A}f_0)(x)$, where

$$(\hat{A}f_0)(x) = aw^{-1}(x)f_0(x) + \frac{b}{\pi}\int_{-1}^{1} w^{-1}(t)\frac{f_0(t)}{t-x}\,dt. \tag{1.6}$$

See [**14**] for more details.

The condition (1.5) determines the constant h_0 in (1.2), i.e.,

$$h_0 = \frac{1}{H}\int_{-1}^{1} w^{-1}(x)f(x)\,dx,$$
$$H = \int_{-1}^{1} w^{-1}(x)\bigg\{a\frac{1+x}{2} - \frac{b}{\pi} - \frac{b}{2\pi}[(1-x)\log(1-x) + (1+x)\log(1+x)]\bigg\}\,dx.$$

For the solution of (1.1) we obtain the expression

$$g(x) = h'(x) + \frac{h_0}{2} = af_0'(x) + \frac{b}{\pi}\frac{d}{dx}\bigg[w(x)\int_{-1}^{1} w^{-1}(t)\frac{f_0(t)}{t-x}\,dt\bigg] + \frac{h_0}{2}. \tag{1.7}$$

There exist many methods for the numerical procedure to solve (1.4). Among others we mention [**9,10**] and the book [**17**] and the references therein. The main idea of this paper is to present a numerical procedure for formula (1.7) applying the Lagrange interpolation and the quadrature rule to the inner Cauchy type singular integral after subtracting the singularity. The paper is organized as follows: in Section 2 we describe the numerical procedure, in Section 3 the error estimates are formulated, in Section 4 we prove the theorem listed in the previous section. Finally in Section 5 some numerical examples are given.

2. The numerical method. In the following, given two expressions A and B depending on some variables, we will write $A \sim B$ if and only if $|AB^{-1}| \leq \text{const}$ and $|A^{-1}B| \leq \text{const}$, uniformly in the variables under consideration.

Consider the case

$$\int_{-1}^{1} w^{-1}(x)f(x)\,dx = 0,$$

that means $h_0 = 0$. Since $\hat{A} : L^2_{w^{-1}} \to L^2_w$ is a singular integral operator with index 1, we have

$$aw^{-1}(x) + \frac{b}{\pi}\int_{-1}^{1}\frac{w^{-1}(t)}{t-x}\,dt = 0. \tag{2.1}$$

With the notation

$$F(f;x)=\int_{-1}^{1}\frac{f(t)-f(x)}{t-x}w^{-1}(t)\,dt \tag{2.2}$$

we obtain from (1.7) (with f instead of f_0 and with $h_0=0$) and from (2.1)

$$\begin{aligned}g(x)&=af'(x)+\frac{b}{\pi}\frac{d}{dx}\left[w(x)\int_{-1}^{1}w^{-1}(t)\frac{f(t)-f(x)}{t-x}\,dt\right]\\&\quad+\frac{d}{dx}\left\{w(x)f(x)\left[\frac{b}{\pi}\int_{-1}^{1}\frac{w^{-1}(t)}{t-x}\,dt+aw^{-1}(x)\right]-af(x)\right\}\\&=\frac{b}{\pi}\frac{d}{dx}[w(x)F(f;x)],\end{aligned}$$

which leads to

$$\varphi^2(x)g(x)=\frac{b}{\pi}(\beta-\alpha-x)w(x)F(f;x)+\frac{b}{\pi}\varphi^2(x)w(x)\frac{d}{dx}F(f;x), \tag{2.3}$$

where, here and in the sequel, $\varphi(x)=\sqrt{1-x^2}$.

Our numerical procedure consists of approximating the function $F(f;x)$, defined in (2.2), by using a Lagrange interpolating polynomial on suitable knots, i.e., of replacing the F by

$$L_M(Ff;x)=\sum_{j=1}^{M}l_{M,j}F(f;\xi_j) \tag{2.4}$$

where $l_{M,j}(x)$ are the fundamental Lagrange polynomials

$$l_{M,j}(x)=\prod_{\substack{k=1\\ j\neq k}}^{M}\frac{x-\xi_k}{\xi_j-\xi_k}.$$

Since

$$F(f;\xi_j)=\int_{-1}^{1}\frac{f(t)-f(\xi_j)}{t-\xi_j}w^{-1}(t)\,dt$$

the real problem is to evaluate $F(f;\xi_j)$. By this reason, we remember some interlacing properties of the zeros of orthogonal polynomials.

Consider the weight $\bar{w}(x) = \varphi^2(x)w^{-1}(x) = (1-x)^{1-\alpha}(1+x)^{1-\beta}$ and denote by $x_{m,j}(\bar{w}) = \cos\tau_{m,j}$, $j = 1,\dots,m$, the zeros of the mth Jacobi polynomial $p_m^{(1-\alpha,1-\beta)}$ orthogonal with respect to the weight $\bar{w}$. Further, denote by $x_{m+1,i}(w^{-1}) = \cos\theta_{m+1,i}$, $i = 1,\dots,m+1$, the zeros of $p_{m+1}^{(-\alpha,-\beta)}$ orthogonal with respect to the weight w^{-1}. It is well known that [**18**]

$$x_{m+1,i}(w^{-1}) < x_{m,i}(\bar{w}) < x_{m+1,i+1}(w^{-1}), \qquad i = 1,\dots,m.$$

Moreover, the relation

$$\min_{i,j} |\tau_{m,j} - \theta_{m+1,i}| \sim m^{-1} \tag{2.5}$$

holds [**3**].

