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TIME ASYMPTOTIC BEHAVIOR OF THE SOLUTION TO A CAUCHY PROBLEM GOVERNED BY A TRANSPORT OPERATOR

AREF JERIBI, KHALID LATRACH AND HATEM MEGDICHE

ABSTRACT. The purpose of this paper is to investigate some spectral properties of a time-dependent linear transport equation with diffusive boundary conditions arising in growing cell populations. After deriving the explicit expression of the strongly continuous semigroup $\{U^K(t) : t \geq 0\}$ generated by the streaming operator, we establish the strict singularity of the operator $BU^K(s)B$, $s > 0$, for a wide class of collision operators B. Making use of the Weis theorem, Theorem 4.1, this enables us to estimate the essential type of the transport semigroup from which the asymptotic behavior of the solution is derived.

1. Introduction. In this paper we deal with the well-posedness and the time asymptotic behavior of solutions to transport equations for a sizable class of scattering operators. More precisely, we are concerned with the following initial boundary value problem

$$(1.1)\quad \begin{cases} (\partial\psi)/(\partial t)(\mu,v,t) = -\,v(\partial\psi)/(\partial\mu)(\mu,v,t) - \sigma(v)\psi(\mu,v,t) \\ +\displaystyle\int_0^c r(\mu,v,v')\psi(\mu,v',t)\,dv' \\ A_K\psi(\mu,v,t) = S_K\psi(\mu,v,t) + B\psi(\mu,v,t), \\ \psi(\mu,v,0) = \psi_0(\mu,v), \end{cases}$$

where $\mu \in [0,a]$, v, $v' \in [0,c]$ with $a > 0$ and $c > 0$, S_K denotes the streaming operator and B stands for the collision one (the integral part of A_K). This model describes the number density $\psi(\mu,v,t)$ of cell population as a function of the degree of maturity $\mu \in [0,a]$, $a > 0$, the maturation velocity $v \in [0,c]$, $c > 0$, and the time t. The degree of maturation is defined so that $\mu = 0$ at the birth and $\mu = c$ at mitosis, i.e. cells born at $\mu = 0$ and divided at $\mu = c$. The kernel $r(\mu,v,v')$ is the transition rate. It specifies the transition of cells from the maturation velocity v' to v while $\sigma(v)$ denotes the total transition

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cross section. At mitosis, the daughter and mother cells are related by a linear reproduction law K which describes the boundary conditions.

This model was introduced by Rotenberg in [**19**] and considered afterwards by many authors. Rotenberg discussed essentially the Fokker-Plank approximation of (1.1) for which he obtained numerical solutions. By eigenfunction expansion techniques, Van der Mee and Zweifel [**22**] obtained analytical solutions for a variety of linear boundary conditions. Using the theory of linear time dependent kinetic equations developed in [**1**], Greenberg, Van der Mee and Protopopeccu [**8**, Chapter 13] and Van der Mee [**20**] proved that the associated Cauchy problem to the Rotenberg model with Lebowitz and Rubinow's boundary conditions, cf. [**16**] or [**19**], is governed by a positive C^0-semigroup. In [**5**], a detailed spectral analysis of the operator A_K supplemented with a general (linear) transition rule relating mother and daughter cells at mitosis, covering, in particular, all classical ones considered in [**8, 19, 22**] was given.

Let X be a Banach space and let $Z : D(Z) \subseteq X \to X$ be the infinitesimal generator of a C^0-semigroup of bounded linear operators $(S(t))_{t\geq 0}$ acting on X. We consider the Cauchy problem

$$\begin{cases} (d\psi)/(dt) = Z\psi + W\psi \\ \psi(0) = \psi_0, \end{cases} \tag{1.2}$$

where $W \in \mathcal{L}(X)$ and $\psi_0 \in X$. Since $A := Z + W$ is a bounded perturbation of Z, by the classical perturbation theory [**7**], it generates a C^0-semigroup $(T(t))_{t\geq 0}$ on X given by the Dyson-Phillips expansion

$$T(t) = \sum_{j=0}^{n-1} S_j(t) + R_n(t),$$

where

$$S_0(t) = S(t), \quad S_j(t) = \int_0^t S(s)WS_{j-1}(t-s)\,ds, \quad j = 1, 2, \dots ,$$

and the nth order remainder term $R_n(t)$ can be expressed by

$$R_n(t) = \int_{s_1+\cdots+s_n\leq t, s_i\geq 0} S(s_1)W\cdots WS(s_n) \\ \times WT(t - s_1 - \cdots - s_n)\,ds_1 \dots ds_n.$$

