JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 6, Number 3, Summer 1994

VARIATIONAL METHOD WITH APPLICATION TO CONVOLUTION EQUATIONS

DANIEL GOELEVEN

ABSTRACT. The aim of this paper is to solve the convolution equation $k * u^2 = |u|$ for k subject to the conditions $k \in L^{3/2}(\mathbf{R})$, $k(x) \geq 0$, $k(x) = k(-x)$ and k symmetrically decreasing. By using a result of the Ljusternik-Schnirelman theory on C^1-manifold due to A. Szulkin we improve some recent results of J.B. Baillon and M. Théra.

1. Introduction. Recently, J.B. Baillon and M. Théra [**1**] introduced a notion of self-adjoint nonlinear operator T with respect to a duality mapping J_θ. Using the properties of such a mapping they studied the optimization problem

$$\text{(P)} \qquad \max\{\langle Tu, J_\theta u\rangle : u \in X, ||u|| = 1\},$$

where X is a reflexive real Banach space equipped with a sufficiently smooth norm.

In their papers [**1, 2, 12**], they showed that problem (P) was very useful to obtain solutions of some convolution equations such as the following:

$$\text{(E)} \qquad k * u^2 = u, \qquad u \in L^3(\mathbf{R}),$$

where $k \in L^{3/2}(\mathbf{R}) \cap L^3(\mathbf{R})$ is assumed to be symmetrically decreasing, even and positive.

In this paper we use a recent result of the Ljusternik-Schnirelman theory on C^1-manifold [**10**] due to A. Szulkin to find critical points of the norm $|| \cdot ||$ on the Banach manifold $M := \{u \in X : \langle Tu, J_\theta u\rangle = 1\}$. By using the Lagrange multiplier theorem we prove that our approach can also be used to find solutions of convolution equations and of some set-valued integral equations.

Received by the editors on September 21, 1993.
AMS *Mathematics Subject Classification.* 45G05, 45G10, 49B99.

2. Preliminaries. Let $\langle X, X^* \rangle$ be a dual system of real reflexive Banach spaces, where $||\cdot||$ will be the norm on X and $||\cdot||_*$ the corresponding dual norm. We write $N(u)$ for $||u||$ and we assume the norm is C^1-*Gâteaux differentiable*, i.e., for each $x \in X\backslash\{0\}$, the Gâteaux directional derivative of the norm given by

$$\langle N'(u), h\rangle = \lim_{t\to 0}(N(u+th) - N(u))/t$$

exists, is linear and continuous in h for all u in a neighborhood of x.

2.1. Ljusternik-Schirelmann result for even functions on C^1-manifold. Let M be a closed symmetric C^1-submanifold. Denote the tangent bundle of M by $T(M)$ and the tangent space of M at x by $T_x(M)$. $T(M)^*$ will be the cotangent bundle and $T_x(M)^*$ the cotangent space of M at x. Let $f \in C^1(M, \mathbf{R})$. $df(x) \in T_x(M)^*$ denotes the differential of f at x and a point $x \in M$ is said to be a *critical point* of f if $df(x) = 0$. The function f is said to satisfy the *Palais-Smale condition* at level $c \in \mathbf{R}$ ($(\mathrm{PS})_c$ for short) if each sequence $\{u_n\} \subset M$ such that $f(u_n) \to c$ in $\mathbf{R}$ and $df(u_n) \to 0$ in $T_x(M)$ has a convergent subsequence [**10**].

In what follows we shall need the notions of Ljusternik-Schnirelmann genus. Let Σ be the collection of all symmetric subsets of $X\backslash\{0\}$ which are closed in X. A nonempty set $A \in \Sigma$ is said to have *genus* k (i.e., $\gamma(A) = k$) if k is the smallest integer with the property that there exists an odd continuous mapping $\eta : A \to \mathbf{R}^k\backslash\{0\}$. If there is no such k, then we say that $\gamma(A) = +\infty$ and if $A = \varnothing$, we set $\gamma(A) = 0$. Properties of the genus may be found in [**3**]. We only recall here that if N is a symmetric and bounded neighborhood of the origin in $\mathbf{R}^k$ and if A is homeomorphic to the boundary of N by an odd homeomorphism, then $\gamma(A) = k$.