We assume now, as interpolation knots ξ_j, the following numbers

$$\xi_1 = \frac{-1 + x_{m,1}(\bar{w})}{2}, \qquad \xi_{j+1} = x_{m,j}(\bar{w}), \qquad j = 1,\dots,m,$$
$$\xi_{m+2} = \frac{+1 + x_{m,m}(\bar{w})}{2}, \qquad M = m+2.$$

This choice of ξ_j allows us to evaluate F by using a Gaussian quadrature rule on $m+1$ points avoiding the numerical cancellation phenomenon. Furthermore, we have

$$\begin{aligned} F(f;\xi_j) &= \int_{-1}^{1} \frac{f(t) - f(\xi_j)}{t - \xi_j} w^{-1}(t)\,dt \\ &= \sum_{i=1}^{m+1} \frac{\lambda_{m+1,i}(w^{-1})}{x_{m+1,i}(w^{-1}) - \xi_j} (f(x_{m+1,i}(w^{-1}) - f(\xi_j)) \\ &\quad + e_{m+1}(Ff;\xi_j) \\ &= F_m(f,\xi_j) + e_{m+1}(Ff;\xi_j) \end{aligned} \tag{2.6}$$

where $\lambda_{m+1,i}(w^{-1})$ denote the Christoffel numbers of the Gaussian quadrature rule with respect to the weight w^{-1} and $e_{m+1}(Ff;\xi_j)$ represents the Gaussian quadrature error with respect to the weight w^{-1} and to the function $\int_{-1}^{1}[(f(t) - f(\xi_j))/(t-\xi_j)]w^{-1}(t)\,dt$. Neglecting the error $e_{m+1}(Ff;\xi_j)$, we replace in (2.4) the $F(f;\xi_j)$ by the value

$F_m(f;\xi_j)$, and we assume, as approximate solution of g, the following expression

$$\varphi^2(x)g_m(x) = \frac{b}{\pi}w(x)(\beta-\alpha-x)L_{m+2}(F_m f;x) + \frac{b}{\pi}w(x)\varphi^2(x)\frac{d}{dx}L_{m+2}(F_m f;x). \tag{2.7}$$

Remarks. i) Note that, as can be observed by the numerical examples given in Section 5, it is possible to evaluate numerically with a stable procedure the derivative of $L_{m+2}(Ff;x)$.

ii) If we need to evaluate the solution $g(x)$ of Equation (1.1) on the knots ξ_j, then we can apply more directly the following relation

$$\varphi^2(\xi_j)\bar{g}_m(\xi_j) = \frac{b}{\pi}w(\xi_j)(\beta-\alpha-\xi_j)F_m(f;\xi_j) + \frac{b}{\pi}w(\xi_j)\varphi^2(\xi_j)\left[\frac{d}{dx}F_m(f;x)\right]_{\xi_j}. \tag{2.8}$$

iii) Equation (1.4) is a classical singular integral equation of Cauchy type. Applying directly the proposed method to equation (1.6), we obtain

$$h_m(x) = \frac{b}{\pi}w(x)L_{m+2}(F_m f;x)$$

as an approximate solution of (1.4).

For this approximation method, we obtain by the estimates (4.7), (4.8) and (4.9) given in Section 4,

$$||h-h_m|| \le \begin{cases} \dfrac{\text{const}\,\log^2 m}{m^{2(1+\lambda-\tau)}}, & \text{for } p=1, \text{ if } 1+\lambda-2\sigma<0; \\ \dfrac{\text{const}\,\log^2 m}{m^{p+\lambda}}, & \text{for } p\ge 2 \text{ and for } p=1, \\ & \text{if } 1+\lambda-2\tau\ge 0; \end{cases}$$

where $\sigma=\min\{\alpha,\beta\}$, $\tau=\max\{\alpha,\beta\}$.

We expressly note that in the second case the result is identical with the result obtained for the collocation method for the Cauchy singular integral equations with index $\chi\in\{0,1\}$ (see [**2**, Theorem 3.1]).

3. The error of the numerical method. In this section we give error estimates for the proposed method that can be used in different occasions.

Since, as we can deduce from Section 1, the solution $g(x)$ of (1.1) is not bounded in ± 1, it does not make sense to estimate the error uniformly on $[-1,1]$. Nevertheless, the following theorem provides a weighted estimate on the whole interval.

Theorem 3.1. *If* $f(x) \in C^{p+\lambda}([-1,1])$, $p \geq 1$, $0 < \lambda \leq 1$, *then*

$$||\varphi^2[g - g_m]|| \leq \frac{\text{const } \log m}{m^{p+\lambda-1}}. \tag{3.1}$$

Remark 1. Sometimes it is sufficient to know the values of the solution in closed subintervals contained in $(-1,1)$. Then, by the previous theorem, the following estimate holds

$$|g(x) - g_m(x)| \leq \frac{\text{const } \log m}{m^{p+\lambda-1}}, \quad \forall\, x \in [a,b] \subset (-1,1). \tag{3.2}$$

Nevertheless, in the case of the closed subintervals, the interpolation knots can be chosen as $\xi_j = x_{m,j}(\bar{w})$, $j = 1, \dots, m$, i.e., without the additional knots $(-1 + x_{m,1}(\bar{w}))/2, (1 + x_{m,m}(\bar{w}))/2$. Finally, when it is sufficient to evaluate the solution $g(x)$ on the points ξ_j, then, as we have just observed in Remark ii) of the previous section, we use the form (2.8), and the following theorem gives us estimates of the error in this case.