It is well known that, for all $\psi_0 \in D(Z)$, the solution of the problem (1.2), given by $\psi(t) = T(t)\psi_0$, exists and is unique.

Let A be a closed, densely defined linear operator on X. We recall that $\lambda \in \mathbf{C}$ is an eigenvalue of finite algebraic multiplicity of A if λ is an isolated point of $\sigma(A)$ and is a pole of the resolvent of A with degenerate associated spectral projection P, see [**7**, p. 247], where $\sigma(A)$ denotes the spectrum of A. If $A \in \mathcal{L}(X)$, then the essential spectral radius of A is defined by

$$\begin{aligned} r_e(A) := \sup\{|\lambda|;\ &\lambda \in \sigma(A) \text{ but } \lambda \text{ is not an eigenvalue of finite} \\ &\text{algebraic multiplicity}\}, \quad \text{see [\textbf{17}] or [\textbf{24}] for details.} \end{aligned}$$

Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on X. Then $r_\sigma(T(t)) = e^{t\omega}$ for every $t > 0$, where $r_\sigma(.)$ denotes the spectral radius and ω is the type of $(T(t))_{t\geq 0}$. Further, there exists $\omega_e(T) \in [-\infty, \omega]$ such that $r_e(T(t)) = e^{t\omega_e(T)}$ for all $(t > 0)$, see [**24**]. The real number $\omega_e(T)$ is called the essential type of $(T(t))_{t\geq 0}$. Let us first recall a known result on the essential type of perturbed C^0-semigroups on Banach spaces.

When dealing with the time asymptotic behavior of $\psi(t)$, a useful technique (initiated by Jörgens [**11**] and Vidav [**23**] and developed afterward by many authors, see, for example, [**7, 13, 17, 24, 27**] and the references therein) consists of studying the asymptotic spectrum of $T(t)$. Developing Jörgens-Vidav's idea, Weis [**27**] established an interesting result, cf. Theorem 4.1, which provides sufficient conditions in terms of the products $WS(s_1)W \cdots WS(s_n)W$, $s_i > 0$, appearing in $R_n(t)$ guaranteeing the strict singularity, cf. Remark 4.1, of some remainder term of the Dyson-Phillips expansion. This implies, see the proof of Theorem 3.1 in [**27**], that the semigroups $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ have the same essential type, say ω_0. Consequently, for $t > 0$, $\sigma(T(t)) \cap \{\alpha \in \mathbf{C} : |\alpha| > e^{\omega_0 t}\}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicity. Therefore, by the spectral mapping theorem for the point spectrum, for any $\nu > \omega_0$ we have that $\sigma(Z + W) \cap \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geq \nu\}$ consists of finitely many isolated eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. If P_i and D_i denote, respectively, the spectral projection and the nilpotent operator associated with λ_i, $1 \leq i \leq n$, then

$$T(t) = (I - P)T(t) + \sum_{i=1}^{n} e^{\lambda_i t} e^{D_i t} P_i \tag{1.3}$$

with $\|(I-P)T(t)\| = o(e^{(\lambda'-\varepsilon)t})$ as $t \to +\infty$, where $P = \sum_{i=1}^{n} P_i$, $\lambda' = \min\{\operatorname{Re}\lambda_i,\ 1 \le i \le p\}$ and $\varepsilon > 0$ small enough, and if D_i is nilpotent of order r, then