The following result is due to A. Szulkin [**10**].

Theorem 2.1. *Suppose that M is a closed symmetric C^1-submanifold of a real Banach space X and $0 \notin M$. Suppose also that $f \in C^1(M, \mathbf{R})$ is even and bounded from below. Define*

$$c_j := \inf_{A\in\Delta_j} \sup_{x\in A} f(x),$$

where $\Delta_j := \{A \subset M, A \in \Sigma, \gamma(A) \geq j$ *and* A *is compact*$\}$. *If* $\Delta_k \neq \varnothing$ *for some* $k \geq 1$ *and if* f *satisfies* $(\mathrm{PS})_c$ *for all* $c = c_j$, $j = 1, \dots, k$, *then* f *has at least* k *distinct pairs of critical points.*

2.2. Self-adjoint operator with respect to a duality mapping. We say that $j : [0, +\infty) \to [0, +\infty)$ is a *gauge function* if j is an increasing continuous function such that $j(0) = 0$ and $j(t) \to +\infty$ as $t \to +\infty$.

The convex continuous function $\theta : [0, +\infty) \to [0, +\infty)$; $t \to \theta(t) := \int_0^t j(s)\, ds$ is called a *potential*. The *duality mapping* $J_\theta : X \to X^*$ associated with θ, is given by $\langle u, J_\theta u\rangle = j(||u||)||u||$ and $||J_\theta u||_* = j(||u||)$ for every $u \in X$. We list the properties of $J_\theta u$ which will be referred to in the following section [**4**].

i) $J_\theta(\lambda u) = j(\lambda ||u||) \cdot J_\theta(u)/j(||u||)$, for all $u \neq 0$, $\lambda > 0$,

ii) J_θ is odd,

iii) $\langle u - v, J_\theta(u) - J_\theta(v)\rangle \geq j(||u||) - j(||v||) \cdot (||u|| - ||v||)$, for all $u, v \in X$,

and, since the norm on X is Fréchet-differentiable,

iv) J_θ is continuous

v) $\langle v, J_\theta u\rangle = j(||u||) \cdot \langle N'(u), v\rangle$ for all $u, v \in X$.

We now consider an operator $T : X \to X$. Following J.B. Baillon and M. Théra [**1**], we say that T is *self-adjoint* with respect to J_θ if

$$\langle Tu, J_\theta(v)\rangle = \langle Tv, J_\theta(u)\rangle, \qquad \forall\, u, v \in X.$$

In [**1**] and [**12**] some properties of such operators are proved. In the following lemma, we only list some of them which will be referred to in the following section.

Lemma 2.2.1 (J.B. Baillon- M. Théra). *Let* T *be a self-adjoint operator with respect to a duality mapping* J_θ.

i) $T(0) = 0$,

ii) $T(\lambda v) = j(\lambda ||v||) \cdot Tv/j(||v||)$, *for all* $v \neq 0$, $\lambda > 0$,

iii) *Let* $\{u_n; n \in \mathbf{N}\}$ *be a sequence such that* $T_n \rightharpoonup Tu$ *and* $J_\theta(u_n) \rightharpoonup \mu$ *then* $\langle Tu, J_\theta(u)\rangle = \langle Tu, \mu\rangle$.

The following mapping will play an important part in the next section. Let $\Phi : X \to \mathbf{R}$; $u \to \Phi(u) := \langle Tu, J_\theta(u)\rangle$. We prove the following:

Proposition 2.2.1. *Let T be a self-adjoint operator with respect to a duality mapping J_θ. If T is compact, then Φ is compact and continuous.*

Proof. a) Let $\{u_n; n \in \mathbf{N}\}$ be a bounded sequence in X. There exist $M > 0$ such that $||u_n|| \le M$ and thus $||J_\theta u_n||_* = j(||u_n||) \le j(M)$. Thus, since X is reflexive, T is compact and $\{J_\theta u_n; n \in \mathbf{N}\}$ is bounded, there exists a subsequence $\{u_n; n \in \mathbf{N}\}$ such that $u_n \rightharpoonup u$, $Tu_n \to T$ and $J_\theta u_n \rightharpoonup \mu$ so that $\Phi(u_n) \to \Phi$ and, as a result, Φ is compact.

