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ON THE THEORY OF PARTIAL INTEGRAL OPERATORS

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The fundamental work of F. Riesz and J. Schauder has shown that the basic facts of the classical theory of integral equations (convergence of iteration methods, Fredholm alternative, bilinear expansions of kernels, etc.) are due to certain functional-analytic and geometric properties of corresponding integral transforms, such as boundedness (continuity), compactness, or weak compactness. With regard to this fact, many authors tried to find conditions for the continuity or compactness of linear integral operators in various function spaces. Now the theory of such operators is rather advanced and complete; the basic results may be found, for example, in the monographs [**9, 20, 22, 25, 38**].

Unfortunately, the operators studied in these monographs do not cover many integral operators arising in mathematical physics. For instance, some problems for elliptic or hyperbolic equations lead to integral equations with the property that the integration is carried out only over some of the variables [**8, 30, 32**]; such equations will be called *partial integral equations* in what follows. For a long time, such equations have not been studied for essentially two reasons. First of all, partial integral equations occur less often than classical integral equations (involving integration with respect to all variables); second, the corresponding operators are not compact, and thus the classical Riesz-Schauder theory does not apply. In recent years, however, it became clear that partial integral equations should be investigated in more detail. In fact, they arise in many fields of current interest, especially in continuum mechanics. Here one could mention, for instance, axial-symmetric contact problems [**1–3, 23, 24, 28, 29**], the theory of thin elastic shells [**32**], and certain problems in aerodynamics [**4**].

It is clear that, for studying partial integral equations, one has to analyze the operators generated by such equations. We mention here the papers [**5–7, 12–19, 27**], where spectral properties of such

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operators have been studied by means of tensor products of function spaces.

The present paper is concerned with general properties of operators generated by partial integral equations: boundedness conditions, norm estimates in various function spaces, adjoint operators, properties of resolvents, and others. In particular, we obtain analogues of Banach's continuity theorem, of the Gribanov-Zabrejko theorem on weak continuity, and of some results of Kantorovich type. Finally, we discuss several applications to integral equations and give a specific example arising in continuum mechanics.

For simplicity, we shall restrict ourselves throughout to spaces of functions of two variables. The operators we shall study in such spaces are of the form

$$K = C + L + M + N, \tag{1}$$

where

$$Cx(t,s) = c(t,s)x(t,s), \tag{2}$$

$$Lx(t,s) = \int_S l(t,s,\sigma)x(t,\sigma)\,d\nu(\sigma), \tag{3}$$

$$Mx(t,s) = \int_T m(t,s,\tau)x(\tau,S)\,d\mu(\tau) \tag{4}$$

and

$$Nx(t,s) = \int_T \int_S n(t,s,\tau,\sigma)x(\tau,\sigma)\,d\mu \times \nu(t,\sigma). \tag{5}$$

Here T and S are arbitrary nonempty sets equipped with σ-algebras $\mathfrak{S}(T)$ and $\mathfrak{S}(S)$, and separable measures μ and ν on $\mathfrak{S}(T)$ and $\mathfrak{S}(S)$, respectively; by $\mu \times \nu$ we mean the product measure on the σ-algebra $\mathfrak{S}(T) \times \mathfrak{S}(S)$. The coefficient $c = c(t,s)$, as well as the kernels $l = l(t,s,\sigma)$, $m = m(t,s,\tau)$, and $n = n(t,s,\tau,\sigma)$ are measurable functions, and the integrals (3)–(5) are meant in the Lebesgue-Radon sense. The whole operator (1) is a *partial integral operator* (PIO) in its most general form.

1. Continuity. In the sequel we use the following notation. By $\mathcal{S} = \mathcal{S}(T \times S)$ we denote the space of all (real or complex) functions on $T \times S$ which are measurable and almost everywhere finite, and by X and Y ideal Banach spaces of functions in $\mathcal{S}$. (Recall that a Banach subspace $Z \subset \mathcal{S}$ is called *ideal* if the relations $x(t,s) \in \mathcal{S}$, $y(t,s) \in Z$, and $|x(t,s)| \leq |y(t,s)|$ a.e. on $T \times S$ imply that also $x(t,s) \in Z$ and $||x||_Z \leq ||y||_Z$.)

Theorem 1. *Suppose that K is a PIO which acts from X into Y. Then K is continuous.*

Proof. Together with (1), consider the operator

$$]K[\, = \,]C[\, + \,]L[\, + \,]M[\, + \,]N[\,, \tag{6}$$

where

$$]C[\, x(t,s) = |c(t,s)| x(t,s), \tag{7}$$

$$]L[\, x(t,s) = \int_T |l(t,s,\sigma)| x(t,\sigma)\, d\nu(\sigma), \tag{8}$$

$$]M[\, x(t,s) = \int_S |m(t,s,\tau)| x(\tau,s)\, d\mu(\tau), \tag{9}$$

and

$$]N[\, x(t,s) = \int_{T\times S}\!\!\int |n(t,s,\tau,\sigma)| x(t,\sigma)\, d\mu \times \nu(\tau,\sigma). \tag{10}$$

Let $x \in X$. By hypothesis, the function $y(t,s) = Kx(t,s)$ belongs to Y. This implies, in particular, that the functions $l(t,s,\sigma)x(t,\sigma)$, $m(t,s,\tau)x(t,s)$, and $n(t,s,\tau,\sigma)x(t,\sigma)$ are integrable, for a.a. $(t,s) \in T \times S$, on S, T, and $T \times S$, respectively. By well-known properties of the Lebesgue-Radon integral, the same is true for the functions $|l(t,s,\sigma)||x(t,\sigma)|$, $|m(t,s,\tau)||x(\tau,s)|$, and $|n(t,s,\tau,\sigma)||x(\tau,\sigma)|$; thus, we may define $]K[\,x$ and get $]K[\,x \in \mathcal{S}$, by Fubini's theorem. In other words, the operator (6) acts from X into $\mathcal{S}$.

We claim that the operator K is closed. Suppose that $x_n \in X$ converges (in X) to $x^* \in X$, and Kx_n converges (in Y) to $y^* \in Y$; we have to show that $Kx^* = y^*$. Choose a sequence n_k of natural numbers such that

$$\sum_{k=0}^{\infty} ||x_{n_k} - x^*|| < \infty.$$

Then, evidently, $x_{n_k}(t,s) \to x^*(t,s)$ a.e. on $T \times S$, and the function

$$z(t,s) = \sum_{k=0}^{\infty} |x_{n_k}(t,s) - x^*(t,s)|$$

belongs to X. Consequently, the sequences $l(t,s,\sigma)x_{n_k}(t,\sigma)$, $m(t,s,\tau)x_{n_k}(\tau,s)$, and $n(t,s,\tau,\sigma)x_{n_k}(\tau,\sigma)$ converge, for a.a. $(t,s) \in T \times S$, to the functions $l(t,s,\sigma)x^*(t,\sigma)$, $m(t,s,\tau)x^*(\tau,s)$, and $n(t,s,\tau,\sigma)\ x^*(\tau,\sigma)$, respectively, and are majorized by the integrable functions $|l(t,s,\sigma)|z(t,\sigma)$, $|m(t,s,\tau)|z(\tau,s)$, and $|n(t,s,\tau,\sigma)|z(\tau,\sigma)$, respectively. By Lebesgue's dominated convergence theorem, we have $Lx_{n_k}(t,s) \to Lx^*(t,s)$, $Mx_{n_k}(t,s) \to Mx^*(t,s)$, and $Nx_{n_k}(t,s) \to Nx^*(t,s)$, for a.a. $(t,s) \in T \times S$, and hence also $Kx_{n_k}(t,s) \to Kx^*(t,s)$ a.e. on $T \times S$. But, by hypothesis, the sequence Kx_{n_k} converges in Y to y^*, and thus $Kx^* = y^*$ as claimed.

We have shown that the operator K is closed between X and Y. The assertion follows now from Banach's closed graph theorem. $\square$

Theorem 1 is an extension of Banach's well-known theorem on the continuity of integral operators (see, e.g., [**20, 25**]) to PIO's. This theorem carries over as well to the case of ideal quasi-Banach spaces (for instance, the space $\mathcal{S}$ itself). It is clear that the operator K maps X into Y if all the operators C, L, M, and N given by (2), (3), (4), and (5), respectively, do so. Interestingly, the converse is not true, at least in case one of the sets T or S contains a countable number of atoms. This problem is related to that of the uniqueness of the representation of the operator K in the form (1). We point out that this representation is unique if the measures μ and ν are continuous (atom-free) on $\mathfrak{S}(T)$ and $\mathfrak{S}(S)$, respectively; this follows from regularity theorems which we shall give in the next section. Such regularity theorems allow us also to conclude the action of all the operators (2)–(5) from the action of the single operator (1) between X and Y.

2. Regularity. As before, the notation $x \le y$ for $x, y \in \mathcal{S}(T \times S)$ means that $x(t,s) \le y(t,s)$ a.e. on $T \times S$.

Recall that a linear operator $K : X \to Y$ is called *regular* if there exists a positive operator $\tilde{K} : X \to Y$ (i.e., $x \ge 0$ implies that $Kx \ge 0$)

such that

$$|Kx| \leq \tilde{K}|x|, \quad x \in X.$$

The classical Kantorovich theorem [**33**] states that a linear operator is regular if and only if it preserves order-boundedness. Moreover, among all positive majorants $\tilde{K}$ of K there exists a minimal one (in the sense of the induced ordering on the space of linear operators); this minimal positive majorant is usually called the *absolute value* of K and denoted by $|K|$.