$$e^{D_i t} = \sum_{j=0}^{r-1} \frac{t^j}{j!}\,(D_i)^j.$$

This approach was used by many authors, in the context of transport theory, to investigate the large time behavior of solutions of transport equations with vacuum boundary conditions in bounded geometry by discussing directly the compactness properties of the second order remainder term of the Dyson-Phillips series [**9, 11, 13, 17, 23, 26, 27**]. So, $R_2(t)$ is strictly singular, cf. Remark 4.1, and therefore the spectral decomposition (1.3) holds true. The possibility of such a result is due to the fact that these remainders are computable because, for vacuum boundary conditions, the semigroup generated by the streaming operator is explicit. Unfortunately, when the boundary conditions are modeled by a bounded abstract operator, the streaming operator generates again a strongly continuous semigroup which is not, in general, explicit, cf. [**8**]. Therefore, to discuss the compactness of some remainder term of the Dyson-Phillips expansion is a very difficult task. This strategy was used recently, see [**3, 15**], to discuss the time structure of solution to the Cauchy problem governed by two classes of transport equations with perfectly reflecting and periodic boundary conditions in slab geometry.

In the present paper we are concerned with the time structure, $t \to \infty$, of the solution (when it exists) to equation (1.1) supplemented with the boundary conditions

$$\psi(0, v) = \int_0^c \kappa(v, v')\ \psi(a, v')\, dv'. \tag{1.4}$$

where $\kappa : [0, c] \times [0, c] \to \mathbf{R}$ is a measurable function.

The outline of this work is as follows. In the next section, we make precise the functional setting of the problem and gather some preparatory facts required in the rest of the paper. Section 3 is devoted to the well-posedness of the Cauchy problem (1.1)–(1.4). We start our analysis by determining the analytical expression of $(e^{tS_K})_{t\geq 0}$, the C_0-semigroup generated by S_K. Next, since the collision operator B is bounded, making use of Phillips's perturbation theorem for C_0-semigroups [**7**, Theorem 1.10, p. 163], one sees that $A_K = S_K + B$ generates a C_0-semigroup $(e^{tA_K})_{t\geq 0}$ on X_p which ensures the well-posedness of the problem (1.1)–(1.4). Finally, in Section 4, we discuss the strict singularity of the operator $Be^{tS_K}B$ $(t > 0)$ on X_p which leads, via Weis's theorem, to the fact that the essential types of $(e^{tA_K})_{t\geq 0}$ and $(e^{t(S_K)})_{t\geq 0}$ are equal. Accordingly, as seen above, the time asymptotic behavior of the solution is then determined by the part of $(e^{tA_K}(t))_{t\geq 0}$ in a finite dimensional subspace of X_p.

2. Preliminaries. In this section we introduce the different notions and notations which we shall need in sequel. Let us first make precise the functional setting of the problem. Let

$$X_p := L_p([0, a] \times [0, c];\, d\mu\, dv)$$

where $a > 0$, $c > 0$ and $1 \leq p < \infty$. We denote by X_p^0 and X_p^1 the following boundary spaces

$$X_p^0 := L_p(\Gamma_0;\, v\, dv) \quad \text{and} \quad X_p^1 := L_p(\Gamma_1;\, v\, dv)$$

endowed with their natural norms, where $\Gamma_0 = \{0\} \times [0, c]$ and $\Gamma_1 = \{a\} \times [0, c]$.

Let W_p be the following partial Sobolev space

$$W_p = \left\{ \psi \in X_p \text{ such that } v\,\frac{\partial \psi}{\partial \mu} \in X_p \right\}.$$

It is well known [**25**], see also [**4, 8**] that any function ψ in W_p possesses traces on the spatial boundary $\{0\}$ and $\{a\}$ which belong, respectively, to the spaces X_p^0 and X_p^1. They are denoted, respectively, by ψ^0 and ψ^1.

The boundary conditions (1.4) may be written abstractly as $\psi^0 = K\psi^1$ where K is defined by

$$\left\{\begin{array}{l} K : X_p^1 \longrightarrow X_p^0 \\ \qquad u \longrightarrow (Ku)(0,v) = \displaystyle\int_0^c \kappa(v,v')\; u(a,v')\,dv'. \end{array}\right. \tag{2.1}$$

We define the streaming operator S_K with domain including the boundary conditions

$$\left\{\begin{array}{l} S_K : D(S_K) \subset X_p \longrightarrow X_p \\ \psi \longrightarrow (S_K\psi)(\mu,v) := -\,v\;(\partial\psi)/(\partial\mu)(\mu,v) - \sigma(v)\psi(\mu,v) \\ D(S_K) = \Big\{\psi \in W_p \text{ such that } \psi^0 = K\psi^1\Big\}, \end{array}\right.$$

where $\sigma(.) \in L^\infty\,[0,c]$.