Theorem 3.2. *If* $f(x) \in C^{p+\lambda}([-1,1])$, $0 < \lambda \leq 1$, *then*

$$|g(\xi_j) - \bar{g}_m(\xi_j)| \leq \begin{cases} \dfrac{\text{const } \log m}{m^{\lambda}}, & \textit{for } p = 1, \\ & \textit{if } 1 + \lambda - 2\sigma < 0; \\ \dfrac{\text{const } \log m}{m^{r}}, & \textit{for } p = 1,\ \textit{if } 1 + \lambda - 2\tau \geq 0, \\ & r = \min\{\lambda, \lambda + 2\sigma - 1\}, \\ & \textit{if } \lambda + 2\sigma - 1 > 0; \\ \dfrac{\text{const } \log m}{m^{q}}, & \textit{for } p \geq 2, \\ & q = \min\{p+\lambda-1, p+\lambda+2\sigma-2\} \end{cases} \tag{3.3}$$

where $\sigma = \min\{\alpha, \beta\}$, $\tau = \max\{\alpha, \beta\}$, $g(x)$ *and* $\bar{g}_m(x)$ *are defined by* (2.3) *and* $(2,8)$, *respectively.*

4. Proofs. First of all, we remember some well-known properties of the Jacobi polynomials $p_m^{(\gamma,\delta)}$ orthogonal with respect to the weight $v(x) = (1-x)^\gamma(1+x)^\delta$, $\gamma, \delta > -1$, which will be used in the following. Denoting by $x_{m,k}(v) = \cos\theta_{m,k}$, $k = 1, \dots, m$ the zeros of $p_m^{(\gamma,\delta)}$ and by $\lambda_{m,k}(v)$, $k = 1, \dots, m$ the Christoffel constants, the following equivalences hold (see [**15**]):

$$\begin{gathered}\theta_{m,k} - \theta_{m,k+1} \sim m^{-1}, \\ \text{uniformly for } 0 \le k \le m, \qquad m \in N,\end{gathered} \tag{4.1}$$

$$\begin{gathered}\lambda_{m,k}(v) \sim m^{-1} v^{\gamma+1/2,\delta+1/2}(x_{m,k}(v)), \\ \text{uniformly for } 1 \le k \le m, \qquad m \in N.\end{gathered} \tag{4.2}$$

We remark that from (4.1) it follows (see [**15**]) that

$$v^{(\gamma,\delta)}(x) \sim v^{(\gamma,\delta)}(x_{m,k}(v)), \qquad x_{m,k-1}(v) \le x \le x_{m,k+1}(v) \tag{4.3}$$

for $k = 2, 3, \dots, m-1$.

Furthermore, since $x_{m,k+1}(v) - x_{m,k}(v) = (\theta_{m,k} - \theta_{m,k+1})\sqrt{1-\cos^2\theta}$ with $x_{m,k}(v) < \cos\theta < x_{m,k+1}(v)$, in view of (3.1) we can state

$$\begin{gathered}x_{m,k+1}(v) - x_{m,k}(v) \sim m^{-1} \begin{cases} \sqrt{1 - x_{m,k}^2(v)}, \\ \sqrt{1 - x_{m,k+1}^2(v)}, \end{cases} \\ k \in \{1, 2, \dots, m-1\}, \qquad m \in N.\end{gathered} \tag{4.4}$$

For given $x \in (-1, 1)$, $m \in N$, we denote by $x_{m,c}(v)$ the closest knot to x, defined by

$$x_{m,c}(v) = \begin{cases} x_{m,d}(v) & \text{if } x - x_{m,d}(v) \le x_{m,d+1}(v) - x, \\ x_{m,d+1}(v) & \text{if } x - x_{m,d}(v) > x_{m,d+1}(v) - x, \end{cases}$$

where $x_{m,d}(v) \le x \le x_{m,d+1}(v)$ for some $d \in \{0, 1, \dots, m\}$ with $x_{m,0}(v) = \cos\theta_{m,0} = -1$, $x_{m,m+1}(v) = \cos\theta_{m,m+1} = 1$.

Now, for any fixed $x \in (-1,1)$, let us consider the set $N^{\sim}(x) = \{m \in N : |\theta_{m,c} - \theta| \sim m^{-1}\}$ where, as before, $x_{m,c}(v) = \cos\theta_{m,c}$, $x = \cos\theta$. We know that this set is infinite (see [**4**, Lemma 3.1, **5**, Lemmas 2.1, 2.2]) and, moreover, there exist some values of x such that $N^{\sim}(x) = N$, (see [**11**, p. 230]).

For the proofs of the theorems stated in the previous section, the following lemmas are needed.

Lemma 4.1. *Let $f(x) \in C^p([-1,1])$. Then, for $m \geq 4p+5$, there exists a sequence of polynomials $\{Q_m\}$ such that, for $|x| \leq 1$ and $j = 0,\dots,p$,*

$$|f^{(j)}(x) - Q_m^{(j)}(x)| \leq \text{const}\left(\frac{\sqrt{1-x^2}}{m}\right)^{p-j} \omega\left(f^{(p)}; \frac{\sqrt{1-x^2}}{m}\right)$$

where $\omega(f;x)$ is the modulus of continuity of the function f and the constant is independent of f and m.

Lemma 4.1 can be found in [**16**, p. 307].

Setting $I_m = [x-(1+x)/(2m^4), x+(1-x)/(2m^4)]$, $I'_m = [-1,1]-I_m$, we have

Lemma 4.2. *Let $v(x) = (1-x)^\gamma(1+x)^\delta$, $\gamma, \delta > -1$, be a Jacobi weight. The inequalities*

$$\int_{I'_m} \frac{v(t)}{|t-x|}\,dt \leq \begin{cases} \text{const}\,\log m, & \text{if } \gamma,\delta \geq 0, \\ \text{const}\,v(x)\log m, & \text{if } -1 < \gamma,\delta < 0, \end{cases} \tag{4.5}$$

hold uniformly for $x \in (-1,1)$.

The proof of this lemma can be found in [**6**, Lemma 3.3, p. 453].

Denote by $\sigma^*_m(x) = \sum_{i=1}^{m'} \lambda_{m,i}(v)/|x_{m,i}(v) - x|$, where, here and in the sequel, the prime denotes a sum in which the term corresponding to the knot $x_{m,c}$ closest to x is omitted; we have

Lemma 4.3. *Let $v(x) = (1-x)^\gamma(1+x)^\delta$, $\gamma, \delta > -1$ be a Jacobi* *weight. The inequalities*

$$\sigma^*_m(x) \le \begin{cases} \text{const } \log m, & \text{if } \gamma,\delta \ge 0, \\ \text{const } v(x)\log m, & \text{if } -1<\gamma,\delta<0, \end{cases} \tag{4.6}$$

hold uniformly for $x \in (-1,1)$.

