

**THE h - p -VERSION
OF SPLINE APPROXIMATION METHODS
FOR MELLIN CONVOLUTION EQUATIONS**

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ABSTRACT. We consider the numerical solution of Mellin convolution equations on an interval by the h - p -version of spline approximation methods. Using a geometric mesh refinement towards the singularity of the integral equation, we prove stability and exponential convergence in the L_q norm, $1 \leq q \leq \infty$, for Galerkin, collocation and Nyström methods based on piecewise polynomials.

1. Introduction. We consider the approximate solution of the one-dimensional Mellin convolution equation

$$(1.1) \quad u(x) - \int_0^1 \kappa(x/y)u(y)y^{-1} dy = f(x), \quad x \in I := (0, 1),$$

where f and κ are given functions and u is the unknown function. Such integral equations having a fixed singularity at the point $x = 0$ arise in a variety of applications; for example, they occur when boundary integral methods are applied to potential problems in plane regions with corners or to crack problems in linear elasticity (see [4, 13] and the references therein). Note that the integral operator in (1.1) is not compact so that standard theories for the numerical analysis of second kind Fredholm integral equations cannot be applied. Nevertheless, using graded meshes and modified spline spaces, results on stability and optimal convergence orders of Galerkin, collocation and quadrature methods for Equation (1.1) which are based on piecewise polynomial basis functions have been obtained in [4, 6, 7, 9]. These papers apply the technique of the traditional h -version of spline approximation methods where accuracy is achieved by decreasing the mesh size h , while keeping the degree p of piecewise polynomials fixed.

In the present paper we study the h - p -version of those approximation methods which is obtained if one simultaneously refines the mesh and

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increases the degree of the splines. For the finite element method on a geometric mesh applied to elliptic boundary value problems in planar polygonal domains, it has been shown in [10] that, if the given data are piecewise analytic, the h - p -version has an exponential rate of convergence with respect to the number of degrees of freedom, whereas the h -version has only a polynomial rate. The state of the art of the p - and h - p -versions of the finite element method is described in [3]. Only recently the h - p -version has been introduced into boundary element methods [2, 15]. These papers analyze the h - p -version of boundary element Galerkin methods with quasiuniform and geometric meshes for some strongly elliptic first-kind integral equations on a polygon; [2] establishes, in particular, the exponential rate of convergence when a geometric mesh refinement towards the vertices is used.

Our aim is to show corresponding results for spline Galerkin methods applied to the second-kind integral equation (1.1). Furthermore, the h - p -version of certain collocation and Nyström quadrature methods is discussed here for the first time. In Section 2 we recall some analytical properties of (1.1) in $L_q(I)$, $1 \leq q \leq \infty$, and in weighted Sobolev spaces. Following [10] we further introduce countable normed spaces $B_q^e(I)$ of real-analytic functions on $(0, 1]$ which are adapted to the singularities of solutions to (1.1) at the origin. As one of our main results we prove that if the right-hand side f of (1.1) belongs to $B_q^e(I)$ and the kernel function κ satisfies appropriate conditions then the solution u is also an element of $B_q^e(I)$.

In Section 3 we define the spline spaces $\mathcal{S}_{n,\sigma,\mu}$, $\sigma \in (0, 1)$, $\mu > 0$, consisting of piecewise polynomials of degree $[\mu n]$ on the subintervals $[\sigma^{n-i}, \sigma^{n-i-1}]$, $i = 1, \dots, n-1$, of the geometric mesh. (Here and in the following $[a]$ denotes the integral part of a .) On the first subinterval $[0, \sigma^{n-1}]$ the splines are assumed to be zero; this corresponds to the simplest modification of the usual spline spaces in order to deal with the singularity of the integral equation (1.1) (cf. [4, 6, 7, 9]). Following [10], we investigate the L_q approximation of functions $u \in B_q^e(I)$ by splines from $\mathcal{S}_{n,\sigma,\mu}$. The underlying approximation theory, however, is simpler than that of [10] and essentially relies on Jackson's theorem. Besides the uniform degree distribution, we also briefly discuss the case of spline spaces with linear degree distribution; cf. (3.4) for definition.

In Section 4 we prove stability and exponential convergence in the L_q norm ($1 \leq q \leq \infty$) of Galerkin and collocation methods with

basis functions from $\mathcal{S}_{n,\sigma,\mu}$. In Section 5 we present the corresponding results for the Nyström method which can be regarded as the discrete iterated collocation method based on the spline spaces $\mathcal{S}_{n,\sigma,\mu}$. Using a somewhat different approach, it is possible to treat Galerkin and collocation methods with splines which are not required to be zero over the first interval $[0, \sigma^{n-1}]$; cf. [7] in case of the h -version.

2. Smoothness of solutions. We first recall some facts about the solvability of Equation (1.1) in weighted Sobolev spaces; see [7]. Let A be the Mellin convolution operator defined by

$$(2.1) \quad Au(x) := (1 - K)u(x), \quad Ku(x) := \int_0^1 \kappa(x/y)u(y)y^{-1} dy.$$

For any interval J and $1 \leq q \leq \infty$, $L_q(J)$ will denote the usual Lebesgue space on J with norm

$$\|u; L_q(J)\| = \left\{ \int_J |u|^q dx \right\}^{1/q}, \quad q < \infty;$$

$$\|u; L_\infty(J)\| = \operatorname{ess\,sup}_J |u(x)|.$$

For any integer $l \geq 1$, we further introduce the weighted Sobolev space

$$L_{q,l}(I) = \{u \in L_q(I) : x^j D^j u \in L_q(I), j = 1, \dots, l\}$$

on the unit interval, which is equipped with the canonical norm

$$\|u; L_{q,l}(I)\| = \sum_{0 \leq j \leq l} \|x^j D^j u; L_q(I)\|, \quad D = d/dx.$$

Let $\tilde{\kappa}$ be the Mellin transform of the kernel function of K :

$$\tilde{\kappa}(z) = \int_0^\infty x^{z-1} \kappa(x) dx.$$

The function $a(z) = 1 - \tilde{\kappa}(z)$ is called the symbol of the Mellin convolution operator A . For fixed q , $1 \leq q \leq \infty$, and $\varrho > 0$, we make the following set of assumptions on κ and a .

$$(A1) \quad \int_0^\infty x^{1/q-1} |\kappa(x)| dx < \infty; \quad a(z) \neq 0, \quad \operatorname{Re} z = 1/q; \\ \{\arg a(1/q + i\xi)\}_{-\infty}^\infty = 0.$$

$$(A2^\varrho) \quad \int_0^\infty x^{1/q-1-\varrho} |\kappa(x)| dx < \infty; \quad a(z) \neq 0, \quad 1/q - \varrho \leq \operatorname{Re} z < 1/q.$$

Here $1/q := 0$ if $q = \infty$, and $\{\arg \cdot\}_{-\infty}^\infty$ denotes the change in argument of $a(1/q + i\xi)$ as ξ runs from $-\infty$ to ∞ .

Note that the L_q operator norm of K is bounded by the integral appearing in (A1), and then the second condition of (A1) is equivalent to the invertibility of A on $L_q(I)$. Under the assumptions (A1) and (A2 $^\varrho$), $u \in L_q(I)$ and $Au \in L_q^\varrho(I)$ imply that $u \in L_q^\varrho(I)$, where $L_q^\varrho(I)$ denotes the weighted space $x^\varrho L_q(I)$. Note that the supremum of the numbers ϱ for which condition (A2 $^\varrho$) is satisfied reflects the principal term of the asymptotics of solutions to (1.1) at the origin; cf. [5].

