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PARABOLIC INTEGRODIFFERENTIAL EQUATIONS WITH SINGULAR KERNELS

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ABSTRACT. We consider a parabolic integrodifferential Volterra equation with nonhomogeneous boundary condition

$$(*)\qquad \begin{cases} u_t(t,x) = (\Delta + c)\int_0^t k(t-s)u(s,x)\,ds + k_0u(t,x) + f(t,x), \\ \qquad\qquad\qquad t\in[0,T], x\in\Omega, \\ u(0,x) = u_0(x), \quad x\in\Omega \\ u(t,x) = \varphi(t,x), \quad t\in[0,T],\ x\in\partial\Omega, \end{cases}$$

where Δ is the Laplace operator and k is a scalar kernel singular at $t = 0$. This assumption on k gives a parabolic character to $(*)$. We state some results about the existence, uniqueness and regularity of the solutions of $(*)$.

0. Introduction. This paper is concerned with a class of parabolic integrodifferential Volterra equations with nonhomogeneous boundary condition

(0.1)
$$\begin{cases} u_t(t,x) = (\Delta + c)\int_0^t k(t-s)u(s,x)\,ds + k_0u(t,x) + f(t,x), \\ \qquad\qquad\qquad t\in[0,T],\ x\in\Omega, \\ u(0,x) = u_0(x), \quad x\in\Omega, \\ u(t,x) = \varphi(t,x), \quad t\in[0,T],\ x\in\partial\Omega, \end{cases}$$

where Ω is a bounded open set in $\mathbf{R}^n$, $n\in\mathbf{N}$, with regular boundary $\partial\Omega$, c and k_0 are real constants, Δ is the Laplace operator and the kernel k is a scalar real function.

Problem (0.1) occurs in the study of heat flow in materials with memory (see [**10, 13, 14**] and references therein).

In the applications one is often concerned with the corresponding problem with infinite delay (that is, with $\int_0^t$ replaced by $\int_{-\infty}^t$), which

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can be written in the form (0.1), provided the history of u up to time $t = 0$ is known. Otherwise, it needs a separate treatment, which will be performed in a forthcoming paper [**17**].

Several papers have been devoted to a problem similar to (0.1):

$$\tag{0.2} \begin{cases} u_t(t,x) = \int_0^t k(t-s)(\Delta + c)u(s,x)\,ds + \Delta u(t,x) + f(t,x), \\ \qquad\qquad\qquad\qquad t \in [0,T],\ x \in \Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ u(t,x) = \varphi(t,x), \quad t \in [0,T],\ x \in \partial\Omega, \end{cases}$$

(see [**7, 11**] for the case $\varphi \equiv 0$ and [**2, 16**] for the inhomogeneous case). Problem (0.2) is easier than (0.1) because the integral term can be considered as a perturbation of Δu.

Problem (0.1) has been studied in the case $\varphi \equiv 0$ in [**4, 5, 6**], with singular kernels of the type $k(t) = t^{-\beta}$, $0 < \beta < 1$. Concerning singular kernels in evolution equations, we quote also [**15**], where the completely monotone kernel $k(t) = \sum_{n=1}^{\infty} e^{-n^\gamma t}$, $\gamma > 1$, has been considered.

Here we assume that $k :]0, +\infty[\to \mathbf{R}$ is a locally integrable function, whose Laplace transform $\hat{k}(\lambda)$ can be analytically extended to a suitable sector S in the complex plane, containing the positive real semiaxis, in such a way that the extension $\hat{k}(\lambda)$ satisfies

$$\tag{0.3} \hat{k}(\lambda) = \bar{k}\lambda^{\beta-1}(1 + O(1/\lambda)), \quad \lambda \in S$$

with $0 < \beta < 1$. Then, one can construct a resolvent operator for problem (0.1) with $\varphi \equiv 0$, in such a way that the solution enjoys many properties of the solutions of parabolic differential equations (see [**6**]). Therefore, the fact that the kernel is singular at $t = 0$ gives a parabolic character to problem (0.1).

Assumption (0.3) is satisfied both in the case $k(t) = t^{-\beta}$ and in the case $k(t) = \sum_{n=1}^{\infty} e^{-n^\gamma t}$.

To solve the nonhomogeneous problem (with $\varphi \not\equiv 0$), we need to introduce the Dirichlet mapping $D : C(\partial\Omega) \to C(\overline{\Omega}) \cap C^2(\Omega)$, where for any $\varphi \in C(\partial\Omega)$, $D\varphi$ is the solution of the problem

$$\tag{0.4} \begin{cases} \Delta z(x) = 0, & x \in \Omega \\ z(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

If (0.1) has a solution u, then the function $v(t,x) = u(t,x) - D\varphi(t,x)$ is the solution of an integrodifferential problem with homogeneous boundary condition
(0.5)
$$\begin{cases} v_t(t,\cdot) = (\Delta + c)\int_0^t k(t-s)v(s,\cdot)\,ds + k_0 v(t,\cdot) + g(t,\cdot), & t \in [0,T], \\ v(0,x) = v_0(x), & x \in \overline{\Omega}, \\ v(t,x) = 0,\ t \in [0,T], & x \in \partial\Omega, \end{cases}$$

and the function g is given by

$$\begin{aligned} g(t,x) = c\int_0^t k(t-s)D\varphi(s,x)\,ds + k_0 D\varphi(t,x) - D\varphi_t(t,x) \\ + f(t,x), \quad t \in [0,T],\ x \in \overline{\Omega}. \end{aligned} \tag{0.6}$$

Setting $v(t) = v(t,\cdot)$, $g(t) = g(t,\cdot)$, problem (0.5) may be rewritten as an abstract integrodifferential equation in the Banach space $X = C(\overline{\Omega})$:

$$\begin{cases} v'(t) = A\int_0^t k(t-s)v(s)\,ds + k_0 v(t) + g(t), & t \in [0,T] \\ v(0) = v_0, \end{cases} \tag{0.7}$$

where

$$\begin{cases} D(A) = \{v \in X : \Delta v \in X,\ v(x) = 0 \text{ for any } x \in \partial\Omega\} \\ Av = (\Delta + c), \quad v \in D(A), \end{cases}$$

generates an analytic semigroup in $C(\overline{\Omega})$.

Therefore, we are led to study regularity properties of the resolvent operator for integrodifferential equations of the form (0.7) in general Banach space X, assuming that $A : D(A) \subset X \to X$ generates an analytic semigroup and k satisfies (0.3). In particular, we find existence, uniqueness and regularity properties of a strict solution (see Theorem 1.7), that is, $v(t) \in D(A)$ for every t, so that we can write $A\int_0^t k(t-s)v(s)\,ds = \int_0^t k(t-s)Av(s)\,ds$. This is important in the applications, where Δv is required to exist and be continuous.