b) Suppose that $u_n \to u$. We have

$$|\langle Tu, J_\theta(v)\rangle - \langle Tu_n, J_\theta u_n\rangle| \le |\langle Tu, J_\theta(u) - J_\theta u_n\rangle| + |\langle Tu, J_\theta u_n\rangle - \langle Tu_n, J_\theta u_n\rangle|.$$

Since T is self-adjoint with respect to J_θ, this yields

$$\begin{aligned}|\langle Tu, J_\theta(v)\rangle - \langle Tu_n, J_\theta u_n\rangle| &\le |\langle Tu, J_\theta(u) - J_\theta(u_n)\rangle| \\ &\quad + |\langle Tu_n, J_\theta u - J_\theta u_n\rangle| \\ &\le ||Tu|| \cdot ||J_\theta(u) - J_\theta u_n||_* \\ &\quad + ||Tu_n|| \cdot ||J_\theta u - J_\theta u_n||_*.\end{aligned}$$

Since T is compact, the set $\{Tu_n; n \in \mathbf{N}\}$ is bounded, and since J_θ is continuous, we get the continuity of Φ. □

Remark 2.2.1. Recall that if Φ is uniformly Fréchet-differentiable and completely continuous ($u_n \rightharpoonup u \Rightarrow \Phi(u_n) \to \Phi(w)$, then Φ' is also completely continuous [**13**].

3. Multiplicity results. In the sequel we assume that the gauge function j is positively homogeneous of order $\alpha/2 > 0$. This assumption is not a real limitation for most of the applications. We also make the following hypotheses:

(H1) X is Kadec,

(H2) Φ is uniformly Fréchet-differentiable,

(H3) T is odd,

(H4) T is self-adjoint with respect to the duality mapping J_θ,

(H5) Φ is completely continuous,

(H6) $\Phi(u) > 0$, for all $u \neq 0$.

We define

$$M := \{u \in X : \Phi(u) = 1\}.$$

For each $u \in M$, we have

$$\begin{aligned}\langle \Phi'(u), u\rangle &= \lim_{\lambda\downarrow 0}[j((1+\lambda)||u||)^2 - j(||u||)^2]/\lambda j(||u||)^2\\ &= \lim_{\lambda\downarrow 0}\alpha(1+\lambda)^{\alpha-1}\\ &= \alpha \neq 0,\end{aligned}$$

so that $\Phi'(u) \neq 0$ on M and thus M is a C^1-submanifold on X. By Assumption (H5), M is closed. By Assumption (H3) and Property ii) of J_θ, Φ is even and M is symmetric. Thus, M is a closed symmetric C^1-submanifold on X.

Theorem 3.2.1. *Under assumptions* (H1)–(H6) *there exist infinitely many distinct pairs of couples* (λ, u), $(\lambda - u) \in \mathbf{R} \times X$ *such that*

i) $\lambda > 0$,

ii) $u \neq 0$,

iii) $\Phi(u) = 1$,

iv) $N'(u) = \lambda\Phi'(u)$.

Proof. Let $J : X \to X^*$ be the duality mapping (i.e., $j(t) = t$) and define the projection mapping $P_u : X \to T_u(M)$ by

$$P_u v = v - J^{-1}\Phi'(u) \cdot \langle \Phi'(u), v\rangle/||\Phi'(u)||_*^2.$$

Put $f := N|_M$. We remark that $\langle N'(u), v\rangle = \langle df(u), v\rangle$ for each $v \in T_u(M)$. Thus

$$\langle df(u), P_u v\rangle = \langle N'(u), v\rangle - \langle N'(u), J^{-1}\Phi'(u)\rangle\langle\Phi'(u), v\rangle/||\Phi'(u)||_*^2$$

so that

$$df(u) = N'(u) - \langle N'(u), J^{-1}\Phi'(u)\rangle \cdot \Phi'(u)/||\Phi'(u)\,||_*^2.$$