The conditions (A1) are of course necessary for the stability in $L_q(I)$ of any projection method applied to (1.1). To obtain sufficient stability conditions, we usually have to require some additional smoothness of the kernel function:

$$(A3_l) \quad \int_0^\infty x^{1/q-1} |x^j D^j \kappa(x)| dx < \infty, \quad j = 0, \dots, l.$$

Condition (A3 $_l$) implies that K is a bounded map of $L_q(I)$ into $L_{q,l}(I)$, and together with (A1) this ensures the invertibility of A on $L_{q,l}(I)$.

To derive exponential convergence rates for our spline approximation methods, the framework of weighted Sobolev spaces does not suffice, and following [10] we therefore introduce appropriate countable normed spaces of real-analytic functions on $(0, 1]$. For $1 \leq q \leq \infty$ and $\varrho \geq 0$, let

$$B_q^\varrho(I) = \{u \in C^\infty(0, 1] : \exists d > 0 \text{ independent of } j \text{ such that} \\ \|x^{j-\varrho} D^j u; L_q(I)\| \leq d^{j+1} j!, \quad j = 0, 1, \dots\}.$$

The functions in $B_q^\varrho(I)$ are characterized by different constants d . If we wish to emphasize the dependence of a function u on d , we shall write $u \in B_{q,d}^\varrho(I)$.

In order to state our result on smoothness of solutions with respect to the scale $B_q^\varrho(I)$, we introduce the hypothesis

$$(A4^\varrho) \quad \kappa|_I \in B_q^\varrho(I).$$

Theorem 2.1. *Assume (A1) and (A4^e), and suppose in addition that (A2^e) holds if $\varrho > 0$. Then $u \in L_q(I)$ and $Au \in B_q^\varrho(I)$ imply $u \in B_q^\varrho(I)$.*

To prove this, we need the following preliminary

Lemma 2.2. (i) *If $u \in B_{q,d}^0(I)$, then*

$$\|(xD)^j u; L_q(I)\| \leq (2ed)^{j+1} j!, \quad j = 0, 1, \dots$$

(ii) *Conversely, if*

$$\|(xD)^j u; L_q(I)\| \leq d^{j+1} j!, \quad j = 0, 1, \dots,$$

then $u \in B_{q,2ed}^0(I)$.

Proof. (i) Using the relation $x^j D^j = xD(xD - 1) \cdots (xD - j + 1)$, one obtains

$$\begin{aligned} \|x^j D^j u; L_q(I)\| &\leq \sum_{i=0}^j \binom{j}{i} j^i \|(xD)^{j-i} u; L_q(I)\| \\ &\leq \sum_{i=0}^j (j!/i!) d^{j+1-i} j^i \leq (j+1) d^{j+1} j^j. \end{aligned}$$

Since by Stirling's formula,

$$(2.2) \quad j^j \leq (2\pi)^{-1/2} e^j j! \leq e^j j!, \quad j = 0, 1, \dots,$$

we finally get

$$\|x^j D^j u; L_q(I)\| \leq (ed)^{j+1} (j+1)! \leq (2ed)^{j+1} j!.$$

(ii) Note that

$$(xD)^j = \sum_{i=0}^j c_i^j x^{j-i} D^{j-i}$$

with certain nonnegative numbers c_i^j . Applying the operator xD to the last relation, we obtain

$$c_i^{j+1} = c_i^j + (j+1-i)c_{i-1}^j, \quad c_{j+1}^j = c_{-1}^j := 0.$$

Now, by induction one can easily show that for any i and j

$$c_i^j \leq \binom{j}{i} j^i.$$

Therefore we obtain the estimate

$$\|(xD)^j u; L_q(I)\| \leq \sum_{i=0}^j \binom{j}{i} j^i \|x^{j-i} D^{j-i} u; L_q(I)\|$$

which implies the assertion as before. \square

Proof of Theorem 2.1. Step 1. We first verify the assertion in the case $\varrho = 0$. We have to show that, under the assumptions (A1) and (A4^o), $u \in L_q(I)$ and $Au \in B_q^o(I)$ imply $u \in B_q^o(I)$. Thus, by Lemma 2.2, one may assume that for some $d \geq 1$

$$(2.3) \quad \begin{aligned} & \|u; L_q(I)\| \leq d, \\ & \|(xD)^j Au; L_q(I)\| + \|(xD)^j \kappa; L_q(I)\| \leq d^{j+1} j!, \quad j = 0, 1, \dots \end{aligned}$$

Note that $u \in L_{q,j}(I)$ for any $j \geq 1$; cf. [7, Theorem 1.10]. Proceeding by induction on j , we suppose that for some $k \geq 1$ and $d_1 \geq d$

$$(2.4) \quad \|(xD)^j u; L_q(I)\| \leq d_1^{j+1} j!, \quad j < k.$$

To show (2.4) for $j = k$, we start with the relation

$$(2.5) \quad xDKu(x) = KxDu(x) - \kappa(x)u(1), \quad u \in L_{q,1}(I)$$

which follows by applying the operator xD to the integral

$$Ku(x) = \int_0^1 \kappa(x/y)u(y)y^{-1} dy = \int_x^\infty \kappa(z)u(x/z)z^{-1} dz.$$

By iteration, (2.5) yields for any $k \geq 1$

$$(2.6) \quad (xD)^k Au = A(xD)^k u + \sum_{j < k} (xD)^{k-1-j} \kappa(x) (xD)^j u(1), \quad u \in L_{q,k}(I).$$

Applying Sobolev's embedding theorem for the interval $[1/2, 1]$, one has the estimate

$$\|(xD)^{k-1}u(1)\| \leq \varepsilon \|(xD)^k u; L_q(I)\| + c(\varepsilon) \|(xD)^{k-1}u; L_q(I)\|$$

for any $\varepsilon > 0$ and $k \geq 1$. Together with (2.3) and (2.6), this implies

$$\begin{aligned} \|A(xD)^k u; L_q(I)\| &\leq \|(xD)^k Au; L_q(I)\| \\ &\quad + d\{\varepsilon \|(xD)^k u; L_q(I)\| + c(\varepsilon) \|(xD)^{k-1}u; L_q(I)\|\} \\ &\quad + \sum_{j \leq k-2} d^{k-j} (k-1-j)! \{ \|(xD)^{j+1}u; L_q(I)\| \\ &\quad \quad \quad + c(1) \|(xD)^j u; L_q(I)\| \}. \end{aligned}$$

Choosing ε sufficiently small and using (2.3), (2.4) and the invertibility of A in $L_q(I)$, we obtain from the last inequality

$$\|(xD)^k u; L_q(I)\| \leq cd^{k+1}k! + \sum_{j \leq k-2} cd_1^{j+2} (j+1)! d^{k-j} (k-j-1)!,$$

where $c \geq 1$ does not depend on k . Using the estimates

$$(j+1)!(k-j-1)! \leq (k-1)!$$

and selecting $d_1 = 2cd^2$, we finally have

$$\|(xD)^k u; L_q(I)\| \leq cd^{k+1}k! + \sum_{j \leq k-2} cd_1^{j+2} d^{k-j} (k-1)! \leq d_1^{k+1}k!.$$

This shows (2.4) for $j = k$ and completes the proof of the assertion in view of Lemma 2.2.