The proof of this lemma can be found in [**6**, Lemma 3.4, p. 454].

Lemma 4.4. *If* $f(x) \in C^{p+\lambda}([-1,1])$, $p \ge 1$, $0 < \lambda \le 1$, *then the following estimates hold*:

(4.7)
$$||wF(r_m)|| \le \frac{\text{const } \log m}{m^{p+\lambda}};$$

(4.8)
$$||wL_m(Fr_m)|| \le \begin{cases} \dfrac{\text{const } \log^2 m}{m^{2(1+\lambda-\tau)}}, & \text{for } p=1, \text{ if } 1+\lambda-2\sigma<0, \\ \dfrac{\text{const } \log^2 m}{m^{p+\lambda}}, & \text{for } p\ge 2, \text{ and} \\ & \text{for } p=1, \text{ if } 1+\lambda-2\tau\ge 0; \end{cases}$$

(4.9)
$$||wL_m(e_m(Ff))|| \le \begin{cases} \dfrac{\text{const } \log^2 m}{m^{2(1+\lambda-\tau)}}, & \text{for } p=1, \text{ if } 1+\lambda-2\sigma<0, \\ \dfrac{\text{const } \log^2 m}{m^{p+\lambda}}, & \text{for } p\ge 2, \text{ and} \\ & \text{for } p=1, \text{ if } 1+\lambda-2\tau\ge 0; \end{cases}$$

(4.10)
$$||w\varphi^2 F'(r_m)|| \le \frac{\text{const } \log m}{m^{p+\lambda-1}};$$

where $r_m(x) = f(x) - Q_m(x)$, $Q_m(x)$ *denotes the polynomial of Lemma* 4.1, $e_m(Ff)$ *is defined by* (2.6), $\sigma = \min\{\alpha,\beta\}$, *and* $\tau = \max\{\alpha,\beta\}$.

Proof. We can write

$$|F(r_m;x)| = \left| \int_{-1}^{1} \frac{r_m(t) - r_m(x)}{t-x} w^{-1}(t)\,dt \right| \\ \le \int_{I_m} \left| \frac{r_m(t) - r_m(x)}{t-x} \right| w^{-1}(t)\,dt + |r_m(x)| \int_{I'_m} \frac{w^{-1}(t)}{|t-x|}\,dt \\ + \int_{I'_m} \frac{|r_m(t)| w^{-1}(t)}{|t-x|}\,dt := A_1 + A_2 + A_3.$$

By Lemma 4.1, since $1 \pm t \sim 1 \pm x$, for $t, x \in I_m$, we have

$$(4.11) \qquad A_1 \le \int_{I_m} |r'_m(y_x)| w^{-1}(t)\,dt \\ \le \frac{(1-x^2)^{(p+\lambda-1)/2} w^{-1}(x)}{m^{p+\lambda-1}} \int_{I_m} dt \\ \le \frac{(1-x^2)^{(p+\lambda-1)/2} w^{-1}(x)}{m^{p+\lambda+3}}.$$

Also by Lemma 4.1 and by (4.5), we find

(4.12)
$$A_2 \le \frac{(1-x^2)^{(p+\lambda)/2} w^{-1}(x) \log m}{m^{p+\lambda}};$$

(4.13)
$$A_3 \le \frac{1}{m^{p+\lambda}} \int_{I'_m} \frac{(1-t^2)^{(p+\lambda)/2} w^{-1}(t)}{|t-x|}\,dt \\ \le \begin{cases} \dfrac{(1-x^2)^{(1+\lambda)/2} w^{-1}(x) \log m}{m^{1+\lambda}}, & \text{for } p = 1, \\ & \text{if } 1+\lambda-2\sigma<0 \\ \dfrac{\log m}{m^{p+\lambda}}, & \text{for } p \ge 2, \text{ and } p = 1, \\ & \text{if } 1+\lambda-2\tau \ge 0, \end{cases}$$

where $\sigma = \min\{\alpha, \beta\}$, $\tau = \max\{\alpha, \beta\}$.

Then, by (4.11)–(4.13) the relation (4.7) holds. To prove the estimate (4.8), we observe that, for $|x| \le 1 - m^{-2}$

$$(4.14) \qquad |F(r_m;x)| \le \begin{cases} \dfrac{\text{const } \log m}{m^{2(1+\lambda-\tau)}}, & \text{for } p=1, \text{ if } 1+\lambda-2\sigma<0, \\ \dfrac{\text{const } \log m}{m^{p+\lambda}}, & \text{for } p \ge 2, \text{ and for } p=1, \\ & \text{if } 1+\lambda-2\tau \ge 0. \end{cases}$$

Therefore, recalling that $|\xi_j| \le 1 - m^{-2}$, $j = 1, \dots, m+2$, we have

$$\begin{aligned} ||wL_m(Fr_m)|| &\le \max_{|x|\le 1} w(x) \sum_{j=1}^{m} |l_{m,j}(x)| \\ &\quad \cdot \left| \int_{-1}^{1} \frac{r_m(t)-r_m(\xi_j)}{t-\xi_j} w^{-1}(t)\,dt \right| \\ &\le \begin{cases} \dfrac{\text{const } \log m ||L_m||}{m^{2(1+\lambda-\tau)}}, & \text{for } p=1, \\ & \text{if } 1+\lambda-2\sigma<0, \\ \dfrac{\text{const } \log m ||L_m||}{m^{p+\lambda}}, & \text{for } p \ge 2, \text{ and for } p=1, \\ & \text{if } 1+\lambda-2\tau \ge 0, \end{cases} \end{aligned}$$

where $||L_m|| = \max_{-1\le x\le 1} \sum_{j=1}^{m} |l_{m,j}(x)|$ is the Lebesgue constant corresponding to the Lagrange interpolation process. Therefore, remembering [**12**, Theorem 3.1] the relation (4.8) holds.