1°) f satisfies $(\mathrm{PS})_c$ at any level $c \in \mathbf{R}$. Indeed, let $\{u_n\} \subset M$ be a sequence such that $N(u_n) \to c$ and $df(u_n) \to 0$. This sequence is bounded and, since X is a reflexive space, $||N'u_n||_* \leq 1$, there exists a subsequence, denoted again by $\{u_n\}$, such that $u_n \rightharpoonup u$, $\Phi'(u_n) \to \Phi'(u)$ and $N'(u_n) \to \beta$. We have

$$\begin{aligned} N'(u_n) &\longrightarrow \langle N'(u_n), J^{-1}\Phi'(u_n)\rangle \Phi'(u_n)/||\Phi'(u_n)||_*, \\ &\longrightarrow \langle \beta, J^{-1}\Phi'(u)\rangle \Phi'(u)/||\Phi'(u)||_*. \end{aligned}$$

Thus,

$$\langle N'(u_n), u_n - u\rangle - \langle N'(u), u_n - u\rangle \to 0,$$

and by property v) of the duality mapping, we also have

$$\langle J_\theta u_n, u_n - u\rangle/j(||u_n||) - \langle J_\theta u, u_n - u\rangle/j(||u||) \to 0.$$

Thus,

$$\begin{aligned} (\langle J_\theta u_n, u_n - u\rangle - \langle J_\theta u, u_n - u\rangle)/j(||u_n||) + \langle J_\theta u, u_n - u\rangle/j(||u_n||) \\ - \langle J_\theta u, u_n - u\rangle/j(||u||) \to 0. \end{aligned}$$

Now there exist c, $C > 0$, such that $c \leq j(||u_n||) \leq C$. The majoration is due to the fact that $\{u_n; n \in \mathbf{N}\}$ is bounded and j increasing. The minoration is due to the fact that $\{u_n; n \in \mathbf{N}\}$ lies in M. Indeed, if such a $c > 0$ does not exist then we can find a subsequence $\{u_{n'}; n' \in \mathbf{N}\}$ such that $u_{n'} \to 0$ and $\Phi(u_{n'}) = 1$ which is absurd since Φ is continuous and $\Phi(0) = 0$. As a consequence, we get

$$\langle J_\theta u_n, u_n - u\rangle - \langle J_\theta u, u_n - u\rangle \to 0.$$

Thus, by property iii) of the duality mapping, we get

$$\lim j(||u_n||) - j(||u||) \cdot (||u_n|| - ||u||) \leq 0.$$

Since j is increasing, it is necessary that

$$\lim j(||u_n||) - j(||u||) \cdot (||u_n|| - ||u||) = 0,$$

and thus $||u_n|| \to ||u||$. From assumption (H1), we get $u_n \to u$ and the Palais-Smale condition is satisfied.

2°) For each $k \geq 1$, $k \in \mathbf{N} : \Delta_k \neq \varnothing$. Let $S^{k-1} := \{x \in \mathbf{R}^k; ||x|| = 1\}$ be the unit-sphere in $\mathbf{R}^k$. S^{k-1} is the boundary of a symmetric and bounded neighborhood of the origin in $\mathbf{R}^k$. Let $\eta : S^{k-1} \to M;\ x \to x \cdot (1/\Phi(x))^{1/\alpha}$. By assumption (H6), η is well defined. $\Phi(x \cdot (1/\Phi(x))^{1/\alpha}) = j(||x||(1/\Phi(x))^{1/\alpha})^2 \cdot \Phi(x) \cdot j(||x||)^2 = 1$, and thus η takes its values in M. Put $A := \eta(S^{k-1})$. It is clear that A is a symmetric compact subset of M, and thus, since $\eta : S^{k-1} \to A$ is an odd homeomorphism, we have $\gamma(A) = k$.

3°) By Theorem 2.1 f has infinitely many distinct pairs $(u, -u)$ of critical points. Since f is convex, each u is a relative minimum for the convex function N, and by the Lagrange multiplier theorem there exists $\lambda \neq 0$, such that $N'(u) = \lambda\Phi'(u)$. We get $||u|| = \langle N'(u), u\rangle = \lambda\langle\Phi'(u), u\rangle = \lambda\alpha$, and thus $\lambda > 0$ and $||u|| \neq 0$. □

Corollary 3.2.1. *Let $\alpha > 0$. Under assumptions* (H1)–(H6)*, there exist infinitely many distinct pairs of couples (μ, u), $(\mu, -u) \in \mathbf{R} \times X$ such that*

i) $\mu > 0$,

ii) $u \neq 0$,

iii) $\Phi(u) = 1$,

iv) $(N^\alpha(u))' = \mu\Phi'(u)$.