Step 2. Passing to the Mellin convolution operator $A_\rho = x^{-\rho}Ax^\rho$ with kernel function $x^{-\rho}\kappa$, the assertion in the general case may be reduced to the situation considered in Step 1. Indeed, using the relations $(xD)^j x^{-\rho}\kappa = x^{-\rho}(xD - \rho)^j \kappa$ and the considerations in the

proof of Lemma 2.2, $\kappa \in B_q^\varrho(I)$ implies that $x^{-\varrho}\kappa \in B_q^0(I)$. Further, by assumption, A_ϱ is invertible on $L_q(I)$ while A is invertible on $L_q(I)$ and $L_q^\varrho(I)$. Thus it suffices to verify that $v = x^{-\varrho}u \in L_q(I)$ and $A_\varrho v = x^{-\varrho}Au \in B_q^0(I)$ imply $v \in B_q^\varrho(I)$, which was already done in Step 1. \square

Remark 2.3. Another analyticity assumption on κ , which is stronger than (A4 $^\varrho$) but usually satisfied in practice, is the following condition

$$(A5^\varrho) \quad \int_0^\infty x^{1/q-1-\varrho} |x^j D^j \kappa(x)| dx \leq d^{j+1} j!, \quad j = 0, 1, \dots$$

This condition implies that

$$(2.7) \quad K(L_q^\varrho(I)) \subset B_q^\varrho(I).$$

Indeed, for the Mellin convolution operator $x^{j-\varrho} D^j K x^\varrho$ with kernel function $x^{j-\varrho} D^j \kappa$, one then obtains the estimate

$$\begin{aligned} \|x^{j-\varrho} D^j K u; L_q(I)\| &= \|x^{j-\varrho} D^j K x^\varrho(x^{-\varrho}u); L_q(I)\| \\ &\leq \left(\int_0^\infty x^{1/q-1-\varrho} |x^j D^j \kappa(x)| dx \right) \|x^{-\varrho}u; L_q(I)\| \\ &\leq c d^{j+1} j!, \quad j = 0, 1, \dots \end{aligned}$$

Notice that (2.7) together with the invertibility of A on $L_q(I)$ and $L_q^\varrho(I)$ yields immediately the assertion of Theorem 2.1 if one replaces condition (A4 $^\varrho$) by (A5 $^\varrho$) there. Moreover, the fact that (A5 $^\varrho$) implies (A4 $^\varrho$) can now easily be verified if we choose in (2.5) a smooth function u , vanishing in a neighborhood of $x = 0$ and such that $u(1) \neq 0$, and observe that $xDKu - KxDu \in B_q^\varrho(I)$.

Example 2.4. Consider the operator (2.1) with kernel function $\kappa(x) = -\chi_I(x)x^\beta$, $\beta > 0$, where χ_I denotes the characteristic function of the interval I (i.e., $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ otherwise). Then the symbol of A takes the form $a(z) = 1 + (\beta + z)^{-1}$, and the conditions (A1), (A2 $^\varrho$) and (A4 $^\varrho$) are satisfied for $1 \leq q \leq \infty$ and $0 \leq \varrho < \beta + 1/q$.

Example 2.5. Let $\kappa(x) = 1/\pi(1+x^2)$. The corresponding equation (1.1) appears as a local model when the single layer potential is

used to solve the exterior Neumann problem for Laplace's equation and the plane domain has a corner with right angle. Then $a(z) = 1 - 1/2 \sin(\pi z/2)$, and the assumptions (A1), (A2^e), (A3_l) and (A5^e) are fulfilled if $1 \leq q < 3$, $0 \leq \varrho < 1/q - 1/3$ and $l = 1, 2, \dots$.

3. Auxiliary results from approximation theory. For $1 \leq q \leq \infty$ and $k = 0, 1, \dots$, let $W_q^k(I)$ denote the usual Sobolev space of order k on the unit interval, where $W_q^0(I) = L_q(I)$. By $E_m(u; W_q^k)$ we denote the error in the best approximation of the function u by polynomials of degree $\leq m$ with respect to the norm in $W_q^k(I)$:

$$E_m(u; W_q^k) = \inf \{ \|u - \varphi; W_q^k(I)\| : \varphi \in \mathbf{P}_m \}.$$

The following proposition, being a simple consequence of Jackson's theorem, is the key to the derivation of exponential convergence rates for our spline approximation methods.

Proposition 3.1. *We have*

- (i) $E_m(u; L_q) \leq c^k (1+m)^{-k} \|D^k u; L_q(I)\|$, $u \in W_q^k(I)$, $m+1 \geq k$;
- (ii) $E_m(u; W_q^1) \leq c^{k+1} (1+m)^{-k} \|D^{k+1} u; L_q(I)\|$, $u \in W_q^{k+1}(I)$, $m \geq k$,

where c does not depend on u, m and k .

Proof. If $k = 0$, then the result is trivially true. Let $k \geq 1$.

- (i) By Jackson's theorem

$$E_m(u; L_q) \leq c(1+m)^{-1} \|Du; L_q(I)\|,$$

and applying successively the relations

$$E_{m-j}(D^j u; L_q) \leq c(1+m-j)^{-1} E_{m-j-1}(D^{j+1} u; L_q), \quad j = 0, 1, \dots$$

and an elementary inequality, one obtains (see [12, Chapter 1.6])

$$E_m(u; L_q) \leq c^k (k^k / k!) (1+m)^{-k} \|D^k u; L_q(I)\|.$$

Together with Stirling's formula (cf. (2.2)), this yields the result.

(ii) By (i) there exists a polynomial $\psi \in \mathbf{P}_{m-1}$ such that

$$\|Du - \psi; L_q(I)\| \leq c^k m^{-k} \|D^{k+1}u; L_q(I)\|, \quad m \geq k \geq 1,$$

and setting $\varphi(x) = \int_0^x \psi(t) dt + u(0)$, the last estimate implies

$$\|u - \varphi; W_q^1(I)\| \leq c_1 \|D(u - \varphi); L_q(I)\| \leq c_1 c^k m^{-k} \|D^{k+1}u; L_q(I)\|,$$

where c, c_1 only depend on q . \square

Let \mathcal{R}_m be the orthogonal projection of $L_2(I)$ onto \mathbf{P}_m . To investigate the L_q convergence of spline Galerkin methods, we need the following simultaneous approximation result.

Proposition 3.2. *For any $u \in W_q^{k+1}(I)$, $1 \leq q \leq \infty$ and $m \geq k$,*

$$\|(1 - \mathcal{R}_m)u; L_q(I)\| \leq c^{k+1} (1 + m)^{-k} \|D^{k+1}u; L_q(I)\|,$$

where c is independent of u, m and k .

Proof. 1. Let $1 \leq q \leq 2$. Then, for any $\varphi \in \mathbf{P}_m$, the estimate

$$\begin{aligned} \|(1 - \mathcal{R}_m)u; L_q(I)\| &\leq \|(1 - \mathcal{R}_m)u; L_2(I)\| \\ &\leq \|u - \varphi; L_2(I)\| \leq c_1 \|u - \varphi; W_q^1(I)\| \end{aligned}$$

holds, where c_1 only depends on q . Here we have used Sobolev's embedding theorem. Now it suffices to apply Proposition 3.1 (ii).