Remark 2.1. It should be noted that the hypothesis $\sigma(.) \in L^\infty\,[0,c]$ is not necessary. The validity of the results of this paper requires only that $\sigma(.)$ to be bounded below on $[0,c]$. □

Consider now the resolvent equation for the operator S_K, $(\lambda - S_K)\psi = \varphi$, where φ is a given function in X_p, $\lambda \in \mathbf{C}$ and the unknown ψ must be sought in $D(S_K)$.

Let $\underline{\sigma}$ be the real number defined by

$$\underline{\sigma} = \text{ess-}\inf\{\sigma(v),\ v \in [0,c]\}.$$

Thus, for $\operatorname{Re}\lambda > -\underline{\sigma}$, a straightforward calculation leads to

$$\begin{aligned} \psi(\mu,v) &= \psi(0,v)\,\exp\left(-\,\frac{\lambda+\sigma(v)}{v}\,\mu\right) \\ &\quad + \frac{1}{v}\int_0^\mu \exp\left(-\,\frac{\lambda+\sigma(v)}{v}\,(\mu-\mu')\right)\varphi(\mu',v)\,d\mu'. \end{aligned} \tag{2.2}$$

Accordingly, for $\mu = a$, we get

$$\begin{aligned} \psi(a,v) &= \psi(0,v)\,\exp\left(-\,\frac{\lambda+\sigma(v)}{v}\,a\right) \\ &\quad + \frac{1}{v}\int_0^a \exp\left(-\,\frac{\lambda+\sigma(v)}{v}\,(a-\mu)\right)\varphi(\mu,v)\,d\mu. \end{aligned} \tag{2.3}$$

In the sequel we shall need the following operators

$$P_\lambda : X_p^0 \longrightarrow X_p^1,\ u \longrightarrow (P_\lambda u)(0,v) := u(0,v)\ \exp\left(-\frac{\lambda+\sigma(v)}{v}\,a\right);$$

$$Q_\lambda : X_p^0 \longrightarrow X_p,\ u \longrightarrow (Q_\lambda u)(0,v) := u(0,v)\ \exp\left(-\frac{\lambda+\sigma(v)}{v}\,\mu\right);$$

$$\begin{cases} \Pi_\lambda : X_p \longrightarrow X_p^1 \\ \varphi \longrightarrow (\Pi_\lambda \varphi)(a,v) \\ := 1/v \displaystyle\int_0^a \exp\left(-\frac{\lambda+\sigma(v)}{v}\,(a-\mu)\right) g(\mu,v)\,d\mu; \end{cases}$$

and finally

$$\begin{cases} R_\lambda : X_p \longrightarrow X_p \\ \varphi \longrightarrow (R_\lambda \varphi)(\mu,v) \\ := 1/v \displaystyle\int_0^\mu \exp\left(-\frac{\lambda+\sigma(v)}{v}\,(\mu-\mu')\right) g(\mu',v)\,d\mu'. \end{cases}$$

Clearly, for λ satisfying $\operatorname{Re}\lambda > -\underline{\sigma}$, the operators P_λ, Q_λ, Π_λ and Ξ_λ are bounded. One checks readily that the norms of P_λ and Q_λ satisfy

$$\begin{aligned} \|P_\lambda\| &\le e^{-a(\operatorname{Re}\lambda+\underline{\sigma})/c}, \\ \|Q_\lambda\| &\le (p\,(\operatorname{Re}\lambda+\underline{\sigma}))^{-1/p}. \end{aligned}$$