Proof. Let (λ, u) satisfy i)–iv) of Theorem 2.2.1. We have

$$(N^\alpha(u))' = \alpha N^{\alpha-1}(u)N'(u) = \alpha\lambda||u||^{\alpha-1}\Phi'(u).$$

Put $\mu := \alpha\lambda||u||^{\alpha-1}$. The conclusion follows from Theorem 2.2.1. □

Remark 3.3.1. By comparison with the work of J.B. Baillon and M. Théra [**1, 2**], we remark that our approach is restricted to gauge functions which are positively homogeneous of order $\alpha > 0$ and to operators T which are odd. On the other hand, the norm is only assumed Fréchet-differentiable and our Theorem 3.2.1 is a multiplicity result. As we shall see in the following section, the field of applications of our approach is larger and include, for instance, some set-valued integral equations.

Remark 3.3.2. Let $I := [-\pi, \pi]$, and let $\Phi(u) := \int_I k * \cos(u^2)u^2\,dx$, where $k \in L^3(I)$ is even and symmetrically decreasing. It is easy to see that all the assumptions of Corollary 3.2.1 are satisfied. Thus, there exist infinitely many distinct pairs of elements $(-u, u) \in L^3(I)\backslash\{0\}$ such that $|u| = k * \cos(u^2) - u \cdot k * \sin(2u)$. In this manner, it is possible to prove the existence of nontrivial solutions for several equations. In the following section we consider a specific problem for which the calculations are related in detail.

4. Applications. We consider the problem

$$(\mathrm{E}') \qquad k * u^2 = |u|, \qquad u \in L^3(\mathbf{R}),$$

where $k \in L^{3/2}(\mathbf{R})$ satisfies $k(-x) = k(x)$ and is symmetrically decreasing. Recall that if $k \in L^{3/2}(\mathbf{R})$ and if $v \in L^{3/2}(\mathbf{R})$, then $k^*v \in L^3(\mathbf{R})$ [**5**].

It is clear that if u is a solution of Problem (E′), then $v := |u|$ is a positive solution of Problem (E).

We take

$$j(t) := t^3, \qquad N(u) := \bigg(\int_{\mathbf{R}} |u|^3\bigg)^{1/3}$$

and

$$Tu := k * u^2 \operatorname{sign}(u)$$

so that

$$J_\theta(u) = u^2 \operatorname{sign}(u), \qquad \Phi(u) = \int_{\mathbf{R}} k * u^2 u^2,$$

and

$$M = \bigg\{u \in L^3(\mathbf{R}) : \int_{\mathbf{R}} k * u^2 u^2 = 1\bigg\}.$$

It is easy to see that

$$(N^3(u))' = 3u|u|$$

and since $k(x) = k(-x)$,

$$\Phi'(u) = 4k * u^2 \cdot u.$$

It is clear that if u is a solution of $(N^3(u))' = \lambda\Phi'(u)$, then $w := 4\lambda u/3$ is a solution of Problem (E$'$).

Assumptions (H1)–(H4) and (H6) are satisfied [**1**]. We now consider the approximate problem

$$(\mathrm{E}'_n) \qquad k * u^2 = |u|, \qquad u \in L^3(\mathbf{R}), \qquad \text{support}\,(u) \subset [-n, +n].$$

We put $X_n := L^3([-n, +n])$ and, to apply Corollary 3.2.1 on this subspace, we still have to prove that Φ is completely continuous. In fact, the result has been proved by J.B. Baillon and M. Théra [**2**]. We can apply Corollary 3.2.1 to Problem E'_n and obtain infinitely many pair of distinct (self up to a translation, since the support is fixed) solutions. We thus have the following result.