2. If $2 \leq q \leq \infty$, we have for any $\varphi \in \mathbf{P}_m$

$$\begin{aligned} \|(1 - \mathcal{R}_m)u; L_q(I)\| &\leq \|u - \varphi; L_q(I)\| + \|\mathcal{R}_m(u - \varphi); L_q(I)\| \\ &\leq \|u - \varphi; L_q(I)\| + c_1 (1 + m)^{1-2/q} \|u - \varphi; L_q(I)\| \\ &\leq (1 + c_1) (1 + m)^{1-2/q} \|u - \varphi; L_q(I)\|, \end{aligned}$$

where c_1 is independent of u, φ and m and the second inequality is a consequence of [14, Theorem 3.3]. It remains to apply Proposition 3.1 (i). \square

To the Chebyshev nodes on I

$$(3.1) \quad \xi_j = \xi_j^{(m)} := \frac{1}{2} \left\{ 1 + \cos \frac{(2j+1)\pi}{2(m+1)} \right\}, \quad j = 0, \dots, m,$$

we now associate the corresponding Lagrange interpolatory projection \mathcal{L}_m onto \mathbf{P}_m . The following result is needed in the convergence analysis of spline collocation methods and is the analogue of the preceding proposition.

Proposition 3.3. *For any $u \in W_q^{k+1}(I)$, $1 \leq q \leq \infty$ and $m \geq k$,*

$$\|(1 - \mathcal{L}_m)u; L_q(I)\| \leq c^{k+1}(1+m)^{-k} \|D^{k+1}u; L_q(I)\|,$$

where c does not depend on u, m and k .

Proof. 1. Let $q < \infty$. By a theorem of Erdős and Feldheim (cf. [16, Chapter 14.3]), we have

$$\|\mathcal{L}_m u; L_q(I)\| \leq c(q) \|u; L_\infty(I)\|$$

for all $m \in \mathbf{N}$ and all continuous functions u on $[0, 1]$. Utilizing Sobolev's embedding theorem, we thus obtain for any $\varphi \in \mathbf{P}_m$

$$\begin{aligned} \|(1 - \mathcal{L}_m)u; L_q(I)\| &\leq \|u - \varphi; L_q(I)\| + \|\mathcal{L}_m(u - \varphi); L_q(I)\| \\ &\leq (1 + c(q)) \|u - \varphi; L_\infty(I)\| \leq c_1(q) \|u - \varphi; W_q^1(I)\| \end{aligned}$$

and it suffices to apply Proposition 3.1 (ii).

2. Let $q = \infty$. By a theorem of Bernstein (see [13, Chapter 3.3.1]), the L_∞ operator norm of \mathcal{L}_m is bounded by $c_1(1 + \log(1 + m))$. Now we obtain as before

$$\|(1 - \mathcal{L}_m)u; L_\infty(I)\| \leq c_1(1 + \log(1 + m)) \|u - \varphi; L_\infty(I)\|$$

for any $\varphi \in \mathbf{P}_m$, and Proposition 3.1 (i) completes the proof. \square

We now introduce the spaces of piecewise polynomials on geometric meshes occurring in the h - p -version of spline approximation methods.

For any $n \in \mathbf{N}$ and fixed $\sigma \in (0, 1)$, we first define the geometric mesh $\Delta_n = \Delta_{n,\sigma} = \{x_i : 0 \leq i \leq n\}$ on I , where

$$(3.2) \quad x_i = x_i^{(n)} = \sigma^{n-i}, \quad i = 1, \dots, n; \quad x_0 = 0.$$

Let $I_i = (x_{i-1}, x_i)$, $h_i = x_i - x_{i-1}$, where the upper index n is omitted for convenience. To the mesh $\Delta_{n,\sigma}$ and a fixed parameter $\mu > 0$, we then associate the spline space

$$(3.3) \quad \mathcal{S}_n = \mathcal{S}_{n,\sigma,\mu} := \{u \in L_\infty(I) : u|_{I_1} = 0, u|_{I_i} \in \mathbf{P}_{[\mu n]}, i \geq 2\}$$

which corresponds to a uniform degree distribution on the subintervals of Δ_n . Sometimes we also consider the case of linear degree distribution

$$(3.4) \quad \mathcal{S}_{n,\sigma,\mu}^{lin} := \{u \in L_\infty(I) : u|_{I_1} = 0, u|_{I_i} \in \mathbf{P}_{[\mu i]}, i \geq 2\}.$$

The following theorem is the crucial approximation property of \mathcal{S}_n ; for $q = 2$ it is of course a special case of the two-dimensional results obtained in [10]. Let P_n denote the orthogonal projection of $L_2(I)$ onto $\mathcal{S}_{n,\sigma,\mu}$.

Theorem 3.4. *Let $u \in B_q^\varrho(I)$, $1 \leq q \leq \infty$ and $\varrho > 0$. Then*

$$(3.5) \quad \|(1 - P_n)u; L_q(I)\| \leq ce^{-bn},$$

where the constants c and b do not depend on n .

Proof. Step 1. On the first subinterval $(0, x_1)$ of the mesh Δ_n , we have

$$(3.6) \quad \|u; L_q(I_1)\| \leq \sigma^{(n-1)\varrho} \|x^{-\varrho}u; L_q(I_1)\| \leq c_1\sigma^{n\varrho}.$$

Here and in the sequel c, c_1, \dots denote various constants not depending on n and μ .

Step 2. Next we prove the estimate

$$(3.7) \quad \|u - P_n u; L_q(x_1, 1)\| \leq c_1\sigma^{n\mu/c}$$

in the case $q = 2$. In order to do so, we choose a polynomial φ_i on each subinterval I_i , $i \geq 2$, such that according to Proposition 3.1 (i)

(with $m = [\mu n]$ and $k = [\nu n]$, $0 < \nu < \mu$) and the scaling argument, the estimate

$$(3.8) \quad \begin{aligned} \|u - \varphi_i; L_2(I_i)\| &\leq \{c_2 h_i / (1 + [\mu n])\}^{[\nu n]} \|D^{[\nu n]} u; L_2(I_i)\| \\ &\leq \{c_2 h_i / x_{i-1} (1 + [\mu n])\}^{[\nu n]} x_i^e \|x^{[\nu n] - e} D^{[\nu n]} u; L_2(I_i)\| \end{aligned}$$

holds. Here c_2 is also independent of i and ν , and ν will be chosen sufficiently small later on. Since $h_i / x_{i-1} = (1 - \sigma) / \sigma$ (cf. (3.2)) and $u \in B_2^g(I)$, the last expression in (3.8) can be bounded by

$$\{c_3 / (1 + [\mu n])\}^{[\nu n]} x_i^e [\nu n]!.$$

Therefore we obtain from (3.8)

$$(3.9) \quad \begin{aligned} \left\{ \sum_{i \geq 2} \|u - \varphi_i; L_2(I_i)\|^2 \right\}^{1/2} &\leq c_4 \{c_3 / (1 + [\mu n])\}^{[\nu n]} [\nu n]! \\ &\leq c_4 \left(c_3 \frac{1 + [\nu n]}{1 + [\mu n]} \right)^{[\nu n]} \end{aligned}$$

since $[\nu n]! (1 + [\nu n])^{-[\nu n]} \leq 1$. Choosing now $\nu = \mu / c$ with sufficiently large c in (3.9), we get (3.7) for $q = 2$. Indeed, if $c = 2c_3 / \sigma$ and $\mu n / c \geq 1$, the last expression in (3.9) can be dominated by $c_4 \sigma^{[\nu n]} \leq c_1^{\nu n}$, whereas for all indices n with $\mu n / c < 1$ it can be bounded by $c_4 \leq c_2 \sigma^{\nu n}$.