Now we observe that, for $x = \xi_j$, $j = 1, \dots, m+2$, we have

$$\begin{aligned} |e_m(Ff;x)| &\le |F(r_m;x)| + |r_m(x)| {\sum_{i=1}^{m}}' \frac{\lambda_{m,i}(w^{-1})}{|x_{m-1}(w^{-1})-x|} \\ &\quad + {\sum_{i=1}^{m}}' \frac{\lambda_{m,i}(w^{-1})|r_m(x_{m,i}(w^{-1}))|}{|x_{m,i}(w^{-1})-x|} \\ &\quad + \lambda_{m,c}(w^{-1}) \frac{|r_m(x_{m,c}(w^{-1}))-r_m(x)|}{|x_{m,c}(w^{-1})-x|} \\ &:= B_1 + B_2 + B_3 + B_4. \end{aligned}$$

B_1 is estimated by (4.14). By Lemma 4.1 and relation (4.6), we have:

$$(4.15) \qquad B_2 \le \frac{(1-x^2)^{(p+\lambda)/2} w^{-1}(x) \log m}{m^{p+\lambda}}.$$

Again, by Lemma 4.1, relations (4.2) and (4.4), we can write

$$B_3 \le \frac{1}{m^{p+\lambda}} \sum_{i=1}^{m}{}' \times \frac{(1-x_{m,i}^2(w^{-1}))^{(p+\lambda)/2} w^{-1}(x_{m,i}(w^{-1}))(x_{m,i+1}(w^{-1})-x_{m,i}(w^{-1}))}{|x_{m,i}(w^{-1})-x|}.$$

Since relation (2.5) holds, we can consider the sum in the righthand side of the estimate written before, as a Riemann sum. Thus, again by (4.5),

(4.16)

$$B_3 \le \frac{1}{m^{p+\lambda}} \int_{I'_m} \frac{(1-t^2)^{(p+\lambda)/2} w^{-1}(t)}{|t-x|}\, dt \le \begin{cases} \dfrac{(1-x^2)^{(1+\lambda)/2} w^{-1}(x) \log m}{m^{1+\lambda}}, & \text{for } p=1, \text{ if } 1+\lambda-2\sigma<0 \\ \dfrac{\log m}{m^{p+\lambda}}, & \text{for } p \ge 2, \text{ and } p=1 \\ & \text{if } 1+\lambda-2\tau \ge 0, \end{cases}$$

with, as before, $\sigma = \min\{\alpha, \beta\}$, $\tau = \max\{\alpha, \beta\}$. By applying Lemma 4.1, relations (4.2) and (4.3), we have

$$B_4 \le \frac{|r'_m(y_c)| w^{-1}(x_{m,c}(w^{-1}))(1-x_{m,c}^2(w^{-1}))^{1/2}}{m} \le \frac{\text{const}\,(1-x^2)^{(p+\lambda)/2} w^{-1}(x)}{m^{p+\lambda}}. \tag{4.17}$$

Therefore, by relations (4.14)–(4.17), we obtain, for $|x| \le 1-m^{-2}$:

$$|e_m(Ff;x)| \le \begin{cases} \dfrac{\text{const } \log m}{m^{2(1+\lambda-\tau)}}, & \text{for } p=1, \text{ if } 1+\lambda-2\sigma<0, \\ \dfrac{\text{const } \log m}{m^{p+\lambda}}, & \text{for } p \ge 2, \text{ and for } p=1 \\ & \text{if } 1+\lambda-2\tau \ge 0. \end{cases} \tag{4.18}$$

Following the same procedure used for the estimate (4.8), we can deduce relation (4.9) taking into account relation (4.18) and again [**12**, Theorem 3.1] for the interpolation process. In order to obtain estimate

(4.10), we consider the two cases $p = 1$ and $p \geq 2$. For $p = 1$, we have (see [**7**, Lemma 9, p. 146])

$$|F'(r_m; x)| \leq \text{const}\, w^{-1}(x) \left(\frac{\varphi(x)}{m}\right)^{\lambda} \log \frac{m}{\varphi(x)}.$$

Therefore

$$\begin{aligned} |w(x)\varphi^2(x)F'(r_m; x)| &\leq \frac{\text{const}\, \varphi^{2+\lambda}(x)}{m^{\lambda}} \left[\log m + \log \frac{1}{\varphi(x)}\right] \\ &\leq \frac{\text{const}\, \log m}{m^{\lambda}}. \end{aligned} \tag{4.19}$$

For $p \geq 2$, setting $I''_m = [x - (1+x)/(2m), x + (1-x)/(2m)]$, $I'''_m = [-1,1] - I''_m$, we have

$$\begin{aligned} |F'(r_m; x)| &= \left|\frac{d}{dx}\int_{-1}^{1} \frac{r_m(t) - r_m(x)}{t - x} w^{-1}(t)\, dt\right| \\ &= \left|\int_{-1}^{1} \frac{d}{dx} \frac{r_m(t) - r_m(x)}{t - x} w^{-1}(t)\, dt\right| \\ &= \left|\int_{-1}^{1} \frac{-r'_m(x)(t-x) + r_m(t) - r_m(x)}{(t-x)^2} w^{-1}(t)\, dt\right| \\ &\leq \int_{I''_m} \left|\frac{-r'_m(x)(t-x) + r_m(t) - r_m(x)}{(t-x)^2}\right| w^{-1}(t)\, dt \\ &\quad + |r'_m(x)| \int_{I'''_m} \frac{w^{-1}(t)}{|t-x|}\, dt \\ &\quad + |r_m(x)| \int_{I'''_m} \frac{w^{-1}(t)}{(t-x)^2}\, dt + \int_{I'''_m} \frac{|r_m(t)| w^{-1}(t)}{(t-x)^2}\, dt \\ &:= C_1 + C_2 + C_3 + C_4. \end{aligned}$$

by Lemma 4.1, since $1 \pm t \sim 1 \pm x$, for $t, x \in I''_m$, we have

$$\begin{aligned} C_1 &\leq \int_{I''_m} |r''_m(y_x)| w^{-1}(t)\, dt \\ &\leq \frac{(1-x^2)^{(p+\lambda-2)/2} w^{-1}(x)}{m^{p+\lambda-2}} \int_{I''_m} dt \\ &\leq \frac{\text{const}\, w^{-1}(x)}{m^{p+\lambda-1}}. \end{aligned} \tag{4.20}$$