Proposition 4.1. *There exist infinitely many pairs of distinct elements* $(-u, u)$ *in* $L^3([-n, +n]) \times L^3([-n, +n])$ *such that* $u \neq 0$, *and*

$$k * u^2 = |u| \quad \text{on } [-n, +n].$$

By applying Corollary 3.2.1 to each X_n as $n \to \infty$, we get a sequence u_n such that $u_n \in M$, u_n positive and $k * u_n^2 = u_n$. Thus, $||u_n|| = 1$ and there exists a subsequence such that $u_n \rightharpoonup z$ in $L^3(\mathbf{R})$, $u_n^2 \rightharpoonup v$ in $L^{3/2}(\mathbf{R})$ and thus $k * u_n^2(x) \to k * v(x)$ for all x. Hence, we get $u_n(x) \to k * v(x)$ for all x. Therefore, $z = k * v$ and $v = z^2$, so that $z = k * z^2$. If we suppose that $z = 0$, then by using an argument of Hardy-Littlewood-Pólya [**7**] as in [**2**], it is possible to prove that $u_n^2 * u_n^2 \rightharpoonup 0$ in $L^3(\mathbf{R})$. We get $1 = \lim_{n\to\infty} \int_{\mathbf{R}} k * u_n^2 u_n^2 = 0$, which is a contradiction. The following result is thus also true.

Proposition 4.2. *There exists at least a pair* $(-u, u) \in L^3(\mathbf{R}) \times L^3(\mathbf{R})$ *such that* $u \neq 0$, *and*

$$k * u^2 = |u| \quad \text{on } \mathbf{R}.$$

If u is a solution of (E$'$), then $v := |u|$ is a positive solution of (E) and we get the existence of at least one nontrivial solution of the convolution equation (E) which is in accordance with the result of J.B. Baillon and M. Théra [**1, 2**]. Moreover, each solution of Problem (E'_T), $T > 0$, is a solution of the set-valued integral equation

$$u(t) \in \int_{-T}^{T} k(t-s) F(u(s))\, ds,$$

where $F : \mathbf{R} \to 2^{\mathbf{R}}$ denotes the set-valued function

$$x \to F(x) := [-x^2, x^2].$$

Such set-valued integral equations are useful for the study of some unilateral problems [**9, 6**].

Remark 4.1. A similar application of Theorem 2.1 to homogeneous second order differential equations can be found in [**8**].

REFERENCES

1. J.B. Baillon and M. Théra, *A new type of convergence with applications to Hammerstein's equation*, Nonlinear Anal. Theory Methods Appl. **20** (1993), 1029–1042.

2. ———, *About the equation* $k^*u^2 = u$, J. Integral Equations **3** (1991), 343–350.

3. S.N. Chow and J.K. Hale, *Methods of bifurcation theory*, Springer Verlag, New York, 1982.

4. I. Ciorănescu, *Geometry of Banach spaces, duality mappings and nonlinear problems*, Kluwer, Dordrecht, 1990.

5. J. Dieudonné, *Eléments d'analyse* 2, Gauthier-Villars, 1968.

6. K. Glashoff and J. Sprekels, *An application of Glicksberg's thoerem to set-valued integral equations arising in the theory of thermostats*, SIAM J. Math. Anal. **12** (1981), 477–486.

7. G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, London (second edition), 1951.

8. A.K. Ben Naoum, C. Troestler and M. Willem, *Existence and multiplicity results for homogeneous second order differential equations*, J. Differential Equations, preprint 1992, to appear.

9. P.D. Panagiotopoulos, *Inequality problems in mechanic and applications: Convex and nonconvex energy functions*, Birkhäuser, 1985.

10. A. Szulkin, *Ljusternik-Schnirelmann theory on* C^1*-manifolds*, Ann. Inst. Henri Poincaré **5** (1988), 119–139.

11. M. Théra, *Contributions à l'analyse non-linéaire*, Thèse d' Etat, Univ. Paris I, Panthéon-Sorbonne, 1988.

12. ———, *Opérateurs non-linéaires auto-adjoints et Φ-convergence*, Séminaire Choquet **9** (1990/1991).

13. M.M. Vainberg, *Variational method and method of monotone operators in the theory of nonlinear equations*, John Wiley & Sons, New York, 1973.

Department of Mathematics, Facultés Universitaires Notre Dame de la Paix. 8, Rempart de la Vierge, B-5000 Namur, Belgique