Moreover, simple calculations using the Hölder inequality show that

$$\begin{aligned} \|\Pi_\lambda\| &\le (q\,(\operatorname{Re}\lambda+\underline{\sigma}))^{-1/q}, \\ \|R_\lambda\| &\le (\operatorname{Re}\lambda+\underline{\sigma})^{-1}, \end{aligned}$$

where q is the conjugate exponent of p. Using the operators above and the fact that ψ must satisfy the boundary conditions, equation (2.3) may be written as

$$\psi^1 = P_\lambda K \psi^1 + \Pi_\lambda \varphi. \tag{2.4}$$

The solution of this equation reduces to the invertibility of the operator $\mathcal{U}(\lambda) := I - P_\lambda K$. Set

$$\lambda_0 := \begin{cases} -\underline{\sigma} & \text{if } \|K\| \le 1, \\ -\underline{\sigma} + (c\log(\|K\|)/a) & \text{if } \|K\| > 1. \end{cases}$$

Clearly the estimate $\|P_\lambda\| \leq e^{-a(\mathrm{Re}\,\lambda+\underline{\sigma})/c}$ shows that, for $\mathrm{Re}\,\lambda > \lambda_0$, $\|P_\lambda K\| < 1$, i.e., $\{\mathcal{U}(\lambda)\}^{-1}$ exists. Therefore, the solution of (2.4) is given by

$$\psi^1 = \big\{\mathcal{U}(\lambda)\big\}^{-1}\,\Pi_\lambda\varphi = \sum_{n\geq 0}(P_\lambda K)^n\,\Pi_\lambda\varphi. \tag{2.5}$$

On the other hand, equation (2.2) can be written as follows

$$\psi = Q_\lambda K\psi^1 + R_\lambda\varphi. \tag{2.6}$$

Substituting (2.5) in (2.6) we obtain

$$\psi = Q_\lambda K\big\{\mathcal{U}(\lambda)\big\}^{-1}\,\Pi_\lambda\varphi + R_\lambda\varphi.$$

Thus

$$(\lambda - S_K)^{-1} = \sum_{n\geq 0} Q_\lambda K(P_\lambda K)^n\,\Pi_\lambda + R_\lambda. \tag{2.7}$$

This shows the half plane $\{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > \lambda_0\}$ is contained in the resolvent set of S_K, $\rho(S_K)$. Generally, the strip $\{\lambda \in \mathbf{C} : -\underline{\sigma} < \mathrm{Re}\,\lambda \leq \lambda_0\}$ is not included in $\rho(S_K)$; this depends on the size of $\|K\|$.

Recall that the operator A_K is a bounded perturbation of S_K, that is, $A_K = S_K + B$ where B is the bounded operator defined by

$$\begin{cases} B : X_p \longrightarrow X_p \\ \psi \longrightarrow (B\psi)(\mu, v) = \displaystyle\int_0^c r(\mu, v, v')\ \psi(\mu, v')\,dv', \end{cases} \tag{2.8}$$

with $\mu \in [0, a]$, $v \in [0, c]$, $a > 0$, $b > 0$ and the kernel $r : (0, a) \times (0, c) \times (0, c) \to \mathbf{R}$ is assumed to be measurable.

Observe that the operator B acts only on the velocity v', so μ may be viewed merely as a parameter in $[0, a]$. Hence, we may consider B as a function

$$B(.) : \mu \in [0, a] \longrightarrow B(\mu) \in \mathcal{L}(L_p([0, c];\, dv))$$

where $\mathcal{L}(L_p([0, c];\, dv))$ denotes the set of all bounded linear operators on $L_p([0, c];\, dv)$.

For our subsequent analysis we need the following hypothesis:

($\mathcal{A}1$) The function $B(.)$ is strongly measurable, there exists a compact subset $\mathcal{C} \subseteq \mathcal{L}(L_p([0,c];\, dv))$ such that $B(\mu) \in \mathcal{C}$ almost everywhere on $[0,a]$, and $B(\mu) \in \mathcal{K}(L_p([0,c];\, dv))$ almost everywhere on $[0,a]$,

where $\mathcal{K}(L_p([0,c];\, dv))$ denotes the set of compact operators on $L_p([0,c]; dv)$.

The interest in operators of the form (2.8) which satisfy ($\mathcal{A}1$) lies in the following lemma due to Mokhtar-Kharroubi [**17**, Proposition 4.1, p. 56], see also [**5**].