Applying the general abstract theory to problem (0.5) is not straightforward: actually, for getting a strict solution one needs that g, given by (0.6), is Hölder continuous. However, for general kernel k, g is not necessarily Hölder continuous because the function $t \to$

$\int_0^t k(t-s)D\varphi(s,\cdot)\,ds$ is merely continuous. We overcome this difficulty in two different ways. First, we assume that $t \to \int_0^t |k(s)|\,ds$ is Hölder continuous (this happens, for instance, in the case $k(t) = t^{-\beta}$, with $0 < \beta < 1$). However, this assumption on kernel k is not sufficient to guarantee the existence of the strict solution: we can only show the existence of a solution v such that $v(t)$ does not belong necessarily to $D(A)$, but $\int_0^t k(t-s)v(s)\,ds$ is in $D(A)$ for every t.

The second, and more fruitful, way is to replace the Dirichlet map defined in (0.4) by the unique solution $z = D\varphi$ of

$$\begin{cases} \Delta z(x) + cz(x) = 0, & x \in \Omega, \\ z(x) = \varphi(x), & x \in \partial\Omega. \end{cases}$$

Obviously, this can be done for any φ, if $-c$ is not an eigenvalue of the Laplace operator. Otherwise, we consider only boundary data φ satisfying suitable compatibility conditions (see (2.6)). With this choice of the operator D, the integral term in (0.6) disappears, so that g is Hölder continuous, provided the data f, φ satisfy suitable regularity assumptions.

Our work is organized as follows. In Section 1 we list some assumptions, which will remain valid throughout the paper, and review some known results about problem (0.7). Moreover, we give the existence and regularity theorem about the strict solution of (0.7). Section 2 is devoted to the study of the existence and regularity of the solutions of (0.1).

We now give some notations, which we will use in the following. Let X be a complex Banach space with norm $||\cdot||$. If Y is another Banach space, we denote by $\mathcal{L}(X;Y)$ the Banach space of all linear bounded operators $T : X \to Y$, endowed with the norm $||T|| = \sup\{||T(x)||, ||x|| \leq 1\}$. We set $\mathcal{L}(X) = \mathcal{L}(X;X)$.

If $T > 0$, we denote by $C([0,T];X)$ the space of all continuous functions $u : [0,T] \to X$, endowed with the norm $||u||_\infty = \{\sup ||u(x)||, x \in [0,T]\}$. Given $\alpha \in]0,1[$, $C^\alpha([0,T];X)$ is the subspace of $C([0,T];X)$ consisting of the α-Hölder continuous functions u, that is,

$$[u]_\alpha \doteq \sup\{|t-s|^{-\alpha}||u(t)-u(s)||; t,s \in [0,T], t \neq s\} < +\infty.$$

It is endowed with the norm $||u||_{C^\alpha([0,T];X)} \doteq ||u||_\infty + [u]_\alpha$. $C^1([0,T];X)$ (respectively, $C^{1,\alpha}([0,T];X)$) is the space of all differentiable functions

u such that u' belongs to $C([0,T];X)$ (respectively, $C^\alpha([0,T];X)$). $h^\alpha([0,T];X)$ is the subspace of $C^\alpha([0,T];X)$ consisting of the functions u such that

$$\lim_{\tau\to 0}\sup_{|x-y|\le\tau}\tau^{-\alpha}||u(x)-u(y)||=0.$$

1. Existence and regularity of the solutions of a parabolic integrodifferential equation with homogeneous boundary condition. Let X be a complex Banach space with norm $||\cdot||$, and let $A : D(A)\subset X\to X$ be a linear operator satisfying:

$$\begin{cases}\text{there exist } C>0, \omega\in\mathbf{R} \text{ and } \theta\in]\pi/2,\pi[\text{ such that :}\\ \text{(i) the resolvent set } \rho(A) \text{ of } A \text{ contains the sector}\\ \quad S_{\theta,\omega}=\{\lambda\in\mathbf{C};\lambda\ne\omega, |\arg(\lambda-\omega)|<\theta\};\\ \text{(ii) for any } \lambda\in S_{\theta,\omega} : ||(\lambda-A)^{-1}||_{\mathcal{L}(X)}\le C|\lambda-\omega|^{-1}.\end{cases} \tag{1.1}$$

Assumption (1.1) means that A generates an analytic semigroup in X. Since A is a closed operator, $D(A)$ is a Banach space, endowed with the graph norm $||x||_{D(A)}=||x||+||Ax||$, $x\in D(A)$.

Let $k :]0,+\infty[\to\mathbf{R}$ be a locally integrable function, whose Laplace transform $\hat{k}(\lambda)$ has the form

$$\hat{k}(\lambda)=\bar{k}\lambda^{\beta-1}(1+O(1/\lambda)),\quad \lambda\in S_{\varphi_0,0}, \tag{1.2}$$

where $\bar{k}>0$, $O(1/\lambda)\to 0$ as $|\lambda|\to+\infty$, $\beta\in(0,1)$ and $\varphi_0\in\mathbf{R}$ satisfy

$$\beta>2(1-\theta/\pi),\quad \pi/2<\varphi_0<\theta/(2-\beta). \tag{1.3}$$

As it has been noticed in [**6**], for any $\lambda\in S_{\varphi_0,0}$, $\lambda/\hat{k}(\lambda)$ belongs to $S_{\theta,0}$, thanks to (1.3). Since the operator $B=A-\omega$ satisfies (1.1) with the same angle θ and $\omega=0$, we have that for any $\lambda\in S_{\varphi_0,0}$ the operator $\lambda-\hat{k}(\lambda)B$ is invertible and

$$||(\lambda-\hat{k}(\lambda)B)^{-1}||\le C|\lambda|^{-1}, \tag{1.4}$$

(see again [**6**]). Therefore, if $k_0\in\mathbf{R}$ one may write

$$\lambda-\hat{k}(\lambda)A-k_0=(\lambda-\hat{k}(\lambda)B)(I-(\lambda-\hat{k}(\lambda)B)^{-1}(\hat{k}(\lambda)\omega+k_0)).$$

From this, taking into account (1.2) and (1.4), it follows that there exist $r_0 > 0$ and $M > 0$ such that for any $\lambda \in S_{\varphi_0,0}$, $|\lambda| \geq r_0$, the operator $\lambda - \hat{k}(\lambda)A - k_0$ is invertible and

$$||(\lambda - \hat{k}(\lambda)A - k_0)^{-1}|| \leq M|\lambda|^{-1}. \tag{1.5}$$

Then, it is possible to construct a resolvent operator $R : [0, +\infty[\to \mathcal{L}(X)$ for the problem

$$\begin{cases} v'(t) = \int_0^t k(t-s)Av(s)\,ds + k_0 v(t), & t > 0, \\ v(0) = v_0, \end{cases} \tag{1.6}$$

given by

$$R(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t} F(\lambda)\,d\lambda, \quad t \geq 0 \tag{1.7}$$

where $F(\lambda) = (\lambda - \hat{k}(\lambda)A - k_0)^{-1}$ and γ is the path $\gamma = \gamma^+ + \gamma^0 + \gamma^-$; $\gamma^\pm = \{\lambda \in \mathbf{C}; \lambda = \rho e^{\pm i\varphi}, \rho \geq r\}$ and $\gamma^0 = \{\lambda \in \mathbf{C}; \lambda = re^{i\eta}, |\eta| < \varphi\}$, $r \geq r_0$, $\pi/2 < \varphi \leq \varphi_0$, are oriented counterclockwise.