Theorem 2. *Suppose that the measures* μ *and* ν *are continuous, and let* $K : X \to Y$ *be a PIO. Then* K *is regular if and only if the operator* $]K[$ *given by* (6) *also maps* X *into* Y*. In this case,*

$$|K| = \,]K[\,. \tag{11}$$

Proof. The sufficiency of (11) follows from the obvious inequality

$$|Kx| \leq \,]K[\,|x|, \quad x \in X, \tag{12}$$

which implies that $]K[$ is a positive majorant of K, and, hence, $|K| \leq \,]K[$. To prove the necessity of (11), let $K : X \to Y$ be regular. For any nonnegative function $x \in X$, we then get (see again [**33**])

$$|K|x = \sup\{|Kz| : |z| \leq x\} \in Y.$$

Since X and Y are K-spaces of countable type [**20**], we find a countable set M which is dense (in measure) in the set $\{z : |z| \leq x\}$ and such that

$$|K|x = \sup\{|Kz| : z \in M\}.$$

By the continuity of the measures μ and ν, we can choose sequences of sets $T_n \subseteq T$ and $S_n \subseteq S$ such that

$$T = \bigcup_{n=1}^{\infty} T_n, \qquad S = \bigcup_{n=1}^{\infty} S_n, \qquad \mu(T_n) \to 0, \qquad \nu(S_n) \to 0$$

and every point $t \in T$ and $s \in S$ belongs to infinitely many subsets T_n and S_n, respectively. Let

$$U_n = T \times S_n, \qquad V_n = T_n \times S, \qquad \Delta_n = U_n \cup V_n, \qquad \nabla_n = U_n \cap V_n,$$

and

$$\begin{aligned} M^* = \{&x(t,s)\operatorname{sign} c(t,s)\chi_{\nabla_n}(t,s) + u(t,s)\chi_{\nabla'_n}(t,s) \\ &+ v(t,s)\chi_{u'_n}(t,s)\chi_{v_n}(t,s) + w(t,s)\chi_{v'_n}(t,s)\chi_{u_n}(t,s)\}, \end{aligned}$$

where $u, v, w \in M$, n runs over all natural numbers, χ_D is the characteristic function of $D \subseteq T \times S$, and D' denotes the complement $(T \times S) \setminus D$. Since $M \subseteq M^* \subseteq \{z : |z| \leq x\}$, we have

$$|K|x = \sup\{|Kz| : z \in M^*\},$$

since M^* is countable, we also have

$$|K|x(t,s) = \sup\{|Kz(t,s)| : z \in M^*\} \tag{13}$$

for a.a. $(t,s) \in T \times S$. Fix $(t,s) \in T \times S$ with (13), and choose sequences n_k and m_k such that $t \in T_{n_k}$ and $s \in S_{m_k}$ for all k. Moreover, let $u_k(t,s)$, $v_k(t,s)$, and $w_k(t,s)$ be functions in M such that $u_k(\tau,\sigma) \to \operatorname{sign} l(t,s,\tau)x(\tau,\sigma)$, $v_k(\tau,\sigma) \to \operatorname{sign} m(t,s,\sigma)x(\tau,\sigma)$, and $w_k(\tau,\sigma) \to \operatorname{sign} n(t,s,\tau,\sigma)x(\tau,\sigma)$ (convergence in measure). Let

$$\begin{aligned} z_k(\tau,\sigma) = {}& x(\tau,\sigma)\operatorname{sign} c(\tau,\sigma)\chi_{\tilde{\nabla}_k}(\tau,\sigma) \\ &+ u_k(\tau,\sigma)\chi_{\tilde{u}'_k}(\tau,\sigma)\chi_{\tilde{v}_k}(\tau,\sigma) \\ &+ v_k(\tau,\sigma)\chi_{\tilde{v}'_k}(\tau,\sigma)\chi_{\tilde{u}_k}(\tau,\sigma) + w_k(\tau,\sigma)\chi_{\tilde{\Delta}'_k}(\tau,\sigma), \end{aligned}$$

where

$$\tilde{U}_k = T \times S_{m_k}, \qquad \tilde{V}_k = T_{n_k} \times S, \qquad \tilde{\Delta}_k = \tilde{U}_k \cup \tilde{V}_k, \qquad \tilde{\nabla}_k = \tilde{U}_k \cap \tilde{V}_k.$$

By Lebesgue's dominated convergence theorem, we conclude that

$$\begin{aligned} &\lim_{k\to\infty} |Kz_k(t,s)| \\ &\quad = |c(t,s)|x(t,s) + \int_S |l(t,s,\sigma)|x(t,\sigma)\, d\nu(\sigma) \\ &\qquad + \int_T |m(t,s,\tau)|x(\tau,s)\, d\mu(\tau) \\ &\qquad + \int_{T\times S}\!\!\int |n(t,s,\tau,\sigma)|x(\tau,\sigma)\, d\mu(\tau)\, d\nu(\sigma) =]K[\, x(t,s). \end{aligned}$$

On the other hand, from $z_k \in M^*$, $k = 1, 2, 3, \dots$, it follows that

$$]K[\, x(t,s) = \lim_{k\to\infty} |Kz_k(t,s)| \le \sup\{|Kz| : z \in M^*\}.$$

This implies, together with (13), that $]K[\, x(t,s) \le |K|x(t,s)$ and, hence, $]K[\, x \le |K|x$. We have shown that the operator $]K[$ maps all nonnegative functions $x \in X$ into Y. But every function in X may be written as a difference of two nonnegative functions, and, thus, $]K[$ acts from X into Y as claimed.

Equality (11) follows from the just established inequality $]K[\, \le |K|$ and from (12). □

The hypothesis on the continuity of the measures μ and ν is essential not just for the proof but also for the statement of Theorem 2. Nevertheless, one may modify Theorem 2 in such a way that its statement remains true also for discrete (purely atomic) measures and even for arbitrary measures. Denote by T_d and S_d the "discrete part," and by T_c and S_c the "continuous part," respectively, of the sets T and S. We say that the representation (1) of the operator K is normal if $c(t,s) = 0$ for $(t,s) \in T_d \times S_d$, $l(t,s,\sigma) = 0$ for $t \in T_d$, and $m(t,s,\tau) = 0$ for $s \in S_d$. It is not hard to see that one can always find a normal representation for a PIO, just by using the "δ-functions" defined by

$$\delta(t,\tau) = \begin{cases} \mu(\tau)^{-1} & \text{for } t = \tau, \\ 0 & \text{for } t \neq \tau, \end{cases}$$

and

$$\delta(s,\sigma) = \begin{cases} \nu(\sigma)^{-1} & \text{for } s = \sigma, \\ 0 & \text{for } s \neq \sigma. \end{cases}$$

In fact, replacing the functions $c(t,s)$, $l(t,s,\sigma)$, $m(t,s,\tau)$, and $n(t,s,\tau,\sigma)$ in (1) by the functions

$$\begin{aligned} \tilde{c}(t,s) &= c(t,s)\chi_{T_c\times S_c}(t,s), \\ \tilde{l}(t,s,\sigma) &= c(t,s)\delta(s,\sigma) + l(t,s,\sigma)\chi_{T_c\times S_d}(t,s), \\ \tilde{m}(t,s,\tau) &= c(t,s)\delta(t,\tau) + m(t,s,\tau)\chi_{T_d\times S_c}(t,s), \end{aligned}$$

and

$$\begin{aligned} \tilde{n}(t,s,\tau,\sigma) = {}& c(t,s)\delta(t,\tau)\delta(s,\sigma) + l(t,s,\sigma)\delta(t,\tau) \\ & + m(t,s,\tau)\delta(s,\sigma) + n(t,s,\tau,\sigma), \end{aligned}$$

respectively, one gets a normal representation for K. We formulate now an analogue of Theorem 2 in this general case; the modification of the proof is straightforward.

Theorem 3. *Let $K : X \to Y$ be a PIO with normal representation* (1). *Then K is regular if and only if the operator $]K[$ given by* (6) *also maps X into Y. In this case, equality* (11) *holds.*

3. Duality theory. We recall some definitions from the general theory of ideal spaces and operators acting between them [**34, 37**]. Let Ω be a set equipped with a σ-algebra $\mathfrak{S}$ of subsets and a measure μ, and let Z be an ideal space over Ω. The *associate space* Z' consists, by definition, of all measurable functions f on Ω which vanish outside $\operatorname{supp} Z$ (see [**37**]) and satisfy

$$|(f,g)| < \infty, \quad g \in Z,$$

where

$$(f,g) = \int_\Omega f(\omega)g(\omega)\,d\mu(\omega).$$

With the usual algebraic operations and the norm

$$||f||_{Z'} = \sup\{|(f,g)| : ||g||_Z \le 1\}, \tag{14}$$

the associate space is also an ideal space, and $\operatorname{supp} Z' = \operatorname{supp} Z$. The associate space Z' is a closed (possibly strict) subspace of the usual dual space Z^*.

Let A be a linear operator between two ideal spaces X and Y. The *associate operator* A' of A is defined by the relation

$$(Ax, y) = (x, A'y), \quad x \in X,\ y \in Y'. \tag{15}$$

Not every operator A admits an associate A'. Obviously, the operator A' is just the restriction of the usual adjoint operator A^* to Y'; consequently, the operator A' exists if and only if $A^*Y' \subseteq X'$.