Step 3. We finally verify (3.7) for arbitrary q . Then (3.6) and (3.7) obviously imply estimate (3.5). Arguing as before, but applying Proposition 3.2 instead of Proposition 3.1 (i), we get for $i \geq 2$

$$(3.10) \quad \|u - P_n u; L_q(I_i)\| \leq c_3 \{c_3 / (1 + [\mu n])\}^{[\nu n]} x_i^e (1 + [\nu n])!.$$

By virtue of the obvious estimate $(1 + [\nu n])! (1 + [\nu n])^{-[\nu n]} \leq 1$, this yields again

$$\|u - P_n u; L_q(x_1, 1)\| \leq c_4 \left(c_3 \frac{1 + [\nu n]}{1 + [\mu n]} \right)^{[\nu n]},$$

hence the result. \square

Remark 3.5. Let N be the number of degrees of freedom (i.e., the dimension) of $\mathcal{S}_{n,\sigma,\mu}$. Then $N \leq \mu n^2$, i.e., $n \geq \sqrt{N/\mu}$, and (3.6) and (3.7) imply the estimate

$$(3.11) \quad \|(1 - P_n)u; L_q(I)\| \leq c_1 \{\sigma^\ell \sqrt{N/\mu} + \sigma \sqrt{N\mu/c}\} \leq c_2 e^{-b\sqrt{N}},$$

where c, c_1 and c_2 do not depend on N and μ , while b is independent of N but depends on μ . Note that a somewhat more precise estimate for $q = 2$ and a discussion of the optimal choice of μ and σ can be found in [10]. The starting point there is the more precise version of Proposition 3.1 (i)

$$E_m(u; L_2)^2 \leq \frac{(m - k + 1)!}{(m + k + 1)!} \|D^k u; L_2(I)\|^2, \quad m + 1 \geq k$$

which is shown by approximating u by the Legendre polynomials.

Remark 3.6. If P_n denotes the orthogonal projection of $L_2(I)$ onto $\mathcal{S}_{n,\sigma,\mu}^{lin}$ (cf. (3.4)), then the assertion of Theorem 3.4 only holds, in general, if μ is sufficiently large. Indeed, instead of estimate (3.10), one obtains from Proposition 3.2 that

$$\|u - P_n u; L_q(I_i)\| \leq c_3 x_i^\ell \{c_3 / (1 + [\mu i])\}^i (1 + i)!, \quad i \geq 2,$$

which gives

$$\|u - P_n u; L_q(I_i)\| \leq c_4 \{c_3 \sigma^{-\ell} (1 + i) / (1 + [\mu i])\}^i \sigma^{\ell n} \leq c_4 2^{-i} \sigma^{\ell n}$$

if μ is large enough. These estimates yield

$$\|u - P_n u; L_q(x_1, 1)\| \leq c_4 \sigma^{\ell n},$$

and consequently

$$(3.12) \quad \|(1 - P_n)u; L_q(I)\| \leq c_1 \sigma^{\ell n} \leq c_2 e^{-b\sqrt{N/\mu}},$$

where $N = \dim \mathcal{S}_{n,\sigma,\mu}^{lin}$ and c_1, c_2, b are independent of n, N and μ . This corresponds to the result in [10] for $q = 2$.

Finally, to the Chebyshev nodes (3.1) and the mesh $\Delta_{n,\sigma}$ we associate the collocation points

$$x_{ij} = x_{ij}^{(n)} = x_{i-1} + \xi_j h_i, \quad (i, j) \in \mathcal{T},$$

where $\mathcal{T} = \{(i, j) : 2 \leq i \leq n, 0 \leq j \leq [\mu n]\}$. For any continuous function u on $(0, 1]$, define the interpolatory projection $Q_n u \in \mathcal{S}_{n,\sigma,\mu}$ by

$$(3.13) \quad (Q_n u)(x_{ij}) = u(x_{ij}), \quad (i, j) \in \mathcal{T}.$$

To prove exponential convergence of spline collocation methods, we need the following analogue of Theorem 3.4.

Theorem 3.7. *If $u \in B_q^\varrho(I)$, $1 \leq q \leq \infty$ and $\varrho > 0$, then*

$$\|(I - Q_n)u; L_q(I)\| \leq ce^{-bn},$$

where c and b are independent of n .

Proof. This follows as in Step 3 of the proof of Theorem 3.4, using of course Proposition 3.3 in place of Proposition 3.2. \square

4. Stability and exponential convergence of Galerkin and collocation methods. Let $A = 1 - K$ be the Mellin convolution operator defined in (2.1). For the approximate solution of Equation (1.1), we first consider the h - p -version of spline Galerkin methods, namely the Galerkin method with splines from $\mathcal{S}_n = \mathcal{S}_{n,\sigma,\mu}$. The Galerkin solution $u_n \in \mathcal{S}_n$ is defined by

$$(4.1) \quad P_n A u_n = u_n - P_n K u_n = P_n f.$$

Theorem 4.1. (i) *Assume (A1), and suppose additionally that condition (A3₂) holds if $q \neq 2$. Then the Galerkin method (4.1) is stable in $L_q(I)$, i.e.,*

$$(4.2) \quad \|P_n A u_n; L_q(I)\| \geq c \|u_n; L_q(I)\|, \quad u_n \in \mathcal{S}_n, \quad n \geq n_0,$$

where n_0 is large enough and c does not depend on u_n and n .

(ii) Under the conditions of (i) and the assumptions $f \in B_q^\varrho(I)$, $\varrho > 0$, (A2 $^\varrho$) and (A4 $^\varrho$), the Galerkin solution u_n of (4.1) converges exponentially to the exact solution u of (1.1) in $L_q(I)$, i.e.,

$$(4.3) \quad \|u - u_n; L_q(I)\| \leq ce^{-b\sqrt{N}},$$

where N is the number of degrees of freedom, c and b are some constants not depending on N (if μ is fixed).

Note that the Galerkin method (4.1) is stable in $L_2(I)$ under the minimal (invertibility) assumptions on A , whereas in our approach the stability in $L_q(I)$, $q \neq 2$, requires some additional smoothness of the kernel function. The proof of Theorem 4.1 (i) is based on the stability of the finite section method for (1.1). Consider the truncation operators

$$(4.4) \quad \pi_n u = \chi_{(x_1, 1)} u, \quad x_1 = x_1^{(n)} = \sigma^{n-1}.$$

Lemma 4.2. *If assumption (A1) is satisfied, then*

$$(4.5) \quad \|\pi_n A \pi_n u; L_q(I)\| \geq c \|\pi_n u; L_q(I)\|, \quad u \in L_q(I), \quad n \geq n_0,$$

where n_0 is sufficiently large and c is independent of u and n .

Proof. Consider the map $\Phi u(x) = e^{-x/q} u(e^{-x})$ which is an isomorphism of $L_q(I)$ onto $L_q(0, \infty)$. Then $\Phi(1-K)\Phi^{-1}$ becomes the Wiener-Hopf integral operator

$$1 - W, \quad Wu(x) := \int_0^\infty w(x-y)u(y) dy,$$

with kernel function $w(x) = e^{-x/q} \kappa(e^{-x}) \in L_1(-\infty, \infty)$. Now (4.5) is a consequence of the stability of the finite section method for Wiener-Hopf equations; see [8] for $q < \infty$ and [1] for $q = \infty$, or [13]. \square

Proof of Theorem 4.1. (i) For any $u_n \in \mathcal{S}_n$, the relation

$$(4.6) \quad P_n(1-K)u_n = \pi_n(1-K)\pi_n u_n + \pi_n(1-P_n)Ku_n$$

holds. Therefore, by virtue of (4.5), for the proof of estimate (4.2), it is sufficient to verify that

$$(4.7) \quad \|\pi_n(1 - P_n)K\|_q \rightarrow 0, \quad n \rightarrow \infty,$$

where $\|\cdot\|_q$ denotes the operator norm on $L_q(I)$. We first consider the case $q = 2$.