This was done in [**4**], for $D(A)$ dense in X and $k_0 = 0$, and it can be proved in the general case as in [**11**].

The resolvent operator has the following properties.

Proposition 1.1. [**4**]. *Let* (1.1)–(1.3) *hold and let* $R(\cdot)$ *be defined by* (1.7). *Then*

(i) $R(\cdot)$ *has an analytical extension to* $S_{\varphi_0-\pi/2,0}$;

(ii) *there exists* $M_1 > 0$ *such that for each* $t > 0$ *we have*

$$||R(t)||_{\mathcal{L}(X)} + ||tR'(t)||_{\mathcal{L}(X)} + ||t^2R''(t)||_{\mathcal{L}(X)} \leq M_1 e^{r_0 t}; \tag{1.8}$$

(iii) *if* $t > 0$ *and* $x \in X$, *then* $\int_0^t k(t-s)R(s)x\,ds$ *belongs to* $D(A)$ *and*

$$R'(t)x = A\int_0^t k(t-s)R(s)x\,ds + k_0R(t)x; \tag{1.9}$$

(iv) *R is Laplace transformable and*

$$\hat{R}(\lambda) = F(\lambda), \quad \lambda \in S_{\varphi_0,0}, \quad |\lambda| \geq r_0.$$

The existence of the resolvent operator for problem (1.6) allows one to solve in a strict and strong sense the inhomogeneous problem

(1.10)
$$\begin{cases} v'(t) = \int_0^t k(t-s)Av(s)\,ds + k_0 v(t) + g(t), \quad t \in [0,T], \ T > 0, \\ v(0) = v_0, \end{cases}$$

where A, k satisfy assumptions (1.1)–(1.3) and $g : [0,T] \to X$ is Hölder continuous.

We recall that a function $v : [0,T] \to D(A)$ is said to be a strict solution of (1.10) in $[0,T]$ if $v \in C([0,T]; D(A)) \cap C^1([0,T]; X)$ and (1.10) holds, while by a strong solution we mean that $v \in C([0,T]; X)$, $v(0) = v_0$ and there exists a sequence $v_n \in C^1([0,T]; X)$, with the functions $\int_0^t k(t-s)v_n(s)\,ds$ belonging to $C([0,T]; D(A))$, such that $v_n \to v$ and $v_n'(t) - A\int_0^t k(t-s)v_n(s)\,ds - k_0 v_n(t) \to g(t)$ in $C([0,T]; X)$.

The connection between existence of a resolvent operator and solvability of problem (1.10) will be stated in the sequel.

To study the Hölder regularity of the solution of (1.10), in [**6**] the authors introduced a class of subsets of X. Such sets are defined for $\alpha > 0$ in the following way: $x \in D_{k,A}(\alpha, \infty)$ if and only if

$$|x|_{D_{k,A}(\alpha,\infty)} \doteq \sup\{\rho^\alpha ||\rho e^{i\sigma} F(\rho e^{i\sigma})x - x||; \ \rho > 1, \ |\sigma| < \varphi_0\} < +\infty.$$

$D_{k,A}(\alpha, \infty)$ is a Banach space under the norm

$$||x||_{D_{k,A}(\alpha,\infty)} \doteq ||x|| + |x|_{D_{k,A}(\alpha,\infty)}. \tag{1.11}$$

We note that the definition of $D_{k,A}(\alpha, \infty)$ is clearly related to that of the Lions and Peetre [**9**] real interpolation spaces $D_A(\alpha, \infty)$ [**3, 12, 18**]. In the case of our class of kernels, the connection is stated in the following proposition (see [**6**, Proposition 3.5]):

Proposition 1.2. *Suppose that k satisfies* (1.2) *and* (1.3). *Then*

$$D_{k,A}(\alpha, \infty) \equiv D_A(\alpha/(2-\beta), \infty). \tag{1.12}$$

Concerning the regularity properties of the resolvent operator, in [**6**, Theorem 3.3] the following has been proved.

Theorem 1.3. *If $x \in D_{k,A}(\alpha, \infty)$, $0 < \alpha < 1$, then the function $t \to R(t)x$ belongs to $C^\alpha([0, +\infty); X)$.*

The following result about the existence and regularity of an intermediate kind of solution between a strict one and a strong one has been proved in [**6**] for $k_0 = 0$. In our case the proof is completely analogous.

Theorem 1.4. *Assume* (1.1)–(1.3). *Let $g \in C^\alpha([0, T]; X)$, $0 < \alpha < 1$, and $v_0 \in D(A)$. Then*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)g(s)\, ds, \quad t \in [0, T], \tag{1.13}$$

is the unique function belonging to $C^1([0, T]; X)$ such that the function $\int_0^t k(t-s)v(s)\, ds$ belongs to $C([0, T]; D(A))$ and

$$\begin{cases} v'(t) = A\int_0^t k(t-s)v(s)\, ds + k_0 v(t) + g(t) \\ v(0) = v_0 \end{cases}$$

holds. Moreover, if

$$g(0) \in D_{k,A}(\alpha, \infty), \qquad v_0 \in D_{k,A}(\alpha + 1, \infty), \tag{1.14}$$

then $v \in C^{1,\alpha}([0, T]; X)$.

From the last theorem we deduce

Corollary 1.5. *Assume* (1.1)–(1.3). *Let $g \in C([0, T]; X)$ and $v_0 \in \overline{D(A)}$. Then the function v given by* (1.13) *is the unique strong solution of* (1.10). *In addition, if $v_0 \in D_{k,A}(\alpha, \infty)$, $0 < \alpha < 1$, then v belongs to $C^\alpha([0, T]; X)$.*

Proof. Let $g_n \in C^1([0, T]; X)$ and $v_{0n} \in D(A)$ such that g_n converges to g in $C([0, T]; X)$ and v_{0n} converges to v_0 in X. By Theorem 1.4 the sequence

$$v_n(t) = R(t)v_{0n} + \int_0^t R(t-s)g_n(s)\,ds, \quad t \in [0,T],$$

and the function v satisfy the conditions of the definition of the strong solution. The uniqueness follows by standard arguments.