By Lemma 4.1, we have

$$C_2 \le \frac{(1-x^2)^{(p+\lambda-1)/2}}{m^{p+\lambda-1}}\bigg\{(1-x)^{-\alpha}\int_{-1}^{x-(1+x)/(2m)} \frac{(1+t)^{-\beta}}{x-t}\,dt + (1+x)^{-\beta}\int_{x+(1-x)/(2m)}^{1} \frac{(1-t)^{-\alpha}}{t-x}\,dt\bigg\}.$$

Using the two linear transformations $(1+t) = (1+x)y$ and $(1-t) = (1-x)y$, respectively, in the two integrals in the righthand side of the previous expression, we obtain

$$\tag{4.21} \begin{aligned} C_2 &\le \frac{(1-x^2)^{(p+\lambda-1)/2}w^{-1}(x)}{m^{p+\lambda-1}}\bigg\{\int_0^{1-1/(2m)} \frac{y^{-\beta}}{1-y}\,dy + \int_0^{1-1/(2m)} \frac{y^{-\alpha}}{1-y}\,dy\bigg\} \\ &\le \frac{\mathrm{const}\, w^{-1}(x)\log m}{m^{p+\lambda-1}}. \end{aligned}$$

Again, by Lemma 4.1 and by the same transformations written before, we have

$$\tag{4.22} \begin{aligned} C_3 &\le \frac{(1-x^2)^{(p+\lambda)/2}w^{-1}(x)}{m^{p+\lambda}}\bigg\{\frac{1}{1+x}\int_0^{1-1/(2m)} \frac{y^{-\beta}}{(1-y)^2}\,dy + \frac{1}{1-x}\int_0^{1-1/(2m)} \frac{y^{-\alpha}}{(1-y)^2}\,dy\bigg\} \\ &\le \frac{(1-x)^{(p+\lambda)/2}(1+x)^{(p+\lambda-2)/2}w^{-1}(x)}{m^{p+\lambda-1}} \\ &\quad + \frac{(1-x)^{(p+\lambda-2)/2}(1+x)^{(p+\lambda)/2}w^{-1}(x)}{m^{p+\lambda-1}} \\ &\le \frac{\mathrm{const}\, w^{-1}(x)}{m^{p+\lambda-1}}; \end{aligned}$$

$$
\begin{aligned}
C_4 &\le \frac{1}{m^{p+\lambda}} \int_{I_m'''} \frac{(1-t^2)^{(p+\lambda)/2} w^{-1}(t)}{(t-x)^2}\, dt \\
&\le \frac{(1+x)^{(p+\lambda-2)/2} w^{-1}(x)}{m^{p+\lambda}} \int_0^{1-1/(2m)} \frac{y^{-\beta}}{(1-y)^2}\, dy \\
&\quad + \frac{(1-x)^{(p+\lambda-2)/2} w^{-1}(x)}{m^{p+\lambda}} \int_0^{1-1/(2m)} \frac{y^{-\alpha}}{(1-y)^2}\, dy \\
&\le \frac{\text{const}\, w^{-1}(x)}{m^{p+\lambda-1}}.
\end{aligned}
\tag{4.23}
$$

Then, by (4.20)–(4.23), relation (4.10) easily follows. Thus, Lemma 4.4 is completely proved. □

Proof of Theorem 3.1. By (2.3) and (2.7) we obtain

(4.24)

$$
\begin{aligned}
\varphi^2(x)[g(x)-g_m(x)] &= \frac{b}{\pi} w(x)(\beta-\alpha-x)[F(f;x)-L_{m+2}(F_m f;x)] \\
&\quad + \frac{b}{\pi} w(x)\varphi^2(x)\frac{d}{dx}[F(f;x)-L_{m+2}(F_m f;x)] \\
&:= \frac{b}{\pi} w(x)(\beta-\alpha-x)A_1(x)+\frac{b}{\pi} w(x)\varphi^2(x)\frac{d}{dx}A_1(x).
\end{aligned}
$$

Denote by $Q_{m+1}(x)$ the $m+1$-th polynomial defined by Lemma 4.1, recall that

$$F(Q_{m+1};x) = F_m(Q_{m+1};x) = L_{m+2}(F_m Q_{m+1};x) = L_{m+2}(F Q_{m+1};x)$$

and that

$$L_{m+2}(F_m f;x) = L_{m+2}(Ff;x) - L_{m+2}(e_{m+1}Ff;x).$$

Adding and subtracting $F(Q_{m+1};x)$ in $A_1(x)$ and also recalling the relation $r_{m+1}(x) = f(x) - Q_{m+1}(x)$, we obtain for $A_1(x)$, defined by (4.24), the following expression:

$$A_1(x) = F(r_{m+1};x) - L_{m+2}(Fr_{m+1};x) + L_{m+2}(e_{m+1}(Ff);x)).$$

Therefore, also by applying the Markov-Bernstein inequality, we obtain

$$\begin{aligned}||\varphi^2[g-g_m]|| \le\ & ||wF(r_{m+1})|| + ||wL_{m+2}(Fr_{m+1})|| \\ & + ||wL_{m+2}(e_{m+1}(Ff))|| \\ & + ||w\varphi^2 F'(r_{m+1})|| + m||w\varphi L_{m+2}(Fr_{m+1})|| \\ & + m||w\varphi L_{m+2}(e_{m+1}(Ff))||.\end{aligned}$$