Lemma 2.1. *Let B be a linear operator defined by* (2.8). *If the hypothesis* ($\mathcal{A}1$) *is satisfied, then B can be approximated, in the uniform topology, by a sequence $(B_n)_n$ of linear operators with kernels of the form*

$$\sum_{i=1}^{n} \eta_i(\mu)\theta_i(v)\beta_i(v')$$

where $\eta_i \in L_\infty([0,a];\, d\mu)$, $\theta_i \in L_p([0,c];\, dv)$, $\beta_i \in L_q([0,c];\, dv)$ and $q = p/(p-1)$.

3. Generation results. The goal of this section is to discuss the well-posedness of the problem (1.1)–(1.4). We first show that the streaming operator S_K generates a strongly continuous semigroup on X_p, $1 \le p < \infty$, and we derive its explicit expression for $t > 0$. More precisely, we have:

Theorem 3.1. *Let $p \in [1,\infty)$, and assume that transition operator K is given by* (2.1). *Then S_K generates a C_0-semigroup given by*

$$U^K(t) = U_0(t) + \sum_{n \ge 0} J_n(t),$$

where

$$U_0(t)\varphi(\mu, v) = \exp(-\sigma(v)t)\varphi(\mu - vt, v)\ \chi_{[0,(\mu/v)]}(t)$$

and

$$
\begin{aligned}
&(J_n(t)\varphi)(\mu, v_0)\\
&\quad = \exp\left(-\frac{\sigma(v_0)}{v_0}\mu\right)\int_0^c\cdots\int_0^c dv_1\ldots dv_{n+1}\,\Pi_{i=1}^{n+1}\,\kappa(v_{i-1}, v_i)\\
&\qquad \times \exp\left(-a\sum_{i=1}^{n+1}\frac{\sigma(v_i)}{v_i} - \sigma(v_{n+1})\left[t - \frac{\mu}{v_0} - \sum_{i=1}^{n+1}\frac{a}{v_i}\right]\right)\\
&\qquad \times \varphi\left(a - v_{n+1}\left[t - \frac{\mu}{v_0} - \sum_{i=1}^{n+1}\frac{a}{v_i} + \frac{a}{v_{n+1}}\right], v_{n+1}\right)\\
&\qquad \times \chi_{[(\mu/v_0)+\sum_{i=1}^{n+1}a/(v_i)-a/(v_{n+1}),\mu/(v_0)+\sum_{i=1}^{n+1}a/(v_i)]}(t)
\end{aligned}
$$

with $t \geq 0$ *and* $\varphi \in X_p$.

Remark 3.1. Note that the semigroup theory of A_K is similar to that of transport operator with abstract boundary conditions in slab geometry. It is well known that, for contractive or conservative and positive boundary conditions ($\|Ku\| = \|u\|$, $0 \leq K$), the theory is a consequence of the Lumer-Phillips theorem, see, for example, [**1, 4, 8, 25**]. For multiplying boundary conditions and $p = 1$, the theory follows from an argument due to Batty and Robinson [**15**, Theorem 5.3]. Actually, the result also holds true for $p \in (1, \infty)$, [**10**]. It was derived by means a renormalization argument as in [**8**, p. 479]. Note however that a semigroup theory is available now for general kinetic equations for non divergence free external forces [**21**]. □

Proof of Theorem 3.1. The fact that S_K generates a C_0-semigroup follows from [**10**]. Our next step is to exhibit its analytic expression. To do so, let $u \in X_p^1$. Using (2.1), we get

$$
(P_\lambda K u)(a, v) = \exp\left(-\frac{\lambda + \sigma(v)}{v}\,a\right)\int_0^c \kappa(v, v')\; u(a, v')\,dv'
$$

then

$$
\begin{aligned}
&(KP_\lambda K u)(0, v)\\
&\quad = \int_0^c dv_1\,\kappa(v, v_1)\,\exp\left(-\frac{\lambda + \sigma(v_1)}{v_1}\,a\right)\int_0^c dv_2\,\kappa(v_1, v_2)\,u(a, v_2)\\
&\quad = \int_0^c dv_1\int_0^c dv_2\,\kappa(v, v_1)\,\kappa(v_1, v_2)\exp\left[\left(-\frac{\lambda}{v_1} - \frac{\sigma(v_1)}{v_1}\right)a\right]u(a, v_2).
\end{aligned}
$$