Let $u \in L_2(I)$. In the sequel c, c_1, \dots denote various constants not depending on u, n and i . Applying Proposition 3.1 (i) (with $k = 1$) and the scaling argument, we obtain for $i \geq 2$

$$(4.8) \quad \begin{aligned} \|(1 - P_n)Ku; L_2(I_i)\| &\leq ch_i(1 + [\mu n])^{-1} \|DKu; L_2(I_i)\| \\ &\leq c(h_i/x_{i-1})(1 + [\mu n])^{-1} \|xDKu; L_2(I_i)\| \leq c_1 n^{-1} \|xDKu; L_2(I_i)\|. \end{aligned}$$

If we assume, in addition, that K is a continuous map of $L_2(I)$ into $L_{2,1}(I)$, then (4.8) implies that

$$\|\pi_n(1 - P_n)Ku; L_2(I)\| \leq c_1 n^{-1} \|xDKu; L_2(I)\| \leq c_2 n^{-1} \|u; L_2(I)\|$$

which proves (4.7) for $q = 2$. Now we observe that, for any $\varepsilon > 0$, K may be approximated by a convolution operator K_ε with kernel κ_ε satisfying condition (A3₁) such that $\|K - K_\varepsilon\|_2 \leq \varepsilon$. Since the projections P_n are, of course, uniformly bounded on $L_2(I)$, it suffices to verify the L_2 stability of the Galerkin method for $1 - K_\varepsilon$ if ε is sufficiently small.

Next we prove (4.7) for arbitrary q . Applying Proposition 3.2 (with $k = 1$) and the scaling argument, we get for any $u \in L_q(I)$ and $i \geq 2$

$$\begin{aligned} \|(1 - P_n)Ku; L_q(I_i)\| &\leq ch_i^2(1 + [\mu n])^{-1} \|D^2Ku; L_q(I_i)\| \\ &\leq c(h_i/x_{i-1})^2(1 + [\mu n])^{-1} \|x^2D^2Ku; L_q(I_i)\| \\ &\leq c_1 n^{-1} \|x^2D^2Ku; L_q(I_i)\|. \end{aligned}$$

Since K is a bounded operator of $L_q(I)$ into $L_{q,2}(I)$ by condition (A3₂), it follows from the last estimate that

$$(4.9) \quad \begin{aligned} \|\pi_n(1 - P_n)Ku; L_q(I)\| &\leq c_1 n^{-1} \|x^2D^2Ku; L_q(I)\| \\ &\leq c_2 n^{-1} \|u; L_q(I)\| \end{aligned}$$

which completes the stability proof.

(ii) By assumption and Theorem 2.1, the exact solution u of (1.1) belongs to $B_q^\varrho(I)$. Therefore, in view of Theorem 3.4 and Remark 3.5, it is now sufficient to verify the estimate

$$\|u - u_n; L_q(I)\| \leq c\|(1 - P_n)u; L_q(I)\|, \quad n \geq n_0.$$

Since $\|u - u_n; L_q(I)\| \leq \|(1 - P_n)u; L_q(I)\| + \|P_n u - u_n; L_q(I)\|$ and the stability estimate (4.2) implies

$$\|P_n u - u_n; L_q(I)\| \leq c\|P_n A(1 - P_n)u; L_q(I)\|, \quad n \geq n_0,$$

it remains to verify that the last expression is bounded by $c\|(1 - P_n)u; L_q(I)\|$. For $q = 2$, this is obvious because of the uniform boundedness of P_n . In the general case, we have by (4.9)

$$\begin{aligned} \|P_n A(1 - P_n)u; L_q(I)\| &\leq \|\pi_n K(1 - P_n)u; L_q(I)\| \\ &\quad + \|\pi_n(1 - P_n)K(1 - P_n)u; L_q(I)\| \\ &\leq c\|(1 - P_n)u; L_q(I)\| \end{aligned}$$

which completes the proof. \square

We now consider the h - p -version of spline collocation methods. We seek an element $u_n \in \mathcal{S}_n$ such that

$$(4.10) \quad Q_n A u_n = u_n - Q_n K u_n = Q_n f,$$

where Q_n denotes the interpolatory projection defined in (3.13). The following result is the analogue of Theorem 4.1 for the collocation method. Its proof follows the same line as there, using of course Proposition 3.3 and Theorem 3.7 instead of Proposition 3.2 and Theorem 3.4.

Theorem 4.3 (i) *Assume (A1) and (A3₂). Then the collocation method (4.10) is stable in $L_q(I)$, i.e., estimate (4.2) holds with P_n replaced by Q_n .*

(ii) *If, in addition, the assumptions $f \in B_q^\varrho(I)$, $\varrho > 0$, (A2^ϑ) and (A4^ϑ) are satisfied, then the collocation method converges exponentially in $L_q(I)$, i.e., estimate (4.3) holds with the collocation solution u_n of (4.10).*

Remark 4.4. For the approximation methods (4.1) and (4.10) with basis functions from $\mathcal{S}_{n,\sigma,\mu}^{in}$ (cf. (3.4)), the results of this section hold if μ is sufficiently large; compare Remark 3.6 and the proof of Theorem 4.1. To derive stability of those methods, one has to use, for example, the estimates

$$\|(1 - P_n)Ku; L_2(I_i)\| \leq ch_i(1 + [\mu i])^{-1} \|DKu; L_2(I_i)\|, \quad i \geq 2$$

instead of (4.8).

Remark 4.5. In contrast to collocation methods based on piecewise polynomials of fixed degree (cf. [4, 7]), for the h - p -version one does not have so much freedom in the choice of collocation points. If we take, for example, equidistant points $\xi_j = \xi_j^{(m)} = j/m$, $j = 0, \dots, m$, on the unit interval, then the L_∞ operator norm of the corresponding Lagrange interpolatory projection \mathcal{L}_m grows exponentially as $m \rightarrow \infty$ (cf., e.g., [11]) so that the results of Proposition 3.3 and Theorem 4.3 cannot be expected in this case. Selecting, however, $\xi_j^{(m)}$ as the Gauss-Legendre points on I , one has the estimate (see [16, Chapter 14.4])

$$\|\mathcal{L}_m u; L_\infty(I)\| \leq c\sqrt{m} \|u; L_\infty(I)\|, \quad m = 1, 2, \dots$$

for any continuous function u on $[0, 1]$, and defining Q_n again by (3.13), Theorem 4.3 holds with this choice of the collocation points x_{ij} . This only requires a slight modification of the corresponding proofs.