Then, by Lemma 4.4, the estimate (3.1) follows and the theorem is completely proved. □

Proof of Theorem 3.2. By (2.3) and (2.8), we obtain

$$|g(x) - \bar{g}_m(x)| \le \text{const}\, w(x)\left\{\frac{e_{m+1}(Ff;x)}{1-x^2} + \frac{d}{dx}e_{m+1}(Ff;x)\right\}, \quad x = \xi_j, \qquad j = 1, m+2.$$

Therefore, remembering our choice of the ξ_j, i.e., $|\xi_j| \le 1 - m^{-2}$, by relations (4.11)–(4.13) and (4.15)–(4.17), we have

(4.25)
$$\frac{w(x)|e_m(Ff;x)|}{1-x^2} \le \begin{cases} \dfrac{\text{const}\,\log m}{m^{2\lambda}}, & \text{for } p=1, \text{ if } 1+\lambda-2\sigma<0; \\ \dfrac{\text{const}\,\log m}{m^{s}}, & \text{for } p=1, \text{ if } 1+\lambda-2\tau\ge 0, \\ & s=\min\{2\lambda, \lambda+2\sigma-1\}, \\ & \text{if } \lambda+2\sigma-1>0; \\ \dfrac{\text{const}\,\log m}{m^{p+\lambda+2\sigma-2}}, & \text{for } p\ge 2, \end{cases}$$

with $\sigma = \min\{\alpha, \beta\}$, $\tau = \max\{\alpha, \beta\}$. Moreover, we have

$$\begin{aligned} w(x)\left|\frac{d}{dx}e_m(Ff;x)\right| \le\ & w(x)\left|\frac{d}{dx}\int_{-1}^{1}\frac{r_m(t)-r_m(x)}{(t-x)}w^{-1}(t)\,dt\right| \\ & + w(x)\left|\frac{d}{dx}{\sum_{i=1}^{m}}' \frac{\lambda_{m,i}(w^{-1}(r_m(x_{m,i}(w^{-1}))-r_m(x))}{x_{m,i}(w^{-1})-x}\right| \\ & + w(x)\left|\frac{d}{dx}\frac{\lambda_{m,c}(w^{-1})(r_m(x_{m,c}(w^{-1}))-r_m(x)}{x_{m,c}(w^{-1})-x}\right| \\ :=\ & D_1 + D_2 + D_3. \end{aligned}$$

Again, we consider the two cases $p = 1$ and $p \geq 2$.

For $p = 1$, by [**7**, Lemma 9, p. 146] and [**7**, Lemma 8, p. 146], we have, for $|x| \leq 1 - m^{-2}$

$$D_1 \leq \frac{\text{const } \log m}{m^\lambda}; \tag{4.26}$$

$$D_2 \leq \frac{\text{const } \log m}{m^\lambda}. \tag{4.27}$$

Now

$$\begin{aligned} D_3 &\leq \frac{w(x)\lambda_{m,c}(w^{-1})|r'_m(x)|}{|x_{m,c}(w^{-1}) - x|} \\ &\quad + \frac{w(x)\lambda_{m,c}(w^{-1})|r_m(x_{m,c}(w^{-1})) - r_m(x)|}{(x_{m,c}(w^{-1}) - x)^2} \\ &\leq \frac{w(x)\lambda_{m,c}(w^{-1})|r'_m(x)|}{|x_{m,c}(w^{-1}) - x|} \\ &\quad + \frac{w(x)\lambda_{m,c}(w^{-1})|r'_m(y_c)|}{|x_{m,c}(w^{-1}) - x|}. \end{aligned}$$

By Lemma 4.1, relations (4.2) and (4.3) and recalling relation (2.5), we obtain

$$D_3 \leq \frac{\text{const}}{m^\lambda}. \tag{4.28}$$

For $p \geq 2$, by relations (4.20)–(4.23), we have

$$D_1 \leq \frac{\text{const } \log m}{m^{p+\lambda-1}}. \tag{4.29}$$

Moreover,

$$\begin{aligned} D_2 &\leq w(x)|r'_m(x)| \sum_{i=0}^{m}{}' \frac{\lambda_{m,i}(w^{-1})}{|x_{m,i}(w^{-1}) - x|} \\ &\quad + w(x)|r_m(x)| \sum_{i=0}^{m}{}' \frac{\lambda_{m,i}(w^{-1})}{(x_{m,i}(w^{-1}) - x)^2} \\ &\quad + w(x) \sum_{i=0}^{m}{}' \frac{\lambda_{m,i}(w^{-1})|r_m(x_{m,i}(w^{-1}))|}{(x_{m,i}(w^{-1}) - x)^2} \\ &:= F_1 + F_2 + F_3. \end{aligned}$$

Recalling relation (4.6) and by Lemma 4.1,

$$F_1 \le \frac{\text{const}\ \log m}{m^{p+\lambda-1}}.$$

Again, by Lemma 4.1, relations (4.2) and (4.4), we can write

$$F_2 \le \frac{w(x)(1-x^2)^{(p+\lambda)/2}}{m^{p+\lambda}} \cdot {\sum_{i=0}^{m}}' \frac{w^{-1}(x_{m,i}(w^{-1}))(x_{m,i+1}(w^{-1})-x_{m,i}(w^{-1}))}{(x_{m,i}(w^{-1})-x)^2}.$$