This leads to

$$(P_\lambda K)^2 u(a,v) = \exp\left(-\frac{\lambda+\sigma(v)}{v}\,a\right)\int_0^c dv_1 \int_0^c dv_2\,\kappa(v,v_1)\,\kappa(v_1,v_2) \times \exp\left[\left(-\frac{\lambda}{v_1}-\frac{\sigma(v_1)}{v_1}\right)a\right]u(a,v_2).$$

Applying again the operator $P_\lambda K$ to the last expression, we get

$$\begin{aligned}((P_\lambda K)^3 u)(a,v) &= \exp\left(-\frac{\lambda+\sigma(v)}{v}\,a\right)\int_0^c\int_0^c\int_0^c dv_1\,dv_2\,dv_3 \\ &\quad\times \kappa(v,v_1)\,\kappa(v_1,v_2)\,\kappa(v_2,v_3) \\ &\quad\times \exp\left(-\lambda\left[\frac{1}{v_1}+\frac{1}{v_2}\right]a - \left[\frac{\sigma(v_1)}{v_1}+\frac{\sigma(v_2)}{v_2}\right]a\right)u(a,v_3).\end{aligned}$$

Let $n \in \mathbf{N}$. By induction, we obtain

$$\begin{aligned}((P_\lambda K)^n u)(a,v_0) &= \exp\left(-\frac{\lambda+\sigma(v_0)}{v_0}\,a\right)\int_0^c\cdots\int_0^c dv_1\ldots dv_n\ \Pi_{i=1}^n\ \kappa(v_{i-1},v_i) \\ &\quad\times \exp\left(-\lambda\left[\sum_{i=1}^n\frac{1}{v_i}-\frac{1}{v_n}\right]a - \left[\sum_{i=1}^n\frac{\sigma(v_i)}{v_i}-\frac{\sigma(v_n)}{v_n}\right]a\right)u(a,v_n).\end{aligned}$$

Now, making the change of variables $t=(a-\mu)/v$, then, for $\varphi \in X_p$, we have

$$\begin{aligned}(\Pi_\lambda\varphi)(a,v) &= \frac{1}{v}\int_0^a \exp\left[-\frac{\lambda+\sigma(v)}{v}\,(a-\mu)\right]\varphi(\mu,v)\,d\mu \\ &= \int_0^{+\infty}\exp(-[\lambda+\sigma(v)]t)\ \varphi(a-vt,v)\ \chi_{[0,(a/v)]}(t)\,dt.\end{aligned}$$

Hence,

$$\begin{aligned}((P_\lambda K)^n\Pi_\lambda\varphi)(a,v_0) &= \exp\left(-\frac{\lambda+\sigma(v_0)}{v_0}\,a\right)\int_0^c\cdots\int_0^c dv_1\ldots dv_n\ \Pi_{i=1}^n\ \kappa(v_{i-1},v_i) \\ &\quad\times \exp\left(-\lambda\left[\sum_{i=1}^n\frac{1}{v_i}-\frac{1}{v_n}\right]a - \left[\sum_{i=1}^n\frac{\sigma(v_i)}{v_i}-\frac{\sigma(v_n)}{v_n}\right]a\right) \\ &\quad\times\int_0^{+\infty}\exp(-[\lambda+\sigma(v_n)]t)\ \varphi(a-v_nt,v_n)\ \chi_{[0,(a/v_n)]}(t)\,dt.\end{aligned}$$