Let us return to the PIO (1). We define the *transposed operator* $K^{\#}$ of K by

$$\begin{aligned}(K^{\#}y)(t,s) = c(t,s)y(t,s) &+ \int_S l^*(t,s,\sigma)y(t,\sigma)\,d\nu(\sigma)\\ &+ \int_T m^*(t,\tau,s)y(\tau,s)\,d\mu(\tau)\\ &+ \int_S\int_T n^*(t,\tau,s,\sigma)y(\tau,\sigma)\,d\mu\times\nu(\tau,\sigma),\end{aligned} \tag{16}$$

where

$$\begin{aligned}l^*(t,s,\sigma) &= l(t,\sigma,s),\\ m^*(t,\tau,s) &= m(\tau,t,s),\\ n^*(t,\tau,s,\sigma) &= n(\tau,t,\sigma,s).\end{aligned}$$

In general, the transposed operator of an arbitrary operator of type (1) may be different from the associate operator, as simple examples show. Nevertheless, a classical result for integral operators (see [**37**]) carries over to PIO's:

Theorem 4. *Let K be a PIO which acts between two ideal spaces X and Y. Then both the associate operator K' and the transposed operator $K^{\#}$ exist and are equal, i.e.,*

$$K'y = K^{\#}y \tag{17}$$

for any $y \in Y'$ with $K^{\#}y \in \mathcal{S}$.

Proof. As was shown in the proof of Theorem 1, the operator $]K[$ defined by (6) acts from X into $\mathcal{S}$ and is continuous. Consequently, the image N of the unit ball $||x||_X \le 1$ of X under $]K[$ is a bounded subset of $\mathcal{S}$, and, hence, so is the set

$$\tilde{N} = \overline{\bigcup\{\{v : -]K[\,x \le v \le\,]K[\,x\} : ||x||_X \le 1\}}. \tag{18}$$

(closure in $\mathcal{S}$). Denote by $\tilde{Y}$ the ideal space whose unit ball coincides with the set (18); such a space may be easily constructed. By construction, the operator $]K[$ acts from X into $\tilde{Y}$ and is continuous.

Let $x \in X$ and $y \in \tilde{Y}'$. By Fubini's theorem, we then have

$$\begin{aligned}
&\int_S \int_T c(t,s)x(t,s)y(t,s)\,d\mu \times \nu(t,s)\\
&\quad + \int_S \int_T \Big(\int_S l(t,s,\sigma)x(t,\sigma)\,d\nu(\sigma)\Big) y(t,s)\,d\mu \times \nu(t,s)\\
&\quad + \int_S \int_T \Big(\int_T m(t,\tau,s)x(\tau,s)\,d\mu(\tau)\Big) y(t,s)\,d\mu \times \nu(t,s)\\
&\quad + \int_S \int_T \Big(\int_S \int_T n(t,\tau,s,\sigma)x(\tau,\sigma)\,d\mu \times \nu(\tau,\sigma)\Big) y(t,s)\,d\mu \times \nu(t,s)\\
&= \int_S \int_T c(t,s)x(t,s)y(t,s)\,d\mu \times \nu(t,s)\\
&\quad + \int_S \int_T x(t,\sigma)\Big(\int_S l(t,s,\sigma)y(t,s)\,d\nu(s)\Big)\,d\mu \times \nu(t,\sigma)\\
&\quad + \int_S \int_T x(\tau,s)\Big(\int_T m(t,\tau,s)y(t,s)\,d\mu(t)\Big)\,d\mu \times \nu(\tau,s)\\
&\quad + \int_S \int_T x(\tau,\sigma)\Big(\int_S \int_T n(t,\tau,s,\sigma)y(t,s)\,d\mu \times \nu(t,s)\Big)\,d\mu \times \nu(\tau,\sigma).
\end{aligned}$$

This shows that

$$(Kx,y) = (x, K^{\#}y), \quad x \in X,\ y \in Y', \tag{19}$$

and thus the operator $K^{\#}$ coincides with the associate operator K' of K if we consider K as an operator from X into $\tilde{Y}$.

Fix a nonnegative function $v_0 \in Y'$ such that $\operatorname{supp} v_0 = \operatorname{supp} \tilde{Y}$. For any $y \in Y'$, we write $y_n = \min\{|y|, nv_0\}\operatorname{sign} y$, and get

$$(Kx, y_n) = (x, K^{\#}y_n), \quad n = 0,1,2,\dots. \tag{20}$$

Since the limit, as $n \to \infty$, of the left-hand side of (20) exists for any $x \in X$, the sequence $K^{\#}y_n$ is X-weakly Cauchy in X'. But the space X' is X-weakly complete (see [**34**, **37**]), and, hence, $K^{\#}y_n \to K'y$ for some $y \in Y'$. Since this function y obviously satisfies

$$(Kx,y) = (x, K'y), \quad x \in X,\ y \in Y',$$

we have shown that the associate operator K' of K also exists, if we consider K as an operator from X into Y.

Now let $y \in Y'$ and $K^{\#}y \in \mathcal{S}$. Then $\left]K^{\#}\right[\,|y| \in \mathcal{S}$ and, by Lebesgue's theorem, the sequence $K^{\#}y_n$ converges (in $\mathcal{S}$) to $K^{\#}y$. But, by what has been proved before, the sequence $K^{\#}y_n$ converges as well X-weakly to $K'y$. Since these two types of convergence are compatible (see, e.g., [**34**]), we conclude that $K'y = K^{\#}y$, and so we are done. □

A particular case of Theorem 4, which is much easier to prove, is

Theorem 5. *Let K be a PIO which acts between two ideal spaces X and Y and is regular. Then both the associated operator K' and the transposed operator $K^{\#}$ exist and are equal.*

4. Algebras of PIO's. Given two ideal spaces X and Y, denote by $\mathcal{L}(X,Y)$ the space of all continuous linear operators and by $\mathcal{L}_r(X,Y)$ the space of all regular linear operators between X and Y; similarly, $\mathfrak{N}(X,Y)$ (respectively, $\mathfrak{N}_r(X,Y)$) is the space of all (regular) operators of the form (1). Theorems 1 and 2 state that $\mathfrak{N}(X,Y) \subseteq \mathcal{L}(X,Y)$ and $\mathfrak{N}_r(X,Y) \subseteq \mathcal{L}_r(X,Y)$. If $\mathcal{L}(X,Y)$ and $\mathcal{L}_r(X,Y)$ are equipped with the usual operator norm, the subspaces $\mathfrak{N}(X,Y)$ and $\mathfrak{N}_r(X,Y)$ are not closed. However, if we consider $\mathcal{L}_r(X,Y)$ with the norm

$$||K||_r = ||\,|K|\,||, \tag{21}$$

$|K|$ given as in Section 2, then $\mathcal{L}_r(X,Y)$ becomes a Banach space in the norm (21), and $\mathfrak{N}_r(X,Y)$ is closed in $\mathcal{L}_r(X,Y)$. In order to state more precise results, some auxiliary definitions are in order. Recall that Y/X denotes the space of all *multiplicators* between X and Y, i.e., of all functions $c(t,s)$ such that $c(t,s)x(t,s) \in Y$ for all $x(t,s) \in X$. This is an ideal space with norm

$$||c(t,s)||_{Y/X} = \sup\{||c(t,s)x(t,s)||_Y : ||x(t,s)||_X \leq 1\}. \tag{22}$$

Further, by $\mathcal{R}_1(X,Y)$, $\mathcal{R}_m(X,Y)$, and $\mathcal{R}_n(X,Y)$ we denote the sets of all measurable functions $l(t,s,\sigma)$ on $T \times S \times S$, $m(t,\tau,s)$ on $T \times T \times S$, and $n(t,\tau,s,\sigma)$ on $T \times T \times S \times S$, respectively, such that $l(t,s,\sigma) = 0$ for $t \in T_d$ and $m(t,\tau,s) = 0$ for $s \in S_d$. All these three sets are ideal Banach spaces equipped with the norms

$$||l(t,s,\sigma)||_{\mathcal{R}_l(X,Y)} = \sup_{||x(t,s)||_X \leq 1} \left\| \int_S |l(t,s,\sigma)x(t,\sigma)|\,d\mu(\sigma) \right\|_Y, \tag{23}$$

$$(24)\quad ||m(t,\tau,s)||_{\mathcal{R}_m(X,Y)} = \sup_{||x(t,s)||_X\le 1}\left\|\int_T m(t,\tau,s)x(\tau,s)|\,d\nu(\tau)\right\|_Y,$$

and

$$(25)\quad ||n(t,\tau,s,\sigma)||_{\mathcal{R}_n(X,Y)} = \sup_{||x(t,s)||_X\le 1}\left\|\int_S\int_T |n(t,\tau,s,\sigma)x(\tau,\sigma)|\,d\mu\times\nu(\tau,\sigma)\right\|,$$

respectively. Consider now the direct sum

$$(26)\qquad \mathcal{R}(X,Y) = \mathcal{R}_c(X,Y)\dot{+}\mathcal{R}_1(X,Y)\dot{+}\mathcal{R}_m(X,Y)\dot{+}\mathcal{R}_n(X,Y),$$

where $\mathcal{R}_c(X,Y)$ is the subspace of Y/X consisting of all functions $c(t,s)$ such that $c(t,s)=0$ for $t\in T_d$ or $s\in S_d$. The space (26) will be equipped with the norm

$$(27)\quad ||(c,l,m,n)||_{\mathcal{R}(X,Y)} = ||c(t,s)||_{\mathcal{R}_c(X,Y)} + ||l(t,s,\sigma)||_{\mathcal{R}_l(X,Y)} + ||m(t,\tau,s)||_{\mathcal{R}_m(X,Y)} + ||n(t,\tau,s,\sigma)||_{\mathcal{R}_n(X,Y)}.$$

Theorem 6. *Let X and Y be two ideal spaces. Then $\mathfrak{N}_r(X,Y)$ is a closed subspace of $\mathcal{L}_r(X,Y)$ which is isomorphic to the space $\mathcal{R}(X,Y)$. More precisely, the two-sided estimate*

$$(28)\qquad ||K||_{\mathcal{L}_r(X,Y)} \le ||(c,l,m,n)||_{\mathcal{R}(X,Y)} \le 4||K||_{\mathcal{L}_r(X,Y)}$$

holds, where

$$\begin{aligned}Kx(t,s) = c(t,s)x(t,s) &+ \int_S l(t,s,\sigma)x(t,\sigma)\,d\mu(\sigma)\\ &+ \int_T m(t,\tau,s)x(\tau,s)\,d\nu(t)\\ &+ \int_S\int_T n(t,\tau,s,\sigma)x(\tau,\sigma)\,d\mu\times\nu(\tau,\sigma)\in\mathcal{L}_r(X,Y).\end{aligned}$$

Proof. The statement is an immediate consequence of the completeness of the space $\mathcal{R}(X,Y)$ and of Theorems 2 and 3. $\square$

In view of applications, the following theorem on the superposition of PIO's is useful, which follows by a standard reasoning from Fubini's theorem.