Concerning the last part, we observe that, by virtue of Theorem 1.3, we have $R(t)v_0 \in C^\alpha([0,T];X)$. Moreover, if $t,\tau \in [0,T]$, $t > \tau$, then

$$\begin{aligned}\int_0^t R(t-s)g(s)\,ds &- \int_0^\tau R(\tau-s)g(s)\,ds \\ &= \int_\tau^t R(t-s)g(s)\,ds + \int_0^\tau \left(\int_{\tau-s}^{t-s} R'(\sigma)\,d\sigma\right)g(s)\,ds,\end{aligned}$$

from which, taking (1.8) into account, it follows that

$$\begin{aligned}&\left\|\int_0^t R(t-s)g(s)\,ds - \int_0^\tau R(\tau-s)g(s)\,ds\right\| \\ &\quad\le M_1e^{r_0T}||g||_\infty(t-\tau) + M_1e^{r_0T}||g||_\infty \int_0^\tau \left(\int_{\tau-s}^{t-s} \frac{1}{\sigma}\,d\sigma\right)ds \\ &\quad\le M_1e^{r_0T}||g||_\infty\left((t-\tau) + \int_0^\tau \frac{ds}{(\tau-s)^\alpha}\int_0^{t-\tau}\sigma^{\alpha-1}\,d\sigma\right) \\ &\quad\le M_1e^{r_0T}||g||_\infty\left((t-\tau) + \int_0^T \frac{ds}{s^\alpha}\frac{(t-\tau)^\alpha}{\alpha}\right).\end{aligned}$$

Therefore, $v(t)$ is α-Hölder continuous. □

We now give an existence, uniqueness and regularity result of the strict solution of (1.10). Nevertheless, we must note that there is a loss of regularity of the solution with respect to the inhomogeneous term. To prove this theorem, we need the following regularity properties of the resolvent operator $R(t)$.

Lemma 1.6. *Assume* (1.1)–(1.3). *Then*:

(i) *there exists a constant* $M_2 > 0$ *such that*

$$||AR(t)||_{\mathcal{L}(X)} \le M_2e^{r_0t}t^{\beta-2}, \quad t \ge 0, \tag{1.15}$$

$$(1.16)\qquad ||AR(t)-AR(\tau)||_{\mathcal{L}(X)} \le M_2 e^{r_0\tau}\int_\tau^t \sigma^{\beta-3}\,d\sigma, \quad t>\tau>0;$$

(ii) *there exists a constant* $M_3>0$ *such that for each* $x\in D_{k,A}(\alpha,\infty)$, $0<\alpha<1$, *and* $\lambda\in S_{\varphi_0,0}$, $|\lambda|\ge r_0$,

$$(1.17)\qquad ||AF(\lambda)x|| \le M_3|\lambda|^{1-\alpha-\beta}||x||_{D_{k,A}(\alpha,\infty)};$$

(iii) *there exists a constant* $M_4>0$ *such that for each* $x\in D_{k,A}(\alpha,\infty)$, $0<\alpha<1$, $\alpha+\beta>1$, *and* $t,\tau>0$, *we have*

$$(1.18)\qquad ||AR(t)x-AR(\tau)x|| \le M_4|t-\tau|^{\alpha+\beta-2}||x||_{D_{k,A}(\alpha,\infty)};$$

(iv) *for each* $x\in D_{k,A}(\alpha,\infty)$, $0<\alpha<1$, $\alpha+\beta>1$, *and* $t>0$ *the integral* $\int_0^t R(s)x\,ds$ *belongs to* $D(A)$ *and*

$$(1.19)\qquad A\int_0^t R(s)x\,ds = \frac{1}{2\pi i}\int_\gamma \frac{e^{\lambda t}-1}{\lambda}AF(\lambda)x\,d\lambda;$$

(v) *there exists a constant* $M_5>0$ *such that for each* $x\in D_{k,A}(\alpha,\infty)$, $0<\alpha<1$ *and* $\alpha+\beta>1$, *we have*

$$(1.20)\qquad \left\|A\int_0^t R(s)x\,ds\right\| \le M_5||x||_{D_{k,A}(\alpha,\infty)}t^{\alpha+\beta-1}, \quad t>0;$$

$$(1.21)\qquad \left\|A\int_\tau^t R(s)x\,ds\right\| \le M_5||x||_{D_{k,A}(\alpha,\infty)}(t-\tau)^{\alpha+\beta-1}, \quad t>\tau>0.$$

Proof. First of all, we observe that for any $\lambda\in S_{\varphi_0,0}$, $|\lambda|\ge r_0$, we have

$$(1.22)\qquad AF(\lambda) = -\hat{k}(\lambda)^{-1}+\hat{k}(\lambda)^{-1}(\lambda-k_0)F(\lambda).$$

Therefore, (1.7) and (1.22) give

$$(1.23)\qquad AR(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\hat{k}(\lambda)^{-1}(\lambda-k_0)F(\lambda)\,d\lambda, \quad t>0,$$

because of $\int_\gamma e^{\lambda t}\hat{k}(\lambda)^{-1}\,d\lambda = 0$. By standard arguments (1.15) and (1.16) follow, taking into account (1.2) and (1.23).

Concerning (1.17), by (1.2) and (1.22), we get

$$||AF(\lambda)x|| \leq C|\lambda|^{1-\beta}(||\lambda F(\lambda)x - x|| + |k_0|\,||F(\lambda)x||), \tag{1.24}$$

where C is a constant not depending on λ. Since $x \in D_{k,A}(\alpha,\infty)$ and taking into account (1.5), (1.24) yields

$$||AF(\lambda)x|| \leq C|\lambda|^{1-\beta-\alpha}|x|_{D_{k,A}(\alpha,\infty)} + CM|k_0|r_0^{\alpha-1}|\lambda|^{1-\beta-\alpha}||x||,$$

that is (1.17).

To prove (1.18), we observe that, for any $t > \tau > 0$,

$$\begin{aligned} AR(t)x - AR(\tau)x &= \frac{1}{2\pi i}\int_\tau^t \left(\int_\gamma e^{\lambda\sigma}\lambda AF(\lambda)x\,d\lambda\right)d\sigma \\ &= \frac{1}{2\pi i}\int_0^{t-\tau}\left(\int_\gamma e^{\lambda(\sigma+\tau)}\lambda AF(\lambda)x\,d\lambda\right)d\sigma, \end{aligned}$$

from which, using standard arguments and (1.17), we get the claim.

In (1.19), we observe that the integral on the right hand side is convergent, thanks to (1.17). Integrating (1.23) and interchanging the integrals, we get (1.19). (1.20) follows from (1.19) and (1.17).

Finally, it remains to prove (1.21). To this end, thanks to (1.19), we may write

$$A\int_\tau^t R(s)x\,ds = \frac{1}{2\pi i}\int_\gamma \frac{e^{\lambda t} - e^{\lambda\tau}}{\lambda}AF(\lambda)x\,d\lambda, \quad t > \tau > 0.$$

Then, again using standard arguments and (1.17), (1.21) easily follows. $\square$

Theorem 1.7. *Assume* (1.1)–(1.3). *If* $v_0 \in D(A)$ *and* g *belongs to* $C^\alpha([0,T];X) \cap C([0,T];D_{k,A}(\alpha,\infty))$ *for some* $\alpha \in]0,1[$ *with* $\alpha+\beta > 1$, *then the function* v *defined by* (1.13) *is the unique strict solution of* (1.10). *Moreover, if* $v_0 \in D_{k,A}(\alpha+1,\infty)$, *then* v *belongs to* $C^{\alpha+\beta-1}([0,T];D(A))$.