5. Nyström methods. We finally study stability and exponential convergence of a quadrature method which can be interpreted as the discrete iterated version of the collocation method (4.10). To define this method, consider the $(m + 1)$ -point interpolatory quadrature rule

$$(5.1) \quad \int_0^1 v dx \sim \sum_{j=0}^m \omega_j^{(m)} v(\xi_j^{(m)}) = \int_0^1 \mathcal{L}_m v dx$$

with weights $\omega_j = \omega_j^{(m)}$, $j = 0, \dots, m$ and the Chebyshev nodes $\xi_j = \xi_j^{(m)}$ introduced in (3.1). The following property which is a consequence of a theorem of Fejer (cf. [12, Chapter 3.6.2]) is of importance for our convergence analysis:

$$(5.2) \quad \omega_j^{(m)} \geq 0 \quad \text{for all } j \text{ and } m.$$

Let Q_n be the interpolatory projection onto $\mathcal{S}_{n,\sigma,\mu}$ defined in (3.13) via the collocation points x_{ij} , $(i,j) \in \mathcal{T}$. Then the composite quadrature rule obtained by shifting (5.1) (with $m = \lfloor \mu n \rfloor$) to each subinterval I_i , $i \geq 2$, of the geometric mesh $\Delta_{n,\sigma}$ and summing over $i \geq 2$ is

$$(5.3) \quad \int_0^1 v \, dx \sim \sum_{(i,j) \in \mathcal{T}} \omega_j v(x_{ij}) h_i = \int_0^1 Q_n v \, dx.$$

Using (5.3) we approximate the integral operator K in (2.1) by

$$(5.4) \quad K_n u(x) = \sum_{(i,j) \in \mathcal{T}} \omega_j l(x, x_{ij}) u(x_{ij}) h_i, \quad x \in I,$$

where $l(x, y) := y^{-1} \kappa(x/y)$. The Nyström solution $u_n(x)$ to the integral equation (1.1) is now defined by

$$(5.5) \quad (1 - K_n) u_n(x) = f(x), \quad x \in I.$$

Note that (5.5) is a linear system in the values $u_n(x_{ij})$, $(i,j) \in \mathcal{T}$, and then $f(x) + K_n u_n(x)$ may be computed giving $u_n(x)$ for all $x \in I$.

To derive a stability result for (5.5), we need the following technical lemmas.

Lemma 5.1. *If $v \in W_1^{k+1}(I)$ and $m \geq k$, then*

$$(5.6) \quad \left| \int_I v \, dx - \sum_{j=0}^m \omega_j^{(m)} v(\xi_j^{(m)}) \right| \leq c^{k+1} (1+m)^{-k} \|D^{k+1} v; L_1(I)\|,$$

where c is independent of v, k and m .

Proof. For any $\varphi \in \mathbf{P}_m$, the left-hand side of (5.6) can be estimated by

$$\begin{aligned} & \left| \int_I (v - \varphi) \, dx \right| + \left| \sum_{j=0}^m \omega_j^{(m)} (v - \varphi)(\xi_j^{(m)}) \right| \\ & \leq \|v - \varphi; L_1(I)\| + \sum_{j=0}^m \omega_j^{(m)} \|v - \varphi; L_\infty(I)\| \leq c \|v - \varphi; W_1^1(I)\|. \end{aligned}$$

Here we have used (5.2) and Sobolev's embedding theorem. Now the assertion follows from Proposition 3.1 (ii). \square

Lemma 5.2. *Under the condition (A3₂), we have*

$$\|(K\pi_n - K_n)u; L_q(I)\| \leq cn^{-1}\|u; L_{q,2}(I)\|, \quad u \in L_{q,2}(I), \quad n \in \mathbf{N},$$

where c does not depend on u and n and π_n denotes the truncation operator (4.4).

Proof. Applying Lemma 5.1 (with $k = 1$) and the scaling argument, we obtain for any $u \in L_{q,2}(I)$, $n \in \mathbf{N}$ and $i \geq 2$

$$(5.7) \quad \left| \int_{I_i} l(x, y)u(y) dy - \sum_{j=0}^{[\mu n]} \omega_j l(x, x_{ij})u(x_{ij})h_i \right| \\ \leq c_1(1 + [\mu n])^{-1}h_i^2 \|D_y^2 l(x, y)u(y); L_1(I_i)\| \\ \leq c_2 n^{-1} \|y^2 D_y^2 l(x, y)u(y); L_1(I_i)\|.$$

Furthermore, the relation

$$y^2 D_y^2 l(x, y)u(y) = \sum_{0 \leq r \leq 2} l_r(x, y) y^r D^r u(y)$$

holds, where $l_r(x, y) = y^{-1} \kappa_r(x/y)$ and the functions κ_r satisfy the first condition of (A1). From (5.7) we obtain for all $x \in I$ and n

$$|(K\pi_n - K_n)u(x)| \leq c_2 n^{-1} \sum_{0 \leq r \leq 2} \int_0^1 |l_r(x, y)| |y^r D^r u| dy$$

which implies the result by passing to L_q norms. \square

We are now in a position to prove stability of the Nyström method in the weighted Sobolev spaces $L_{q,2}(I)$.

Theorem 5.3. *Assume (A1) and (A3₄). Then the Nyström method (5.5) is stable in $L_{q,2}(I)$, i.e.,*

$$(5.8) \quad \|(1 - K_n)u; L_{q,2}(I)\| \geq c\|u; L_{q,2}(I)\|, \quad u \in L_{q,2}(I), \quad n \geq n_0,$$

where n_0 is sufficiently large and c does not depend on u and n .

Proof. Step 1. First we verify that the operators $1 - K\pi_n$ are stable in $L_{q,2}(I)$. Lemma 4.2 implies the stability of these operators in $L_q(I)$ since one has the matrix representation

$$1 - K\pi_n = \begin{pmatrix} \pi_n(1 - K)\pi_n & 0 \\ -\pi_n K(1 - \pi_n) & 1 \end{pmatrix}$$

with respect to the direct sum $L_q(I) = \pi_n(L_q(I)) \oplus (1 - \pi_n)(L_q(I))$. Moreover, for $j = 1, 2$,

$$\|x^j D^j u; L_q(I)\| \leq \|x^j D^j (1 - K\pi_n)u; L_q(I)\| + \|x^j D^j K\pi_n u; L_q(I)\|$$

and by (A3₂) and the stability of $1 - K\pi_n$ in $L_q(I)$

$$\begin{aligned} \|x^j D^j K\pi_n u; L_q(I)\| &\leq c\|\pi_n u; L_q(I)\| \leq c\|u; L_q(I)\| \\ &\leq c_1\|(1 - K\pi_n)u; L_q(I)\|. \end{aligned}$$

These estimates obviously yield (5.8) with $K\pi_n$ in place of K_n .

Step 2. We show that for any $n \in \mathbf{N}$, $u \in L_{q,2}(I)$ and $j = 0, 1, 2$, the estimate

$$(5.9) \quad \|x^j D^j (K\pi_n - K_n)u; L_q(I)\| \leq cn^{-1}\|u; L_{q,2}(I)\|$$

holds. Then the stability of (5.5) follows from that of $1 - K\pi_n$ by small perturbation with respect to the $L_{q,2}$ operator norm. Note that Lemma 5.2 implies (5.9) if $j = 0$. For $j = 1, 2$, we use the fact that $K^{(j)} = x^j D^j K$ is the Mellin convolution operator with kernel $x^j D^j \kappa$ and that $K_n^{(j)} = x^j D^j K_n$ is the approximate operator (5.4) corresponding to $K^{(j)}$. Since, by (A3₄), $K^{(j)}$ satisfies condition (A3₂), it suffices to apply Lemma 5.2 to those operators. \square

Remark 5.4. Using the more traditional approach of showing that $\|(K\pi_n - K_n)K_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$, one can also prove the stability of (5.5) in $L_\infty(I)$; cf. [9] in the case of the h -version of the Nyström method.