Theorem 7. *Let X, Y and Z be three ideal spaces, and let*

$$\begin{aligned} K_j x(t,s) = c_j(t,s)x(t,s) + \int_S l_j(t,s,\sigma)x(t,\sigma)\,d\mu(\sigma) \\ + \int_T m_j(t,\tau,s)x(\tau,s)\,d\nu(t) \\ + \int_S\int_T n_j(t,\tau,s,\sigma)x(\tau,\sigma)\,d\mu\times\nu(\tau,\sigma), \end{aligned} \tag{29}$$

$j = 1,2$, be two PIO's such that $K_1 \in \mathcal{L}_r(X,Y)$ and $K_2 \in \mathcal{L}(X,Y)$. Then the linear operator $K = K_2K_1$ is also a PIO with coefficient

$$c(t,s) = c_2(t,s)c_1(t,s) \tag{30}$$

and kernels

$$\begin{aligned} l(t,s,\sigma) = c_2(t,s)l_1(t,s,\sigma) + l_2(t,s,\sigma)c_1(t,\sigma) \\ + \int_S l_2(t,s,\xi)l_1(t,\xi,\sigma)\,d\mu(\xi), \end{aligned} \tag{31}$$

$$\begin{aligned} m(t,\tau,s) = c_2(t,s)m_1(t,\tau,s) + m_2(t,\tau,s)c_1(\tau,s) \\ + \int_T m_2(t,\mu,s)m_1(\mu,\tau,s)\,d\nu(\eta), \end{aligned} \tag{32}$$

and

$$\begin{aligned} n(t,\tau,s,\sigma) = c_2(t,s)n_1(t,\tau,s,\sigma) + n_2(t,\tau,s,\sigma)c_1(\tau,\sigma) \\ + l_2(t,s,\sigma)m_1(t,\tau,s) + m_2(t,\tau,s)l_1(t,s,\sigma) \\ + \int_S l_2(t,s,\xi)n_1(t,\tau,\xi,\sigma)\,d\mu(\xi) \\ + \int_T m_2(t,\eta,s)n_1(\eta,\tau,s,\sigma)\,d\nu(\eta) \\ + \int_S n_2(t,\tau,s,\xi)l_1(t,\xi,\sigma)\,d\mu(\xi) \\ + \int_T n_2(t,\eta,s,\sigma)m_1(\eta,s,\sigma)\,d\nu(\eta) \\ + \int_S\int_T n_2(t,\eta,s,\xi)n_1(\eta,\tau,\xi,\sigma)\,d\mu\times\nu(\xi,\eta). \end{aligned} \tag{33}$$

Theorem 7 allows us to give some category-type inclusions between the classes introduced so far, which we state as

Theorem 8. *Let X, Y, and Z be three ideal spaces. Then the following inclusions hold*:

$$\mathcal{R}_c(X,Y)\cdot\mathcal{R}_c(Y,Z)\subseteq\mathcal{R}_c(X,Z), \tag{34}$$

$$\mathcal{R}_c(X,Y)\cdot\mathcal{R}_1(Y,Z),\qquad \mathcal{R}_1(X,Y)\cdot\mathcal{R}_c(Y,Z)\subseteq\mathcal{R}_1(X,Z), \tag{35}$$

$$\mathcal{R}_c(X,Y)\cdot\mathcal{R}_m(Y,Z),\qquad \mathcal{R}_m(X,Y)\cdot\mathcal{R}_c(Y,Z)\subseteq\mathcal{R}_m(X,Z), \tag{36}$$

$$\mathcal{R}_1(X,Y)\cdot\mathcal{R}_m(Y,Z),\qquad \mathcal{R}_m(X,Y)\cdot\mathcal{R}_1(Y,Z)\subseteq\mathcal{R}_n(X,Z), \tag{37}$$

$$\mathcal{R}(X,Y)\cdot\mathcal{R}_n(Y,Z),\qquad \mathcal{R}_n(X,Y)\cdot\mathcal{R}(Y,Z)\subseteq\mathcal{R}_n(X,Z). \tag{38}$$

In particular, $\mathcal{R}_c$, $\mathcal{R}_1$, $\mathcal{R}_m$, and $\mathcal{R}_n$ are subalgebras of the algebra $\mathcal{R}$, and $\mathcal{R}_n$ and $\mathcal{R}_1 \dotplus \mathcal{R}_m \dotplus \mathcal{R}_n$ are ideals in $\mathcal{R}$.

The basic Theorem 6 is not only of theoretical interest. It implies, in fact, that showing that a PIO (1) belongs to $\mathcal{L}_r(X,Y)$ reduces to proving the four relations

$$\begin{aligned} c(t,s)\in\mathcal{R}_c(X,Y),&\qquad l(t,s,\sigma)\in\mathcal{R}_1(X,Y),\\ m(t,\tau,s)\in\mathcal{R}_m(X,Y),&\qquad n(t,\tau,s,\sigma)\in\mathcal{R}_n(X,Y).\end{aligned} \tag{39}$$

The verification of the first relation in (39) reduces to a simple application of results on multiplicator spaces, while that of the last relation may be carried out by means of the theory of Zaanen spaces of kernel functions for linear integral operators (see, e.g., [**34, 37, 38**]). Both procedures have been studied extensively for general spaces, as well as for special (e.g., Lebesgue and Orlicz) spaces. The problem of verifying the second and third relation in (39), however, is harder and has not been given much attention yet.

In general, it seems to be difficult to give a fairly explicit description of the kernel classes $\mathcal{R}_1$, $\mathcal{R}_m$, and $\mathcal{R}_n$. Such a description, apparently, depends heavily on specific properties of the spaces X and Y, and the problems are due to the lack of symmetry in the variables s and t. Some of the most important special cases, in which more information may be obtained, will be considered in subsequent sections.

5. PIO's in spaces with mixed norm. Let U and V be perfect ideal spaces over S and T, respectively [**34**]. By $[U\to V]$ and $[U\leftarrow V]$

we denote the corresponding spaces with mixed norm, i.e., the set of all functions for which the norms

$$||x(t,s)||_{[U\to V]} = || \, ||x(t,s)||_U||_V \tag{40}$$

and

$$||x(t,s)||_{[U\leftarrow V]} = || \, ||x(t,s)||_V||_U, \tag{41}$$

respectively, are finite. Both $[U \to V]$ and $[U \leftarrow V]$ are ideal spaces; they are also examples of tensor products of U and V, since for any $u(s) \in U$ and $v(t) \in V$ one has $w(t,s) = v(t)u(s) \in [U \to V]$, $[U \leftarrow V]$ and

$$||w(t,s)||_{[U\to V]} = ||w(t,s)||_{[U\leftarrow V]} = ||u(s)||_U||v(t)||_V. \tag{42}$$

Let us return to the PIO's

$$Lx(t,s) = \int_S l(t,s,\sigma)x(t,\sigma)\,d\mu(\sigma) \tag{43}$$

and

$$Mx(t,s) = \int_T m(t,\tau,s)x(\tau,s)\,d\nu(\tau). \tag{44}$$

In what follows, we shall describe three approaches to action conditions for these operators in spaces with mixed norm. Consider the families $L(t)$, $t \in T$, and $M(s)$, $s \in S$, of linear integral operators defined by

$$L(t)u(s) = \int_S l(t,s,\sigma)u(\sigma)\,d\mu(\sigma), \quad t \in T, \tag{45}$$

$$M(s)v(t) = \int_T m(t,\tau,s)v(\tau)\,d\nu(\tau), \quad s \in S. \tag{46}$$

Theorem 9. *Let U_1 and U_2 be two ideal spaces over S, and V_1 and V_2 two ideal spaces over T. Suppose that the linear integral operator* (45) *maps U_1 into U_2, for each $t \in T$, and that $||L(t)||_{\mathcal{L}(U_1,U_2)} \in V_2/V_1$.*

Then the PIO (43) *acts between the spaces* $X = [U_1 \to V_1]$ *and* $Y = [U_2 \to V_2]$ *and satisfies*

$$||L||_{\mathcal{L}(X,Y)} \le ||\,||L(t)||_{\mathcal{L}(U_1,U_2)}||_{V_2/V_1}. \tag{47}$$

Similarly, if the linear integral operator (46) *maps* V_1 *into* V_2*, for each* $s \in S$*, and* $||M(s)||_{\mathcal{L}(V_1,V_2)} \in U_2/U_1$*, the PIO* (44) *acts between the spaces* $X = [U_1 \leftarrow V_1]$ *and* $Y = [U_2 \leftarrow V_2]$ *and satisfies*

$$||M||_{\mathcal{L}(X,Y)} \le ||\,||M(s)||_{\mathcal{L}(V_1,V_2)}||_{U_2/U_1}. \tag{48}$$