Proof. Thanks to Theorem 1.4, we need only to prove $v(t) \in D(A)$ to show that v is a strict solution of (1.10). To this end, we may write $v(t)$ in the form

$$\begin{aligned} v(t) = R(t)v_0 + \int_0^t R(t-s)(g(s)-g(t))\,ds \\ + \int_0^t R(s)g(t)\,ds, \quad t \in [0,T]. \end{aligned} \tag{1.25}$$

Taking into account (1.15) and A being a closed operator, the first integral on the right hand side of (1.25) belongs to $D(A)$, while $\int_0^t R(s)g(t)\,ds \in D(A)$ in virtue of our hypothesis $g(t) \in D_{k,A}(\alpha,\infty)$ and point (iv) of Lemma 1.6. Therefore, $v(t)$ belongs to $D(A)$ for any $t \in [0,T]$. The uniqueness follows from the uniqueness of the function verifying the conditions of Theorem 1.4.

To prove the last statement of the theorem, first of all observe that in virtue of (1.18), the function $t \to AR(t)v_0$ is $(\alpha+\beta-1)$-Hölder continuous. Moreover, for any $t,\tau \in [0,T]$, $t > \tau$,

$$\begin{aligned} &\int_0^t AR(t-s)(g(s)-g(t))\,ds - \int_0^\tau AR(\tau-s)(g(s)-g(\tau))\,ds \\ &\qquad + A\int_0^t R(s)g(t)\,ds - A\int_0^\tau R(s)g(\tau)\,ds \\ &= \int_\tau^t AR(t-s)(g(s)-g(t))\,ds \\ &\qquad + \int_0^\tau \left[AR(t-s) - AR(\tau-s)\right](g(s)-g(\tau))\,ds \\ &\qquad + \int_0^{t-\tau} AR(s)(g(t)-g(\tau))\,ds + \int_\tau^t AR(s)g(\tau)\,ds. \end{aligned}$$

Taking the norm in the previous identity and using, respectively, (1.15), (1.16), (1.20) and (1.21), we find

$$\begin{aligned}
&\Big\| \int_0^t AR(t-s)(g(s)-g(t))\,ds - \int_0^\tau AR(\tau-s)(g(s)-g(\tau))\,ds \\
&\qquad + A\int_0^t R(s)g(t)\,ds - A\int_0^\tau R(s)g(\tau)\,ds \Big\| \\
&\le M_2 e^{r_0 T}[g]_\alpha \bigg\{ \int_\tau^t (t-s)^{\alpha+\beta-2}\,ds + \int_0^\tau \bigg(\int_{\tau-s}^{t-s} \sigma^{\beta-3}\,d\sigma \bigg)(\tau-s)^\alpha\,ds \bigg\} \\
&\qquad + 3\sup_{s\in[0,T]} ||g(s)||_{D_{k,A}(\alpha,\infty)} M_5 (t-\tau)^{\alpha+\beta-1} \\
&\le M_2 e^{r_0 T}[g]_\alpha \bigg\{ (\alpha+\beta-1)^{-1} + (\beta-2)^{-1} \int_0^{+\infty} y^\alpha ((1+y)^{\beta-2} - y^{\beta-2})\,dy \bigg\} \\
&\qquad (t-\tau)^{\alpha+\beta-1} + 3\sup_{s\in[0,T]} ||g(s)||_{D_{k,A}(\alpha,\infty)} M_5 (t-\tau)^{\alpha+\beta-1}.
\end{aligned}$$

Therefore, $Av(t)$ is $(\alpha+\beta-1)$-Hölder continuous, and the theorem is completely proved. □

2. Existence and regularity of the solutions of a parabolic integrodifferential equation with nonhomogeneous boundary condition. Let Ω be a bounded open set in $\mathbf{R}^n$, $n \in \mathbf{N}$, with boundary $\partial\Omega$ of class C^1.

In this section we shall study several properties (existence, uniqueness and regularity) of the solutions of the parabolic integrodifferential equation with nonhomogeneous boundary condition

(2.1)
$$\begin{cases} u_t(t,x) = (\Delta + c)\int_0^t k(t-s)u(s,x)\,ds + k_0 u(t,x) + f(t,x), \\ \qquad\qquad t \in [0,T],\ x \in \Omega, \\ u(0,x) = u_0(x), \qquad x \in \Omega, \\ u(t,x) = \varphi(t,x), \qquad t\in[0,T],\ x \in \partial\Omega, \end{cases}$$

where $T > 0$, c and k_0 are real constants and f, u_0, φ are functions verifying suitable assumptions. We set

$$\begin{cases} X = C(\overline{\Omega}); ||v|| = \sup_{x\in\overline{\Omega}} |v(x)|, \quad v \in X, \\ D(A) = \{v \in X; \Delta v \in X, v(x) = 0 \text{ for any } x \in \partial\Omega\}, \\ Av = (\Delta + c)v, \quad v \in D(A), \end{cases} \tag{2.2}$$

where Δ is the Laplace operator in the distributional sense. Then A satisfies (1.1) (see [**19, 20**]). In addition, we recall that $\overline{D(A)} = \{v \in X; v(x) = 0 \text{ for any } x \in \partial\Omega\}$. Concerning the kernel k, we assume that (1.2) and (1.3) hold.

To solve problem (2.1), we can revert to an analogous problem with homogeneous boundary condition, using Dirichlet map. We recall that the Dirichlet map is the function $D : C(\partial\Omega) \to C(\overline{\Omega}) \cap C^2(\Omega)$ defined for any $\varphi \in C(\partial\Omega)$, $D\varphi = z$, where z is the solution of the Dirichlet problem

$$\begin{cases} \Delta z(x) = 0, & x \in \Omega, \\ z(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{2.3}$$

Thanks to the regularity of $\partial\Omega$, for any $\varphi \in C(\partial\Omega)$, there exists a unique solution of problem (2.3).

Define $v(t,x) = u(t,x) - D\varphi(t,x)$, $t \in [0,T]$, $x \in \overline{\Omega}$; then v is the solution of problem (1.10), where A is the operator defined by (2.2) and the inhomogeneous term g is given by

$$\begin{aligned} g(t,x) = c\int_0^t k(t-s)D\varphi(s,x)\,ds + k_0 D\varphi(t,x) - D\varphi_t(t,x) \\ + f(t,x), \quad t \in [0,T],\ x \in \overline{\Omega}. \end{aligned} \tag{2.4}$$

To apply Theorem 1.4 and the others of Section 1, the function g must be Hölder continuous. However, that is not the case, in general, because the function $t \to \int_0^t k(t-s)D\varphi(s,\cdot)\,ds$ is not Hölder continuous, even if $D\varphi(t,\cdot)$ is.