We are now ready to state our result on exponential convergence of the Nyström method.

Theorem 5.5. *Assume $f \in B_q^\varrho(I)$, $\varrho > 0$, (A1), (A2^e), (A4^e) and (A5^o). Then the Nyström method converges with the error bound (4.3) where u_n denotes the solution of (5.5).*

Proof. Using the relation $(1 - K_n)u_n = (1 - K)u$ and Theorem 5.3, we obtain

$$\begin{aligned} \|u - u_n; L_{q,2}(I)\| &\leq c\|(1 - K_n)(u - u_n); L_{q,2}(I)\| \\ &= c\|(K - K_n)u; L_{q,2}(I)\|. \end{aligned}$$

Recall that condition (A5^o) is stronger than (A3₄). Furthermore, by assumption and Theorem 2.1, we have $u \in B_q^\varrho(I)$ so that it remains to apply Lemma 5.7 below to the operators $x^j D^j (K - K_n)$, $j = 0, 1, 2, \dots$. \square

Remark 5.6. Under the conditions of the preceding theorem, we obtain the exponential convergence rate

$$\|u - u_n; L_{q,j}(I)\| \leq ce^{-bn}, \quad c, b \text{ independent of } n,$$

for any $j \in \mathbf{N}$. Indeed, if we apply the operators $x^j D^j$ to the relation

$$u - u_n = Ku - K_n u_n = (K - K_n)u + K_n(u - u_n),$$

we obtain with the notation in Step 2 of the proof of Theorem 5.3

$$(5.10) \quad x^j D^j (u - u_n) = (K^{(j)} - K_n^{(j)})u + K_n^{(j)}(u - u_n),$$

where the kernel of $K^{(j)}$ also satisfies the assumption (A5^o). Lemma 5.7 below yields the exponential rate for the first term in (5.10). For the second term, this is a consequence of Theorem 5.5 and the uniform boundedness of $K_n^{(j)}$, the latter following from Lemma 5.2 applied to $K^{(j)}$ and the continuity of $K^{(j)} : L_2(I) \rightarrow L_{2,k}(I)$ for any j and k .

Lemma 5.7. *If the assumptions $u \in B_q^\varrho(I)$, $\varrho > 0$ and (A5^o) are fulfilled, then*

$$(5.11) \quad \|(K - K_n)u; L_q(I)\| \leq ce^{-bn},$$

where the constants c and b do not depend on n .

Proof. We proceed similarly as in the proof of Theorem 3.4. First we have

$$\begin{aligned} \|K(1 - \pi_n)u; L_q(I)\| &\leq c\|(1 - \pi_n)u; L_q(I)\| \\ &\leq c\sigma^{(n-1)\varrho}\|x^{-\varrho}u; L_q(I)\| \leq c\sigma^{n\varrho}. \end{aligned}$$

Here and in the sequel c and b denote various constants which are independent of n . It remains to show the estimate

$$(5.12) \quad \|(K\pi_n - K_n)u; L_q(I)\| \leq c\sigma^{bn}.$$

Applying Lemma 5.1 (with $k = [\nu n]$, $m = [\mu n]$) and the scaling argument, we obtain for all $i \geq 2$ and $\nu \in (0, \mu)$

$$\begin{aligned} (5.13) \quad &\left| \int_{I_i} l(x, y)u(y) dy - \sum_{j=0}^{[\mu n]} \omega_j l(x, x_{ij})u(x_{ij})h_i \right| \\ &\leq c\{c/(1 + [\mu n])\}^{[\nu n]} h_i^{1+[\nu n]} \|D_y^{1+[\nu n]} l(x, y)u(y); L_1(I_i)\| \\ &\leq c\{c/1 + [\mu n]\}^{[\nu n]} x_i^\varrho \|y^{1+[\nu n]-\varrho} D_y^{1+[\nu n]} l(x, y)u(y); L_1(I_i)\|. \end{aligned}$$

Define the functions κ_j and l_j by

$$(5.14) \quad y^{-1}\kappa_j(x/y) = l_j(x, y) = y^j D_y^j l(x, y).$$

Recall that $y^{-1}\kappa(x/y) = l(x, y)$. Then

$$(5.15) \quad \int_0^\infty x^{1/q-1} |\kappa_j(x)| dx \leq d^{j+1} j!, \quad j = 0, 1, \dots,$$

where d is independent of j . This follows easily from condition (A5°), using Lemma 2.2 and the relations $(xD)^j = (Dx - 1)^j$,

$$(D_y y)^j l(x, y) = (-1)^j y^{-1} ((xD)^j \kappa)(x/y).$$

Now let K_j be the Mellin convolution operator with kernel function

$$\{c/(1 + [\mu n])\}^{[\nu n]} |\kappa_j|(x),$$

and define the functions v_j by

$$v_j = 0 \quad \text{on } I_1, \quad v_j(x) = x_i^e |x^{j-e} D^j u(x)| \quad \text{on } I_i, \quad i \geq 2.$$

Since $u \in B_q^e(I)$, we then have

$$(5.16) \quad \|v_j; L_q(I)\| \leq d^{j+1} j!, \quad j = 0, 1, \dots$$

Furthermore, with the notation of (5.14)

$$y^{m-e} D_y^m l(x, y) u(y) = \sum_{j=0}^m \binom{m}{j} l_j(x, y) y^{m-j-e} D^{m-j} u(y)$$

for any m , and combining the estimates (5.13) we can write

$$|(K\pi_n - K_n)u(x)| \leq \sum_{j=0}^{1+[\nu n]} \binom{1+[\nu n]}{j} K_j v_{1+[\nu n]-j}(x), \quad x \in I.$$

Using (5.15) and (5.16), this implies

$$(5.17) \quad \begin{aligned} & \| (K\pi_n - K_n)u; L_q(I) \| \\ & \leq \sum_{j=0}^{1+[\nu n]} c \left\{ \frac{c}{1+[\mu n]} \right\}^{[\nu n]} \binom{1+[\nu n]}{j} d^{j+1} j! d^{2+[\nu n]-j} (1+[\nu n]-j)! \\ & \leq c \{c/(1+[\mu n])\}^{[\nu n]} (2+[\nu n])! \leq c \{c(1+[\nu n])/(1+[\mu n])\}^{[\nu n]}. \end{aligned}$$

Here we have used the estimate

$$(2+[\nu n])!/(1+[\nu n])^{[\nu n]} \leq 2^{1+[\nu n]}.$$

As in the proof of Theorem 3.4, the last quantity in (5.17) can be estimated by $c\sigma^{b\nu}$ if $\nu \in (0, \mu)$ is chosen sufficiently small. This completes the proof of (5.12). \square

Remark 5.8. All results of this section extend with the same proofs to the case where the collocation points x_{ij} are defined by means of the Gauss-Legendre nodes $\xi_j^{(m)}$ on I , since the corresponding weights $\omega_j^{(m)}$ satisfy condition (5.2) again.

Finally we note that the results on the Nyström method remain valid if we choose a composite quadrature rule based on a linear degree distribution on the mesh $\Delta_{n,\sigma}$ (i.e., a suitable interpolatory projection Q_n onto $\mathcal{S}_{n,\sigma,\mu}^{lin}$ is taken in (5.3)) and if μ is sufficiently large.

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