Proof. We prove only the first statement. Let $x(t,s) \in X$. For each $t \in T$ we then have

$$||Lx(t,s)||_{U_2} \le ||L(t)||_{\mathcal{L}(U_1,U_2)}||x(t,s)||_{U_1};$$

hence, by the definition of the multiplicator space V_2/V_1,

$$\begin{aligned} ||Lx(t,s)||_Y &\le ||\,||L(t)||_{\mathcal{L}(U_1,U_2)} \cdot ||x(t,s)||_{U_1}||_{V_2} \\ &\le ||\,||L(t)||_{\mathcal{L}(U_1,U_2)}||_{V_2/V_1} \cdot ||x||_X. \end{aligned}$$

This shows that the operator (43) acts between X and Y and satisfies (47). □

Interestingly, in the case $V_2/V_1 = L_\infty$, the conditions of Theorem 9 are also necessary for the operator (43) to act between $X = [U_1 \to V_1]$ and $Y = [U_2 \to V_2]$. In fact, considering the operator (43) on the "separated" functions $x(t,s) = u(s)v(t)$, $u \in U$, $v \in V$, we conclude that, by the obvious relation $Lx(t,s) = v(t)L(t)u(s)$,

$$\sup_{||u||_{U_1},||v||_{V_1} \le 1} ||v(t)||\,||L(t)u(s)||_{[U_2 \to V_2]} \le ||L||_{\mathcal{L}(X,Y)}. \tag{49}$$

In case $V_2/V_1 = L_\infty$ this means exactly that

$$||\,||L(t)||_{\mathcal{L}(U_1,U_2)}||_{V_2/V_1} \le ||L||_{\mathcal{L}(X,Y)}. \tag{50}$$

Analogous statements hold, of course, for the operator (44) in case $U_2/U_1 = L_\infty$.

Generally speaking, the estimates (49) and (50) are not equivalent. Nevertheless, (49) implies that

$$\sup_{||u||_{U_1}\le 1} ||\,||L(t)u(s)||_{U_2}||_{V_2/V_1} < \infty, \tag{51}$$

which thus is necessary for the operator (43) to act between $X = [U_1 \to V_1]$ and $Y = [U_2 \to V_2]$. The analogous condition for the operator (44) reads

$$\sup_{||v||_{V_1}\le 1} ||\,||M(s)v(t)||_{V_2}||_{U_2/U_1} < \infty. \tag{52}$$

Observe that we did not suppose in Theorem 9 that the operators (43) and (44) be regular. Applying this theorem to the kernels $|l(t,s,\sigma)|$ and $|m(t,\tau,s)|$, rather than to $l(t,s,\sigma)$ and $m(t,\tau,s)$, we get a refinement of Theorem 9. To this end, we denote by $\mathcal{Z}(W_1,W_2)$ (with W_1 and W_2 being two ideal spaces over some set Ω) the space of all *Zaanen kernels* $z(\xi,\eta)$, defining regular linear integral operators from W_1 into W_2, with the norm

$$||z(\xi,\eta)||_{\mathcal{Z}(W_1,W_2)} = \sup_{||w||_{W_1}\le 1}\left\|\int_\Omega |z(\xi,\eta)w(\eta)|\,d\eta\right\|_{W_2}.$$

Theorem 10. *Let U_1 and U_2 be two ideal spaces over S, and V_1 and V_2 two ideal spaces over T. Suppose that $l(t,s,\sigma) \in \mathcal{Z}(U_1,U_2)$ for* a.a. *$t \in T$, and $||l(t,s,\sigma)||_{\mathcal{Z}(U_1,U_2)} \in V_2/V_1$. Then the PIO* (43) *acts between the spaces $X = [U_1 \to V_1]$ and $Y = [U_2 \to V_2]$, is regular, and satisfies*

$$||l(t,s,\sigma)||_{\mathcal{R}_1(X,Y)} \le ||\,||l(t,s,\sigma)||_{\mathcal{Z}(U_1,U_2)}||_{V_2/V_1}. \tag{53}$$

Similarly, if $m(t,\tau,s) \in \mathcal{Z}(V_1,V_2)$ for a.a. *$s \in S$, and $||m(t,\tau,s)||_{\mathcal{Z}(V_1,V_2)} \in U_2/U_1$, then the PIO* (44) *acts between the spaces $X = [U_1 \leftarrow V_1]$ and $Y = [U_2 \leftarrow V_2]$, is regular, and satisfies*

$$||m(t,\tau,s)||_{\mathcal{R}_m(X,Y)} \le ||\,||m(t,\tau,s)||_{\mathcal{Z}(V_1,V_2)}||_{U_2/U_1}. \tag{54}$$

Consider now the two integral operators

$$\tilde{L}u(s) = \int_S ||l(t,s,\sigma)||_{V_2/V_1} u(\sigma)\,d\mu(\sigma), \tag{55}$$

$$\tilde{M}v(t) = \int_T ||m(t,\tau,s)||_{U_2/U_1} v(\tau)\,d\nu(\tau), \tag{56}$$

generated by the kernels $\tilde{l}(s,\sigma) = ||l(t,s,\sigma)||_{V_2/V_1}$ and $\tilde{m}(t,\tau) = ||m(t,\tau,s)||_{U_2/U_1}$, respectively.

Theorem 11. *Let U_1 and U_2 be two ideal spaces over S, and V_1 and V_2 two ideal spaces over T. Suppose that the linear integral operator* (55) *maps U_1 into U_2. Then the PIO* (43) *acts between the spaces $X = [U_1 \leftarrow V_1]$ and $Y = [U_2 \leftarrow V_2]$, is regular, and satisfies*

$$||l(t,s,\sigma)||_{\mathcal{R}_l(X,Y)} \in ||\,||l(t,s,\sigma)||_{V_2/V_1}||_{\mathcal{Z}(U_1,U_2)}. \tag{57}$$

Similarly, if the linear integral operator (56) *maps V_1 into V_2, then the PIO* (44) *acts between the spaces $X = [U_1 \to V_1]$ and $Y = [U_2 \to V_2]$ is regular, and satisfies*

$$||m(t,\tau,s)||_{\mathcal{R}_m(X,Y)} \le ||\,||m(t,\tau,s)||_{U_2/U_1}||_{\mathcal{Z}(V_1,V_2)}. \tag{58}$$

Proof. We prove again only the first statement. Obviously, for $x(t,s) \in X$, we have

$$\begin{aligned}\left\|\int_S l(t,s,\sigma)x(t,\sigma)\,d\mu(\sigma)\right\|_{V_2} &\le \int_S ||l(t,s,\sigma)||_{V_2/V_1}||x(t,\sigma)||_{V_1}\,d\mu(\sigma)\\ &= \tilde{L}u(s), \qquad u(s) = ||x(t,s)||_{V_1},\end{aligned}$$

hence

$$\begin{aligned}\left\|\int_S l(t,s,\sigma)x(t,\sigma)\,d\mu(\sigma)\right\|_{Y} &\le ||\tilde{L}u(s)||_{U_2} \le ||\tilde{L}||_{\mathcal{L}(U_1,U_2)}||u(s)||_{U_1}\\ &= ||L||_{\mathcal{L}(U_1,U_2)}||x(t,s)||_X.\end{aligned}$$

This shows that the operator (43) acts between X and Y and satisfies (57). $\square$

We conclude this section with two more action conditions for PIO's in spaces with mixed norm; these conditions build on multiplicator spaces of functions of two variables.

Theorem 12. *Let U_1 and U_2 be two ideal spaces over S, and V_1 and V_2 two ideal spaces over T. Suppose that the function $||l(t,\cdot,\sigma)||_{U_2}$ belongs to the multiplicator space $[L_1 \to V_2]/[U_1 \leftarrow V_1]$. Then the PIO* (43) *acts between the spaces $X = [U_1 \leftarrow V_1]$ and $Y = [U_2 \to V_2]$, is regular, and satisfies*

$$||l(t,s,\sigma)||_{\mathcal{R}_l(X,Y)} \leq ||\,||l(t,s,\sigma)||_{U_2}||_{[L_1\to V_2]/[U_1\leftarrow V_1]}. \tag{59}$$

Similarly, if the function $||m(\cdot,\tau,s)||_{V_2}$ belongs to the multiplicator space $[U_2 \to L_\infty]/[U_1 \leftarrow V_1]$, then the PIO (44) *acts between the spaces $X = [U_1 \to V_1]$ and $Y = [U_2 \leftarrow V_2]$, is regular, and satisfies*

$$||m(t,\tau,s)||_{\mathcal{R}_m(X,Y)} \leq ||\,||m(t,\tau,s)||_{V_2}||_{[U_2\leftarrow L_\infty]/[U_1\to V_1]}. \tag{60}$$

Proof. Obviously,

$$\left\|\int_S l(t,s,\sigma)x(t,\sigma)\,d\mu(\sigma)\right\|_{U_2} \leq \int_S ||l(t,s,\sigma)||_{U_2}|x(t,\sigma)|\,d\mu(\sigma);$$

hence,

$$\begin{aligned}\left\|\int_S l(t,s,\sigma)x(t,\sigma)\,d\mu(\sigma)\right\|_Y &\\ &\leq ||\,||l(t,s,\sigma)||_{U_2}|x(t,\sigma)|\,||_{[L_1\to V_2]}\\ &\leq ||\,||l(t,s,\sigma)||_{U_2}||_{[L_1\to V_2]/[U_1\leftarrow V_1]}||x(t,s)||_{[U_1\leftarrow V_1]}.\end{aligned}$$

This proves the first statement. The second statement is proved analogously. $\square$