Recalling relation (2.5), we can consider the sum in the righthand side of the previous expression as a Riemann sum. Therefore, recalling also estimate (4.22), we obtain

$$F_2 \le \frac{w(x)(1-x^2)^{(p+\lambda)/2}}{m^{p+\lambda}} \int_{I_m'''} \frac{w^{-1}(t)}{(t-x)^2}\,dt \le \frac{\text{const}}{m^{p+\lambda-1}}.$$

Similarly, by estimate (4.23), we have

$$F_3 \le \frac{w(x)}{m^{p+\lambda}} \int_{I_m'''} \frac{(1-t^2)^{(p+\lambda)/2} w^{-1}(t)}{(t-x)^2}\,dt \le \frac{\text{const}}{m^{p+\lambda-1}}.$$

Then

$$D_2 \le \frac{\text{const}}{m^{p+\lambda-1}}. \tag{4.30}$$

By Lemma 4.1 and by relations (4.2) and (4.3), we have

$$\begin{aligned} D_3 &\le w(x)\lambda_{m,c}(w^{-1}) \cdot \left| \frac{-r_m'(x_{m,c}(w^{-1}))(x_{m,c}(w^{-1})-x)+r_m(x_{m,c}(w^{-1}))-r_m(x)}{(x_{m,c}(w^{-1})-x)^2} \right| \\ &\le w(x)\lambda_{m,c}(w^{-1})|r_m''(y_c)| \le \frac{\text{const}}{m^{p+\lambda-1}}. \end{aligned} \tag{4.31}$$

Thus, by (4.26)–(4.28) for $p = 1$ and by (4.29)–(4.31) for $p \ge 2$, we have

$$w(x)\frac{d}{dx}|e_m(Ff;x)| \le \frac{\text{const}\ \log m}{m^{p+\lambda-1}}. \tag{4.32}$$

Therefore, by (4.25) and (4.32) we easily obtain estimate (3.3). Thus, the theorem is completely proved. □

5. Numerical examples. Let us consider the integral equation

$$a\int_{-1}^{x} g(t)\,dt + \frac{b}{\pi}\int_{-1}^{1} g(t)\log|x-t|\,dt = w(x)x(1-x^2),$$
$$-1 < x < 1,$$

where $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ and α and b are defined in Section1.

In this case the analytical solution is

$$\begin{aligned}(1-x^2)g(x) &= aw(x)(1-x^2)[-4x^2 + (\beta-\alpha)x + 1] \\ &\quad + \frac{b}{\pi}w(x)(-x+\beta-\alpha)\left[x(1-x^2)\log\frac{1-x}{1+x} - 2x^2 + \frac{4}{3}\right] \\ &\quad + \frac{b}{\pi}w(x)(1-x^2)\left[-6x + (1-3x^2)\log\frac{1-x}{1+x}\right].\end{aligned}$$

We consider the approximate solution $(1-x^2)g_m(x)$, defined by (2.7) of Section 2, and evaluate this by choosing various classes of points.

In Table 1 we evaluate the approximate solution and the maximum error on the points ξ_j, $j = 1, \dots, m+2$, defined in Section 2. In Table 2 we consider the class of points $x_{m,i}(w)$, $i = 1, \dots, m$, i.e., the zeros of the m-th orthogonal polynomial $p_m^{(\alpha,\beta)}(x)$. In Table 3 we consider the equispaced points $\pm 2j/m$, $j = 1, \dots, [m/2]-1$ and the point 0. In Table 4 the points are $(\xi_j + \xi_{j+1})/2$, $j = 1, \dots, m+1$, i.e., the class of the midpoints of the ξ_j.

TABLE 1. $x = \xi_j$, $j = 1, m+2$.

m	$\|(1-x^2)(g-g_m)\|_\infty$
25	0.129594D-02
50	0.550330D-03
100	0.159134D-03
150	0.773219D-04

TABLE 2. $x = x_{m,i}(w),\ i = 1, m.$

m	$\|(1-x^2)(g-g_m)\|_\infty$
25	0.914160D-03
50	0.391249D-03
100	0.112876D-03
150	0.547938D-04

TABLE 3. $x = \pm 2j/m,\ j = 1, [m/2] - 1,\ x = 0.$

m	$\|(1-x^2)(g-g_m)\|_\infty$
25	0.120603D-02
50	0.546662D-03
100	0.158775D-03
150	0.772006D-04

TABLE 4. $x = (\xi_j + \xi_{j+1})/2,\ j = 1, m+1.$

m	$\|(1-x^2)(g-g_m)\|_\infty$
25	0.124798D-03
50	0.235395D-04
100	0.432316D-05
150	0.158661D-05

All of these numerical results confirm the theoretical result given by Theorem 3.1. Moreover, it seems that, for the class of the midpoints of the ξ_j, as we show in Table 4, the numerical result is better than the analytical one.

Moreover, for calculating the approximate solution $(1-x^2)g_m(x)$ at the points ξ_j, $j = 1, \dots, m+2$, the relation (2.8) can be used. The numerical results in this case are shown in Table 5. They are also in accordance to the theoretical result given by Theorem 3.2. Finally, in Table 6 we also consider the approximate solution $(1-x^2)g_m(x)$

given by relation (2.8), evaluated on the point $x_{m,i}(w)$. In this case we can obtain the same numerial results as that in Table 5. Therefore, we think that also the points $x_{m,i}(w)$ verify the interlacing properties. (For the interlacing properties of the zeros of orthogonal polynomials, see also [**3, 13, 18**].)

TABLE 5. $x = \xi_j,\ j = 1, m+1.$

m	$\|(1-x^2)(g-g_m)\|_\infty$
25	0.113792D-04
50	0.211094D-05
100	0.382438D-06
150	0.139931D-06

TABLE 6. $x = x_{m,i}(w),\ i = 1, m.$

m	$\|(1-x^2)(g-g_m)\|_\infty$
25	0.113030D-04
50	0.209717D-05
100	0.379960D-06
150	0.139025D-06

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