To circumvent this problem, we may adopt two different strategies. One way is to suppose that the function $t \to \int_0^t |k(s)|\,ds$ is α-Hölder continuous for some $\alpha \in]0,1[$; this condition is verified, for instance, when $k(t) = t^{-\beta}$, $\beta \in]0,1[$. Making such an assumption on kernel k, we cannot prove an existence theorem for the strict solution of (2.1).

To obtain the existence and maximal regularity of the strict solution, we must proceed another way and consider the Dirichlet map relative to the operator $\Delta + c$. The integral term in expression (2.4) for g then disappears, and we get g Hölder continuous by making suitable assumptions on φ and f.

We need only be careful when c is an eigenvalue of $-\Delta$; in that case the Dirichlet problem

$$\begin{cases} \Delta z(x) + cz(x) = 0, & x \in \Omega, \\ z(x) = \varphi(x), & x \in \partial\Omega, \end{cases} \tag{2.5}$$

has a (nonunique) solution if and only if φ satisfies the conditions

$$\int_{\partial\Omega} \varphi(\sigma) \frac{\partial e}{\partial \nu}(\sigma)\, d\sigma = 0, \tag{2.6}$$

where e is any eigenfunction relative to eigenvalue c and $\partial/\partial\nu$ denotes the normal derivative.

We recall a result (see [**16**, Lemma 2.1]) on the regularity of the solution of (2.3) that will be useful in the sequel.

Lemma 2.1. *If φ belongs to $C^{1,\alpha}([0,T];C(\partial\Omega))$, $0 < \alpha < 1$ and $T > 0$ (respectively, $h^\alpha([0,T];C(\partial\Omega))$), then $D\varphi$ belongs to $C^{1,\alpha}([0,T];C(\overline{\Omega}))$ (respectively, $h^\alpha([0,T];C(\overline{\Omega}))$), where D is the Dirichlet map.*

We recall the characterization of the interpolation spaces $D_A(\alpha,\infty)$ when A is the operator defined by (2.2). For any $\alpha \in]0,1[$, $\alpha \neq 1/2$, we have [**12**]

$$D_A(\alpha,\infty) = C_0^{2\alpha}(\overline{\Omega}) \doteq \{v \in C^{2\alpha}(\overline{\Omega}); v(x) = 0,\ x \in \partial\Omega\}. \tag{2.7}$$

Let's first suppose that the function $t \to \int_0^t |k(s)|\, ds$ is α-Hölder continuous for some $\alpha \in]0,1[$. We may then state and prove the following theorems.

Theorem 2.2. *Let $f \in C^\alpha([0,T];C(\overline{\Omega}))$, $\varphi \in C^{1,\alpha}([0,T];C(\partial\Omega))$ and $u_0 \in C(\overline{\Omega})$ be such that $\Delta u_0 \in C(\overline{\Omega})$ and the compatibility condition*

$$u_0(x) = \varphi(0,x), \quad \textit{for every } x \in \partial\Omega, \tag{2.8}$$

holds. Then there exists a unique function $u(t,x)$ belonging to $C^1([0,T];C(\overline{\Omega}))$ such that $\Delta \int_0^t k(t-s)u(s,\cdot)\, ds \in C([0,T]\times\overline{\Omega})$ and (2.1) is satisfied.

Moreover, if we assume $\beta(\alpha+2)<2$, $\partial\Omega$ of class $C^{2,\alpha}$ and

$$f(0,\cdot)\in C^{\frac{2\alpha}{2-\beta}}(\overline{\Omega}), \tag{2.9}$$

$$\varphi_t(0,\cdot)\in C^{\frac{2\alpha}{2-\beta}}(\partial\Omega), \tag{2.10}$$

$$f(0,x)=\varphi_t(0,x)-k_0\varphi(0,x), \quad \text{for any } x\in\partial\Omega, \tag{2.11}$$

$$\varphi(0,\cdot)\in C^{\frac{2(\alpha+1)}{2-\beta}}(\partial\Omega), \tag{2.12}$$

$$u_0\in C^{\frac{2(\alpha+1)}{2-\beta}}(\overline{\Omega}), \tag{2.13}$$

then the function $u(t,x)$ belongs to $C^{1,\alpha}([0,T];C(\overline{\Omega}))$.

Proof. First of all, we observe that by Lemma 2.1 we have $D\varphi\in C^{1,\alpha}([0,T];C(\overline{\Omega}))$, where D is the Dirichlet map. We set

$$\begin{aligned} g(t,x)=c\int_0^t k(t-s)D\varphi(s,x)\,ds+k_0D\varphi(t,x)-D\varphi_t(t,x)\\ +f(t,x), \quad t\in[0,T],\ x\in\overline{\Omega}, \end{aligned} \tag{2.14}$$

$$v_0(x)=u_0(x)-D\varphi(0,x), \quad x\in\overline{\Omega}. \tag{2.15}$$

With the convention $g(t)=g(t,\cdot)$, we consider the abstract problem

$$\begin{cases} v'(t)=A\int_0^t k(t-s)v(s)\,ds+k_0v(t)+g(t), & t\in[0,T],\\ v(0)=v_0, \end{cases} \tag{2.16}$$

where A is the operator defined by (2.2). Since the functions $t\to\int_0^t|k(s)|\,ds$ and $D\varphi(t)$ are α-Hölder continuous, the same holds for the function $t\to\int_0^t k(t-s)D\varphi(s)\,ds$, and hence g belongs to $C^\alpha([0,T];X)$, where $X=C(\overline{\Omega})$. By (2.15) we get $\Delta v_0=\Delta u_0$, because $D\varphi(0,\cdot)$ is a harmonic function. In addition, by (2.8) we have $v_0(x)=0$ for any

$x \in \partial\Omega$, so that $v_0 \in D(A)$. Therefore, we may apply Theorem 1.4: the function

$$v(t,\cdot) = R(t)v_0 + \int_0^t R(t-s)g(s,\cdot)\,ds, \quad 0 \le t \le T, \tag{2.17}$$

belongs to $C^1([0,T];X)$, $\int_0^t k(t-s)v(s)\,ds \in C([0,T];D(A))$ and (2.16) holds. It is easy to check that

$$u(t,x) = v(t,x) + D\varphi(t,x), \quad t \in [0,T],\ x \in \overline{\Omega}, \tag{2.18}$$

is the required function. The uniqueness follows from (2.18) and from the uniqueness of the solution of (2.16).

To prove the last part of the theorem, we begin to show that (1.14) holds. First of all, we observe that, due to (2.10), (2.12) and Schauder's theorem (see [**8**]),

$$k_0 D\varphi(0,\cdot) - D\varphi_t(0,\cdot) \in C^{\frac{2\alpha}{2-\beta}}(\overline{\Omega}), \qquad D\varphi(0,\cdot) \in C^{\frac{2(\alpha+1)}{2-\beta}}(\overline{\Omega}),$$

since $2\alpha/(2-\beta) < 2+\alpha$ and $2(\alpha+1)/(2-\beta) < 2+\alpha$. Therefore, by (2.9), (2.11), (2.13) and (2.8), we get $g(0) \in C_0^{2\alpha/(2-\beta)}(\overline{\Omega})$ and $v_0 \in C_0^{2(\alpha+1)/(2-\beta)}(\overline{\Omega})$, from which, taking into account (2.7) and (1.12), (1.14) follows. By Theorem 1.4, the function $v(t,x)$ defined by (2.17) belongs to $C^{1,\alpha}([0,T];C(\overline{\Omega}))$, and then by (2.18) we get that u belongs to $C^{1,\alpha}([0,T];C(\overline{\Omega}))$. $\square$

We now give a representation formula for the solution of (2.1).