Theorem 13. *Let U_1 and U_2 be two ideal spaces over S, and V_1 and V_2 two ideal spaces over T. Suppose that the function $||l(t,s,\sigma)||_{U_1}$ belongs to the multiplicator space $[V_1 \leftarrow L_1]/[U_2' \to V_2']$ (see* (14)*). Then* *the PIO* (43) *acts between the spaces* $X = [U_1 \to V_1]$ *and* $Y = [U_2 \leftarrow V_2]$ *is regular and satisfies*

$$||l(t,s,\sigma)||_{\mathcal{R}_1(X,Y)} \leq ||\,||l(t,s,\sigma)||_{U_1'}||_{[V_1 \leftarrow L_1]/[U_2' \to V_2']}. \tag{61}$$

Similarly, if the function $||m(t,\tau,s)||_{V_1}$ *belongs to the multiplicator space* $[U_1 \leftarrow L_1]/[U_2' \to V_2']$, *then the PIO* (44) *acts between the spaces* $X = [U_1 \leftarrow V_1]$ *and* $Y = [U_2 \to V_2]$ *is regular, and satisfies*

$$||m(t,\tau,s)||_{\mathcal{R}_m(X,Y)} \leq ||\,||m(t,\tau,s)||_{V_1'}||_{[U_1 \leftarrow L_1]/[U_2' - V_2']}. \tag{62}$$

Proof. The proof is completely analogous to that of Theorem 12; one has just to pass to the corresponding associate operators, see (15). ◻

All sufficient conditions of Theorem 9–13 are different for the PIO (43) and the PIO (44) and refer to different kernel spaces with mixed norm. In this way, the above theorems contain eight statements which guarantee the acting (and, except for Theorem 9, also the regularity) of PIO's between four possible combinations of spaces with mixed norm.

6. Special cases. To verify the conditions of Theorems 9–13, one has to show that certain functions of two variables, constructed by means of the kernels $l(t,s,\sigma)$ and $m(t,\tau,s)$, belong to certain ideal spaces, constructed by means of the spaces U_1, U_2, V_1, and V_2. Since these ideal spaces are rather complicated, however, the natural problem arises to replace them by simpler and more tractable ones. One possibility to do so is to introduce ideal spaces of functions defined either on $T \times S \times S$, or on $T \times T \times S$, or on some permutation of these Cartesian products. Denote by $\tau = (\tau_1, \tau_2, \tau_3)$ an arbitrary permutation of the arguments $(t,s,\sigma) \in T \times S \times S$, or $(t,\tau,s) \in T \times T \times S$. By $[W_1, W_2, W_3; \tau]$ we denote the ideal space of all functions for which the norm

$$||w(t,s,\sigma)||_{[W_1,W_2,W_3;\tau]} = ||\,||\,||w(t,s,\sigma)||_{W\tau_1}||_{W\tau_2}||_{W\tau_3},$$
$$||w(t,\tau,s)||_{[W_1,W_2,W_3;\tau]} = ||\,||\,||w(t,\tau,s)||_{W\tau_1}||_{W\tau_2}||_{W\tau_3}$$

is defined and finite. Using classical results on linear integral operators, from Theorems 10–13, we get

Theorem 14. *Let U_1 and U_2 be two ideal spaces over S, and V_1 and V_2 two ideal spaces over T. Suppose that $l(t,s,\sigma) \in [V_2/V_1, U_2, U_1'; \tau]$ for some $\tau = (\tau_1, \tau_2, \tau_3)$. Denote by X the space $[U_1 \rightarrow V_1]$ (respectively, $[U_1 \leftarrow V_1]$) if σ precedes t (respectively t precedes σ), and by Y the space $[U_2 \rightarrow V_2]$ (respectively, $[U_2 \leftarrow V_2]$) if s precedes t (respectively, t precedes s). Then the PIO* (43) *acts between X and Y, is regular, and satisfies*

$$||l(t,s,\sigma)||_{\mathcal{R}_1(X,Y)} \leq ||l(t,s,\sigma)||_{[V_2/V_1,U_2,U_1';\tau]}. \tag{63}$$

Similarly, if $m(t,\tau,s) \in [U_2/U_1, V_2, V_1'; \tau]$ for some $\tau = (\tau_1,\tau_2,\tau_3)$, and we denote by X the space $[U_1 \rightarrow V_1]$ (respectively, $[U_1 \leftarrow V_1]$) if s precedes τ (respectively, τ precedes s), and by Y the space $[U_2 \rightarrow V_2]$ (respectively, $[U_2 \leftarrow V_2]$) if s precedes t (respectively, t precedes s), then the PIO (44) *acts between X and Y, is regular, and satisfies*

$$||m(t,\tau,s)||_{\mathcal{R}_m(X,Y)} \leq ||l(t,\tau,s)||_{[U_2/U_1,V_2,V_1';\tau]}. \tag{64}$$

To illustrate Theorem 14, consider the special case of the Lebesgue spaces $U_1 = L_{p_1}$, $U_2 = L_{p_2}$, $V_1 = L_{q_1}$, and $V_2 = L_{q_2}$. The classical Minkowski inequality (see, e.g., [**10, 38**]) implies that $[L_p \rightarrow L_q] \subseteq [L_p \leftarrow L_q]$ for $p \geq q$, and $[L_p \leftarrow L_q] \subseteq [L_p \rightarrow L_q]$ for $p \leq q$. Thus, from Theorem 14, we get

Theorem 15. *Let $p_1, p_2, q_1, q_2 \in [0,\infty]$. Suppose that $l(t,s,\sigma) \in [L_{q_1q_2/(q_2-q_1)}, L_{p_2}, L_{p_1/(p_1-1)}; \tau]$, $q_1 \leq q_2$. Then the PIO* (43) *acts between the spaces X and Y is regular and satisfies*

$$\begin{aligned} &||l(t,s,\sigma)||_{\mathcal{R}_l}(X,Y) \\ &\leq ||l(t,s,\sigma)||_{[L_{q_1q_2/(q_2-q_1)},L_{p_2},L_{p_1/(p_1-1)}};\tau], \end{aligned} \tag{65}$$

provided one of the conditions of Table 1 *below holds. Similarly, if $m(t,\tau,s) \in [L_{q_2}, L_{q_1/(q_1-1)}, L_{p_1p_2/(p_1-p_2)}; \tau]$, $p_1 \geq p_2$, then the PIO* (44) *acts between the spaces X and Y, is regular, and satisfies*

$$\begin{aligned} &||m(t,\tau,s)||_{\mathcal{R}_m(X,Y)} \\ &\leq ||m(t,\tau,s)||_{[L_{q_2},L_{q_1/(q_1-1)},L_{p_1p_2/(p_1-p_2)}};\tau] \end{aligned} \tag{66}$$

provided one of the conditions of Table 2 *below holds.*

TABLE 1.

	$X = [L_{p_1} - L_{q_1}]$ $Y = [L_{p_2} \to L_{q_2}]$	$X = [L_{p_1} \to L_{q_2}]$ $Y = [L_{p_2} \leftarrow L_{q_2}]$	$X = [L_{p_1} \leftarrow L_{q_1}]$ $Y = [L_{p_2} \to L_{q_2}]$	$X = [L_{p_1} \leftarrow L_{q_1}]$ $Y = [L_{p_2} \leftarrow L_{q_2}]$
(t, s, σ)	$p_1 \geq q_1$, $p_2 \leq q_2$	$p_1 \geq q_1$	$p_2 \leq q_2$	
(t, σ, s)	$p_1 \geq q_1$, $p_2 \leq q_2$	$p_1 \geq q_1$	$p_2 \leq q_2$	
(s, t, σ)	$p_1 \geq q_1$	$p_1 \geq q_1$, $p_2 \geq q_2$		$p_2 \geq q_2$
(s, σ, t)		$p_2 \geq q_2$	$p_1 \leq q_1$	$p_1 \leq q_1$, $p_2 \geq q_2$
(σ, t, s)	$p_2 \leq q_2$		$p_1 \leq q_1$, $p_2 \leq q_2$	$p_1 \leq q_1$
(σ, s, t)		$p_2 \geq q_2$	$p_1 \leq q_1$	$p_1 \leq q_1$, $p_2 \geq q_2$

TABLE 2.

(t, τ, s)	$p_1 \geq q_1$, $p_2 \leq q_2$	$p_1 \geq q_1$	$p_2 \leq q_2$	
(t, s, τ)	$p_2 \leq q_2$		$p_1 \leq q_1$, $p_2 \leq q_2$	$p_1 \leq q_1$
(τ, t, s)	$p_1 \geq q_1$, $p_2 \leq q_2$	$p_1 \geq q_1$	$p_2 \leq q_2$	
(τ, s, t)	$p_1 \geq q_1$	$p_1 \geq q_1$, $p_2 \geq q_2$		$p_2 \geq q_2$
(s, t, τ)		$p_2 \geq q_2$	$p_1 \leq q_1$	$p_1 \leq q_1$, $p_2 \geq q_2$
(s, τ, t)		$p_2 \geq q_2$	$p_1 \leq q_1$	$p_1 \leq q_1$, $p_2 \geq q_2$

The most interesting and important case in the preceding theorem is when the operators (43) and (44) are considered in the single Lebesgue space $L_p = L_p(S \times T)$. Since $[L_p \to L_p] = [L_p \leftarrow L_p] = L_p$, Theorem 15 reads in this case as follows:

Theorem 16. *Let $1 \leq p \leq \infty$. Suppose that $l(t, s, \sigma) \in [L_\infty, L_p, L_{p/(p-1)}; \tau']$ for some τ', and $m(t, \tau, s) \in [L_p, L_{p/(p-1)}, L_\infty; \tau'']$ for some τ''. Then the PIO's* (43) *and* (44) *are regular in the space L_p and*

$$(67) \qquad ||l(t, s, \sigma)||_{\mathcal{R}_l(L_p, L_p)} \leq ||l(t, s, \sigma)||_{[L_\infty, L_p, L_{p/(p-1)}; \tau']},$$

$$(68) \qquad ||m(t, \tau, s)||_{\mathcal{R}_m(L_p, L_p)} \leq ||m(t, \tau, s)||_{[L_p, L_{p/(p-1)}, L_\infty; \tau'']}.$$

Apart from the acting and regularity conditions for the PIO's (43) and (44), many other statements may be formulated. For instance, one may obtain further statements by means of interpolation theory (applied to either classical or partial integral operators), of Kantorovich type theorems, or of other methods. We point out that Theorem 9 and its partial converse considered above imply the following useful criterion: the PIO (43) (respectively, (44)) acts in L_p if and only if all operators of the family (45) (respectively, (46)) act in L_p for each $t \in T$ (respectively, $s \in S$) and have uniformly bounded norms.

The statements of Theorems 15 and 16, referring to the Lebesgue type spaces $[L_p \to L_q]$ and $[L_p \leftarrow L_q]$, carry over as well to the Orlicz type spaces $[L_M \to L_N]$ and $[L_M \leftarrow L_N]$.

7. PIO's of Volterra type. Consider the linear integral equation

$$\begin{aligned} x(t,s) = \int_S l(t,s,\sigma)x(t,\sigma)\,d\mu(\sigma) + \int_T m(t,\tau,s)x(\tau,s)\,d\nu(\tau) \\ + \int_S\int_T n(t,\tau,s,\sigma)x(\tau,\sigma)\,d\mu\times\nu(\tau,\sigma) + f(t,s) \end{aligned} \tag{69}$$

which may be written concisely as operator equation

$$(I - L - M - N)x = f, \tag{70}$$

where L, M, and N are given by (3), (4), and (5). We are interested in the question as to what extent the basic results for classical integral equations carry over to equation (69). We shall discuss some results which are related to specific properties of the operator $K = I - L - M - N$. Obviously,

$$K = (I-L)(I-M) - (N+LM) = (I-M)(I-L) - (N+ML); \tag{71}$$

hence, in case the operators $I - L$ and $I - M$ are invertible, equation (70) is equivalent to both equations

$$\begin{aligned} (I - (I-M)^{-1}(I-L)^{-1}(N+LM))x = f, \\ (I - (I-L)^{-1}(I-M)^{-1}(N+ML))x = f. \end{aligned} \tag{72}$$

Observe that the operators $N + LM$ and $N + ML$ are (compositions of) integral operators, and, hence, one may apply classical results on integral equations to the operator equations (72). The invertibility of the operators $I-L$ and $I-M$ is related to the solvability of the equation

$$u(t,s) = \int_S l(t,s,\sigma)u(t,\sigma)\,d\mu(\sigma) + f(t,s) \tag{73}$$

(which is a classical integral equation containing a parameter t), and the equation

$$v(t,s) = \int_T m(t,\tau,s)v(\tau,s)\,d\nu(\tau) + f(t,s) \tag{74}$$

(which is a classical integral equation containing a parameter s). The study of equations (73) and (74), in turn, reduces to analyzing the operators (45) and (46).

As a matter of fact, under natural hypotheses on the kernels $l(t,s,\sigma)$ and $m(t,\tau,s)$, the linear integral equations

$$u(s) = \int_S l(t,s,\sigma)u(\sigma)\,d\mu(\sigma) + g(s), \quad t \in T, \tag{75}$$

and

$$v(t) = \int_T m(t,\tau,s)v(\tau)\,d\nu(\tau) + h(t), \quad s \in S, \tag{76}$$

admit unique solutions in U and V for arbitrary functions $g(s)$ and $h(t)$, respectively. Moreover, these solutions are given by

$$u(s) = g(s) + \int_S \phi(t,s,\sigma)g(\sigma)\,d\mu(\sigma), \quad t \in T, \tag{77}$$

and

$$v(t) = h(t) + \int_T \psi(t,\tau,s)h(\tau)\,d\nu(\tau), \quad s \in S \tag{78}$$

involving the resolvent kernels $\phi(t,s,\sigma)$ and $\psi(t,\tau,s)$. In case U and V are ideal spaces, and the spectral radii of the operators $]L(t)[$ and $]M(s)[$ (see (45), (46), (8) and (9)) satisfy

$$\rho(]L(t)[) < 1, \quad t \in T, \qquad \rho(]M(s)[) < 1, \quad s \in S, \tag{79}$$

the resolvent kernels may be represented as a series of iterated kernels

$$\phi(t,s,\sigma) = \sum_{k=1}^{\infty} l^{(k)}(t,s,\sigma), \quad s,\sigma \in S,\ t \in T, \tag{80}$$

and

$$\psi(t,\tau,s) = \sum_{k=1}^{\infty} m^{(k)}(t,\tau,s), \quad t,\tau \in T,\ s \in S, \tag{81}$$

which converge in the Zaanen kernel spaces $\mathcal{Z}(U,U)$ and $\mathcal{Z}(V,V)$, respectively (see Section 5). If, in addition, we have

$$\phi(t,s,\sigma) \in \mathcal{R}_1(U,U), \qquad \psi(t,\tau,s) \in \mathcal{R}_m(V,V), \tag{82}$$

it is natural to expect that

$$x(t,s) = f(t,s) + \int_S \phi(t,s,\sigma) f(t,\sigma)\, d\mu(\sigma) \tag{83}$$

and

$$x(t,s) = f(t,s) + \int_T \psi(t,\tau,s) f(\tau,s)\, d\nu(\tau) \tag{84}$$

are solutions of (73) and (74), respectively. In particular, this is true if

$$\rho(L) < 1, \qquad \rho(M) < 1; \tag{85}$$

in our case, the spectral radii in (79) may be calculated by the formulas

$$\begin{aligned} \rho(]L(t)[) &= \lim_{k\to\infty} \sqrt[k]{||l^{(k)}(t,s,\sigma)||_{\mathcal{R}_l(X,X)}}, \\ \rho(]M(s)[) &= \lim_{k\to\infty} \sqrt[k]{||m^{(k)}(t,\tau,s)||_{\mathcal{R}_m(X,X)}}. \end{aligned} \tag{86}$$

We summarize with

Theorem 17. *Suppose that the linear operator $D = L + M + N$ acts in some ideal space X and is regular. Assume that the estimates* (79) *hold, that the corresponding resolvent kernels satisfy* (82) (*with $X \subseteq [U \to S] \cap [S \leftarrow V]$*), *and that one of the operators $N + LM$ or $N + ML$ is compact. Then the linear integral equation* (69) *satisfies the Fredholm alternative in the space X*; *in particular,* (69) *admits a unique solution $x(t,s) \in X$ for any $f(t,s) \in X$ if and only if* (69) *admits only the trivial solution for $f(t,s) \equiv 0$.*

Consider now the special case when the sets T and S are intervals and (69) is a partial integral equation of Volterra type

$$\begin{aligned} x(t,s) = \int_0^s l(t,s,\sigma)x(t,\sigma)\,d\mu(\sigma) + \int_0^t m(t,\tau,s)x(\tau,s)\,d\nu(\tau) \\ + \int_C^s \int_C^t n(t,\tau,s,\sigma)x(\tau,\sigma)\,d\mu\times\nu(\tau,\sigma) + f(t,s). \end{aligned} \tag{87}$$

In this case, (79) holds true (see, e.g., [**38**]) if the operators (45) and (46) are compact in the spaces U and V, respectively, and U and V are *regular* (i.e., all elements in U and V have absolutely continuous norm, see [**34**]). Moreover, if the resolvent kernels $\phi(t,s,\sigma)$ and $\psi(t,\tau,s)$ satisfy (82), the investigation of equation (87) reduces to that of equations (72) (which are then classical Volterra integral equations, of course). Finally, if at least one of the operators $N+LM$ or $N+ML$ is compact, then, again by the regularity of the space X (see [**38**]), the spectral radius of the corresponding operator is zero. We summarize again with

Theorem 18. *Let X, U, and V be regular ideal spaces, and let the operators $L(t)$, $t\in T$, and $M(s)$, $s\in S$, be compact in U and V, respectively. Suppose that the resolvent kernels $\phi(t,s,\sigma)$ and $\psi(t,\tau,s)$ satisfy* (82). *Assume, moreover, that at least one of the operators $N+LM$ or $N+ML$ is compact in X. Then the linear Volterra equation* (87) *has a unique solution $x(t,s)\in X$ for any $f(t,s)\in X$.*

A simple example illustrating Theorems 17 and 18 is that of the Lebesgue spaces $U=L_p(S)$, $V=L_p(T)$, and $X=L_p(T\times S)$. In this case, condition (82) reduces to

$$\phi(t,s,\sigma)\in\mathcal{L}(X,X), \qquad \psi(t,\tau,s)\in\mathcal{L}(X,X), \tag{88}$$

which, by the results of Section 5, is also necessary. In this way, verifying the hypotheses of Theorems 17 and 18 means, here, simply studying families of linear integral operators in Lebesgue spaces.