Corollary 2.3. *Under the assumptions of the first part of Theorem* 2.2*, the solution of* (2.1) *is given by the formula*

$$u(t,\cdot) = R(t)u_0 + \int_0^t R(t-s)h(s,\cdot)\,ds - \int_0^t R'(t-s)D\varphi(s,\cdot)\,ds, \quad t \in [0,T], \tag{2.19}$$

where $h(t,x)$ is the function defined by

$$h(t,x) = c\int_0^t k(t-s)D\varphi(s,x)\,ds + k_0 D\varphi(t,x) + f(t,x), \quad t \in [0,T],\ x \in \overline{\Omega}. \tag{2.20}$$

Proof. The proof is completely analogous to that of Corollary 2.4 of [**16**], and so we omit it. □

We now define the strong and the strict solution of problem (2.1).

Definition 2.4. *A function $u(t,x) \in C([0,T] \times \overline{\Omega})$ is said to be a strong solution for problem* (2.1) *if*

$$u(0,x) = u_0(x), \quad x \in \overline{\Omega}, \tag{2.21}$$

$$u(t,x) = \varphi(t,x), \quad t \in [0,T],\ x \in \overline{\Omega}, \tag{2.22}$$

and there exists a sequence of functions $\{u_n(t,x)\}_{n\in\mathbf{N}}$ such that for any $n \in \mathbf{N}$, $u_n \in C^1([0,T];C(\overline{\Omega}))$, $\Delta \int_0^t k(t-s)u_n(s,x)\,ds \in C([0,T] \times \overline{\Omega})$ and fulfills

(i) $\lim_{n\to+\infty} u_n = u$ *in* $C([0,T] \times \overline{\Omega})$;

(ii) $\lim_{n\to+\infty}(\frac{\partial u_n}{\partial t}(t,x) - (\Delta+c)\int_0^t k(t-s)u_n(s,x)\,ds - k_0u_n(t,x)) = f(t,x)$ *in* $C([0,T] \times \overline{\Omega})$.

On the other hand, a function $u : [0,T] \times \overline{\Omega} \to \mathbf{R}$ is said to be a strict solution of (2.1) *if $u \in C^1([0,T];C(\overline{\Omega}))$, $\Delta u \in C([0,T] \times \overline{\Omega})$ and satisfies* (2.1).

Therefore, if u is a strict solution of (2.1), then the Laplace operator and the integral commute, and hence u fulfills the equation

$$u_t(t,x) = \int_0^t k(t-s)(\Delta + c)u(s,x)\,ds + k_0u(t,x) + f(t,x) \qquad t \in [0,T],\ x \in \Omega. \tag{2.23}$$

We shall prove a result about the existence and regularity of the strong solution of (2.1).

Theorem 2.5. *Let $f \in C([0,T] \times \overline{\Omega})$, $\varphi \in h^\alpha([0,T];C(\partial\Omega))$ and $u_0 \in C(\overline{\Omega})$ verify the compatibility condition* (2.8). *Then the function $u(t,x)$ given by* (2.19) *is the unique strong solution of* (2.1). *In addition,* *if $\partial\Omega$ is of class $C^{2,\alpha}$, $u_0 \in C^{2\alpha/(2-\beta)}(\overline{\Omega})$ and $\varphi(0,\cdot) \in C^{2\alpha/(2-\beta)}(\partial\Omega)$, then u belongs to $C^\alpha([0,T];C(\overline{\Omega}))$.*

Proof. First of all, we observe that u given by (2.19) is well defined thanks to (1.8). Set $v_0(x) = u_0(x) - D\varphi(0,x)$, $x \in \overline{\Omega}$. By (2.8) we have $v_0 \in \overline{D(A)}$, where A is the operator defined by (2.2). Therefore, there exists a sequence $\{v_{0_n}\}$ in $D(A)$ such that $\{v_{0_n}\}$ converges to v_0 in $C(\overline{\Omega})$ as $n \to +\infty$.

Let $\{f_n\}$ be a sequence in $C^1([0,T];C(\overline{\Omega}))$ such that f_n converges uniformly to f. Moreover, let $\{\varphi_n\}$ be a sequence in $C^2([0,T];C(\partial\Omega))$ such that φ_n converges to φ in $C^\alpha([0,T];C(\partial\Omega))$. Set, for any $n \in \mathbf{N}$,

$$u_{0n}(x) = v_{0n}(x) + D\varphi_n(0,x), \quad x \in \overline{\Omega}; \tag{2.24}$$

it is easy to prove that u_{0n} converges to u_0 in $C(\overline{\Omega})$. In addition, since $v_{0n} \in D(A)$, any u_{0n} fulfills the compatibility condition (2.8), when φ is replaced by φ_n.

For any $n \in \mathbf{N}$, we consider the problem

(2.25)

$$\begin{cases} \frac{\partial u_n}{\partial t}(t,x) = (\Delta + c)\int_0^t k(t-s)u_n(s,x)\,ds + k_0 u_n(t,x) + f_n(t,x), \\ \qquad\qquad\qquad\qquad t \in [0,T],\ x \in \Omega, \\ u_n(0,x) = u_{0n}(x), \quad x \in \Omega, \\ u_n(t,x) = \varphi_n(t,x), \quad t \in [0,T],\ x \in \partial\Omega. \end{cases}$$

In view of Theorem 2.2 and Corollary 2.3, there exists a unique solution $u_n(t,x)$ of (2.25) given by

$$u_n(t,\cdot) = R(t)u_{0n} + \int_0^t R(t-s)h_n(s,\cdot)\,ds - \int_0^t R'(t-s)D\varphi_n(s,\cdot)\,ds$$

where

$$h_n(t,x) = c\int_0^t k(t-s)D\varphi_n(s,x)\,ds + k_0 D\varphi_n(t,x) + f_n(t,x).$$

Therefore, thanks to the properties of the resolvent operator $R(t)$ (see Section 1), it is easy to check that u_n and u verify the conditions of definition of the strong solution (see Definition 2.4).

If $\bar{u}$ is another strong solution of (2.1), it is clear that the function $u - \bar{u}$ is the strong solution of the problem with $u_0 = 0$, $\varphi = 0$ and $f = 0$, and hence, by Corollary 1.5, we have $u = \bar{u}$.