8. PIO's and tensor products. In this section, we shall be concerned with integral equations of the form

$$
\begin{aligned}
x(t,s) = &\int_S l(s,\sigma)x(t,\sigma)\,d\mu(\sigma) + \int_T m(t,\tau)x(\tau,s)\,d\nu(\tau) \\
&+ \int_S \int_T n(t,\tau,s,\sigma)x(\tau,\sigma)\,d\mu\times\nu(\tau,\sigma) + f(t,s),
\end{aligned} \tag{89}
$$

where the operators

$$
Lx(t,s) = \int_S l(s,\sigma)x(t,\sigma)\,d\mu(\sigma) \tag{90}
$$

and

$$
Mx(t,s) = \int_T m(t,\tau)x(\tau,s)\,d\nu(\tau) \tag{91}
$$

commute. The operators (90) and (91) are parameter-dependent families of the corresponding integral operators

$$
\tilde{L}u(s) = \int_S l(s,\sigma)u(\sigma)\,d\mu(\sigma) \tag{92}
$$

and

$$
\tilde{M}v(t) = \int_T m(t,\tau)v(\tau)\,d\nu(\tau). \tag{93}
$$

We shall suppose that the operators $\tilde{L}$ and $\tilde{M}$ act in the spaces $L_p(S)$ and $L_p(T)$, respectively, and the operator

$$
Nx(t,s) = \int_S \int_T n(t,\tau,s,\sigma)x(\tau,\sigma)\,d\mu\times\nu(\tau,\sigma) \tag{94}
$$

acts in the space $L_p(T\times S)$ and is compact. It is well known (see, e.g., [**26**]) that $L_p(T\times S)$ is a tensor product of $L_p(S)$ and $L_p(T)$ with respect to the Levin cross-norm. Consequently, for studying analytical properties of equation (89) (solvability, Fredholm type, index, etc.), one may make use of the theory of tensor products. Results in this spirit are most easily formulated in terms of the spectra of the operators L, M, N, and $D = L + M + N$. We recall some basic notation (for definitions, see e.g. [**21**]): $\sigma(D)$ is the spectrum of D; $\sigma_{\mathrm{ew}}(D)$ (respectively, $\sigma_{\mathrm{es}}(D)$ and $\sigma_{\mathrm{eb}}(D)$) is the essential spectrum of D in Weyl's sense (respectively in Fredholm's and Browder's sense); $\sigma_\pi(D)$ is the limit spectrum of D; $\sigma^\pi(D)$ is the closure of $\sigma_\pi(D^*)$; $\sigma_+(D)$ and $\sigma_-(D)$ are the domains of normal solvability of D in case of a finite-dimensional null space and finite-codimensional range, respectively; $n_+(D)$ and $n_-(D)$ are the corresponding finite dimension and codimension; $\operatorname{ind} D$ is the index of D; $\kappa(D, \lambda)$ is the algebraic multiplicity of the eigenvalue λ of D.

The following two theorems are consequences of general results on the spectra of operators on tensor products.

Theorem 19. *Suppose that the operator $\tilde{L}$ acts in $L_p(S)$, the operator $\tilde{M}$ acts in $L_p(T)$, and the operator N acts in $L_p(T \times S)$ and is compact, $1 \le p \le \infty$. Then*

$$\sigma_{\mathrm{ew}}(D) = (\sigma_{\mathrm{ew}}(\tilde{L}) + \sigma(\tilde{M})) \cup (\sigma(\tilde{L}) + \sigma_{\mathrm{ew}}(\tilde{M})) \tag{95}$$

and for $\lambda \notin \sigma_{\mathrm{ew}}(D)$, one has

$$\begin{aligned}\operatorname{ind}(\lambda I - D) &= \sum_{(\alpha,\beta)\in E'} \operatorname{ind}(\beta I - \tilde{M}) \sum_{j=1}^{\infty} (n_-(\alpha I - \tilde{L})^j - n_-(\alpha I - \tilde{L})^{j-1}) \\ &\quad + \sum_{(\alpha,\beta)\in E''} \operatorname{ind}(\alpha I - \tilde{L}) \sum_{j=1}^{\infty} (n_-(\beta I - \tilde{M}) - n_-(\beta I - \tilde{M})^{j-1}),\end{aligned} \tag{96}$$

where

$$E' = \{(\alpha, \beta) : \alpha + \beta = \lambda,\ \alpha \in \sigma(\tilde{L}) \backslash \sigma_{\mathrm{eb}}(\tilde{L}),\ \beta \in \sigma(\tilde{M}) \backslash \sigma_{\mathrm{ew}}(\tilde{M})\}$$

and

$$E'' = \{(\alpha, \beta) : \alpha + \beta = \lambda,\ \alpha \in \sigma(\tilde{L}) \backslash \sigma_{\mathrm{ew}}(\tilde{L}), \beta \in \sigma(\tilde{M}) \backslash \sigma_{\mathrm{eb}}(\tilde{M})\}.$$

Moreover, one has

$$\sigma_{\mathrm{es}}(D) = \sigma_{\mathrm{ew}}(D) \cup \{\lambda \in \sigma_{\mathrm{ew}}(D) : \operatorname{ind}(\lambda I - D) \ne 0\}, \tag{97}$$

$$\sigma_+(D) = (\sigma_+(\tilde{L}) + \sigma_\pi(\tilde{M})) \cup (\sigma_\pi(\tilde{L}) + \sigma_+(\tilde{M})), \tag{98}$$

$$\sigma_-(D) = (\sigma_-(\tilde{L}) + \sigma^\pi(\tilde{M})) \cup (\sigma^\pi(\tilde{L}) + \sigma_-(\tilde{M})). \tag{99}$$

Finally, if, also, $\tilde{L}$ *and* $\tilde{M}$ *are compact, then*

$$\sigma_{\mathrm{es}}(D) = \sigma_{\mathrm{ew}}(D) = \sigma_+(D) = \sigma_-(D) = \sigma(\tilde{L}) \cup \sigma(\tilde{M}). \tag{100}$$

Theorem 20. *Suppose that the operator* $\tilde{L}$ *acts in* $L_p(S)$*, the operator* $\tilde{M}$ *acts in* $L_p(T)$*, and* $N = 0$*. Then*

$$\sigma(D) = \sigma(\tilde{L}) + \sigma(\tilde{M}), \tag{101}$$

$$\sigma_{\mathrm{eb}}(D) = (\sigma_{\mathrm{eb}}(\tilde{L}) + \sigma(\tilde{M})) \cup (\sigma(\tilde{L}) + \sigma_{\mathrm{eb}}(\tilde{M})), \tag{102}$$

and, for $\lambda \notin \sigma_{\mathrm{eb}}(D)$*, one has*

$$\kappa(D, \lambda) = \sum_{(\alpha,\beta)\in E} \kappa(\tilde{L}, \alpha) \cdot \kappa(\tilde{M}, \beta), \tag{103}$$

where

$$E = \{(\alpha, \beta) : \alpha + \beta = \lambda,\ \alpha \in \sigma(\tilde{L})\backslash\sigma_{\mathrm{eb}}(\tilde{L}),\ \beta \in \sigma(\tilde{M})\backslash\sigma_{\mathrm{eb}}(\tilde{M})\}.$$

Moreover, one has

$$\begin{aligned} n_-(\lambda I - D) = \sum_{\alpha+\beta=\lambda} \sum_{j=1}^{\infty} &(n_-((\alpha I - \tilde{L})^j) - n_-((\alpha I - \tilde{L})^{j-1})) \\ &\cdot (n_-((\beta I - \tilde{M})^j) - n_-((\beta I - \tilde{M})^{j-1})), \end{aligned} \tag{104}$$

$$\begin{aligned} n_+(\lambda I - D) = \sum_{\alpha+\beta=\lambda} \sum_{j=1}^{\infty} &(n_+((\alpha I - \tilde{L})^j) - n_+((\alpha I - \tilde{L})^{j-1})) \\ &\cdot (n_+((\beta I - \tilde{M})^j) - n_+((\beta I - \tilde{M})^{j-1})) \end{aligned} \tag{105}$$

for $\lambda \notin \sigma_{\mathrm{ew}}(D)$.

To illustrate Theorems 19 and 20, consider the linear integral equation

$$\lambda x(t,s) + \int_{-1}^{1} l(s-\sigma)x(t,\sigma)\,d\sigma + \int_0^t m(t,\tau)x(\tau,s)\,d\tau = g(t,s), \tag{106}$$

which occurs in the mechanics of continuous media (see, e.g., [**1**–**3,** **23**]). Here $S=[-1,1]$, $T=[0,a]$, the kernel $l(\xi)$ has the form

$$l(\xi) = \frac{1}{2}\int_{-\infty}^{+\infty} \tilde{l}(z)\exp(i\theta z\xi)\,dz,$$

where θ is a parameter (which has a mechanical meaning), and $\tilde{l}(z)$ is a positive, even continuous function satisfying

$$\tilde{l}(z) = A + O(z^2), \quad z \to 0, \qquad |z|\tilde{l}(z) = B + o(z^{-1}), \quad z \to \infty;$$

moreover, the kernel $m(t,\tau)$ is either continuous or weakly singular, and the function $g(t,s)$ has the form

$$g(t,s) = g_1(s) + g_2(t) + sg_3(t), \qquad g_1 \in L_p(S),\ g_2, g_3 \in L_p(T).$$

Equation (106) may be studied in the space $L_p(T\times S)$. Under the hypotheses given above, the operators $\tilde{L}$ and $\tilde{M}$ are compact, and thus Theorems 19 and 20 apply. The spectrum of $\tilde{L}$ consists of 0 and a finite number (or a sequence converging to 0) of Fredholm points λ, while the spectrum of $\tilde{M}$ contains only 0. Consequently, the spectrum of the operator D coincides with that of the operator $\tilde{L}$; in particular, all elements of $\sigma(D)$ are Fredholm points.

Altogether, *equation* (106) *admits, for each* λ *with* $-\lambda \notin \sigma(D)$, *a unique solution* $x(t,s) \in L_p(T\times S)$ *for any* $g(t,s) \in L_p(T\times S)$.

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