Finally, we must prove the last statement of the theorem. First of all, we may write the function u in the form

$$
\begin{aligned}
u(t,\cdot) &= R(t)(u_0 - D\varphi(0,\cdot)) + \int_0^t R(t-s)h(s,\cdot)\,ds \\
&\quad - R(t)(D\varphi(t,\cdot) - D\varphi(0,\cdot)) + D\varphi(t,\cdot) \\
&\quad - \int_0^t R'(t-s)(D\varphi(s,\cdot) - D\varphi(t,\cdot))\,ds, \quad t \in [0,T].
\end{aligned}
\tag{2.26}
$$

Since $\varphi(0,\cdot) \in C^{2\alpha/(2-\beta)}(\partial\Omega)$, by Schauder's theorem we have $D\varphi(0,\cdot) \in C^{2\alpha/(2-\beta)}(\overline{\Omega})$. Therefore, by (1.12), (2.7) and (2.8), $u_0 - D\varphi(0,\cdot)$ belongs to $D_{k,A}(\alpha,\infty)$. We may apply Corollary 1.5: the function $R(t)(u_0 - D\varphi(0,\cdot)) + \int_0^t R(t-s)h(s,\cdot)\,ds$ belongs to $C^\alpha([0,T];C(\overline{\Omega}))$. Using standard arguments, we might prove that the remaining terms in (2.26) are α-Hölder continuous too (see [**16**, Theorem 2.7]). This completes the proof of the theorem. □

In the case of interest $k(t) = t^{-\beta}$, $\beta \in]0,1[$, the function $t \to \int_0^t |k(s)|\,ds$ is α-Hölder continuous, with $\alpha = 1-\beta$. However, Theorem 1.7 does not apply in this case, as $\alpha + \beta = 1$, so that we cannot prove the existence of the strict solution.

From now on, we consider the Dirichlet map D relative to the operator $\Delta + c$. If c is an eigenvalue of $-\Delta$, a solution (in fact, an infinity of solutions) of the Dirichlet problem (2.5) exists if and only if $\varphi \in C(\partial\Omega)$ satisfies condition (2.6). In this case, we will denote by $D\varphi$ one such solution and assume that (2.6) is fulfilled.

Theorem 2.6. *Let $f \in C^\alpha([0,T];C(\overline{\Omega}))$, $\varphi \in C^{1,\alpha}([0,T];C(\partial\Omega))$, $0 < \alpha < 1$, and $u_0 \in C(\overline{\Omega})$ be such that $\Delta u_0 \in C(\overline{\Omega})$ and the compatibility condition* (2.8) *holds. Then, there exists a unique function $u(t,x)$ belonging to $C^1([0,T];C(\overline{\Omega}))$ such that $\Delta \int_0^t k(t-s)u(s,x)\,ds \in C([0,T] \times \overline{\Omega})$ and* (2.1) *is satisfied.*

In addition, if we assume $\beta(\alpha+2) < 2$, $\partial\Omega$ of class $C^{2,\alpha}$ and (2.9)–(2.13) *hold, then the function $u(t,x)$ belongs to $C^{1,\alpha}([0,T];C(\overline{\Omega}))$.*

Proof. Taking into account that a result similar to Lemma 2.1 holds for the Dirichlet map D relative to the operator $\Delta + c$, we have $D\varphi \in C^{1,\alpha}([0,T];C(\overline{\Omega}))$, and hence the function

$$g(t,x) = k_0 D\varphi(t,x) - D\varphi_t(t,x) + f(t,x), \quad t \in [0,T],\ x \in \overline{\Omega}, \tag{2.27}$$

belongs to $C^{\alpha}([0,T];C(\overline{\Omega}))$. The remainder of the proof is completely analogous to that of Theorem 2.2. $\square$

The following result about the existence of the strong solution of (2.1) can be proved by repeating the same arguments used in the proof of Theorem 2.5.

Theorem 2.7. *Let $f \in C([0,T] \times \overline{\Omega})$, $\varphi \in h^{\alpha}([0,T];C(\partial\Omega))$, $0 < \alpha < 1$, and $u_0 \in C(\overline{\Omega})$ such that the compatibility condition* (2.8) *holds. Then the function $u(t,x)$ defined by*

$$\begin{aligned} u(t,\cdot) = R(t)u_0 &+ \int_0^t R(t-s)(k_0 D\varphi(s,\cdot) + f(s,\cdot))\,ds \\ &- \int_0^t R'(t-s) D\varphi(s,\cdot)\,ds, \quad t \in [0,T] \end{aligned} \tag{2.28}$$

is the unique strong solution of (2.1).

Finally, we shall prove that if the terms u_0, f and φ are more regular, then the solution of (2.1), whose existence has been stated by Theorem 2.6, is the strict solution of (2.1).

Theorem 2.8. *Assume that $\partial\Omega$ is of class $C^{2,\alpha}$ for some $\alpha \in]0,1[$, $\alpha + \beta > 1$. Let $f \in C^{\alpha}([0,T];C(\overline{\Omega})) \cap C([0,T];C^{2\alpha/(2-\beta)}(\overline{\Omega}))$, $\varphi \in C^{1,\alpha}([0,T];C(\partial\Omega)) \cap C^1([0,T];C^{2\alpha/(2-\beta)}(\partial\Omega))$ and $u_0 \in C(\overline{\Omega})$ be such that $\Delta u_0 \in C(\overline{\Omega})$. In addition, f, φ and u_0 verify condition* (2.8) *and*

$$f(t,x) = \varphi_t(t,x) - k_0\varphi(t,x), \quad t \in [0,T],\ x \in \partial\Omega. \tag{2.29}$$

Then the function u given by (2.28) *is the unique strict solution of* (2.1). *Moreover, if u_0 belongs to $C^{2(\alpha+1)/(2-\beta)}(\overline{\Omega})$, then Δu belongs to $C^{\alpha+\beta-1}([0,T];C(\overline{\Omega}))$.*

Proof. First of all, we observe that, by Schauder's theorem, we have $D\varphi \in C^1([0,T]; C^{2\alpha/(2-\beta)}(\overline{\Omega}))$. Let $v(t,x)$ be the solution of problem (2.16) with the inhomogeneous term g given by (2.27). It is enough to prove that v is the strict solution of (2.16), and then (2.18) gives that u is the strict solution of (2.1). To this end, we must apply Theorem 1.7. Taking into account our assumptions on f and φ, (2.29), (1.12) and (2.7), we find that the function g belongs to $C^\alpha([0,T];X) \cap C([0,T]; D_{k,A}(\alpha,\infty))$. Therefore, by Theorem 1.7, v is the strict solution of (2.16).

In addition, if $u_0 \in C^{2(\alpha+1)/(2-\beta)}(\overline{\Omega})$, then, by (2.8), (2.7) and (1.12), $v_0 \in D_{k,A}(\alpha+1,\infty)$, and hence, again by Theorem 1.7, Δv belongs to $C^{\alpha+\beta-1}([0,T]; C(\overline{\Omega}))$. Since $\Delta u = \Delta v - cD\varphi$, it follows that Δu belongs to $C^{\alpha+\beta-1}([0,T]; C(\overline{\Omega}))$, too.

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