This will be written in the form

$$\begin{aligned}
&((P_\lambda K)^n \,\Pi_\lambda \varphi)(a, v_0) \\
&\quad = \int_0^{+\infty} dt' \exp(-\lambda t') \exp\left(- \frac{\lambda + \sigma(v_0)}{v_0}\, a\right) \\
&\qquad \times \int_0^c \cdots \int_0^c dv_1 \dots dv_n \, \Pi_{i=1}^n \, \kappa(v_{i-1}, v_i) \\
&\qquad \times \exp\left(- \lambda \left[\sum_{i=1}^n \frac{1}{v_i} - \frac{1}{v_n} \right] a - \left[\sum_{i=1}^n \frac{\sigma(v_i)}{v_i} - \frac{\sigma(v_n)}{vn} \right] a \right) \\
&\qquad \times \exp(-\sigma(v_n) t') \, \varphi(a - v_n t', v_n) \, \chi_{[0, (a/v_n)]}(t').
\end{aligned}$$

Through making the change of variables $t = t' + \sum_{i=0}^{n-1} a/v_i$, we get

$$\begin{aligned}
&((P_\lambda K)^n \,\Pi_\lambda \varphi)(a, v_0) \\
&\quad = \int_0^{+\infty} dt \exp(-\lambda t) \exp\left(- \frac{\sigma(v_0)}{v_0}\, a\right) \\
&\qquad \times \int_0^c \cdots \int_0^c dv_1 \dots dv_n \, \Pi_{i=1}^n \, \kappa(v_{i-1}, v_i) \\
&\qquad \times \exp\left(- \left[\sum_{i=1}^n \frac{\sigma(v_i)}{v_i} - \frac{\sigma(v_n)}{v_n} \right] a - \sigma(v_n) \left[t - \sum_{i=0}^{n-1} \frac{a}{v_i} \right] \right) \\
&\qquad \times \varphi\left(a - v_n \left[t - \sum_{i=0}^{n-1} \frac{a}{v_i} \right], v_n \right) \chi_{[\sum_{i=0}^{n-1} (a/v_i),\, \sum_{i=0}^{n} (a/v_i)]}(t).
\end{aligned}$$

Now, applying the operator $Q_\lambda K$ to $((P_\lambda K)^n \Pi_\lambda)$, easy calculations give

$$\begin{aligned}
(3.1)\quad &(Q_\lambda K (P_\lambda K)^n \,\Pi_\lambda \varphi)(\mu, v_0) \\
&\quad = \int_0^{+\infty} dt \exp(-\lambda t) \exp\left(- \frac{\sigma(v_0)}{v_0}\, \mu\right) \\
&\qquad \times \int_0^c \cdots \int_0^c dv_1 \dots dv_{n+1} \, \Pi_{i=1}^{n+1} \, \kappa(v_{i-1}, v_i) \\
&\qquad \times \exp\left(- a \sum_{i=1}^{n+1} \frac{\sigma(v_i)}{v_i} - \sigma(v_{n+1}) \left[t - \frac{\mu}{v_0} - \sum_{i=1}^{n+1} \frac{a}{v_i} \right] \right)
\end{aligned}$$

$$\times \varphi\left(a - v_{n+1}\left[t - \frac{\mu}{v_0} - \sum_{i=1}^{n+1}\frac{a}{v_i} + \frac{a}{v_{n+1}}\right], v_{n+1}\right)$$
$$\times \chi_{[(\mu/v_0)+\sum_{i=1}^{n+1}(a/v_i)-(a/v_{n+1}),(\mu/v_0)+\sum_{i=1}^{n+1}(a/v_i)]}(t).$$

Arguing as above, we also obtain

$$(3.2)\quad (R_\lambda\varphi)(\mu, v) = \int_0^{+\infty} dt\, \exp(-[\lambda+\sigma(v)]t)\varphi(\mu - vt, v)\chi_{[0,(\mu/v)]}(t).$$

Now, equations (2.7), (3.1) and (3.2) together with the uniqueness of the Laplace transform, cf. Lemma 15 in [**6**, p. 626], give, by identification, the desired result. $\square$

The last result says that S_K generates a strongly continuous semigroup on X_p with $1 \le p < +\infty$. Since B is assumed to be bounded, it follows from Phillips's perturbation theorem for strongly continuous semigroups [**7**] that $A_K = S_K + B$ generates also a strongly continuous semigroup $(V^K(t))_{t\ge 0}$ given by the Dyson-Phillips expansion. This ensures the well-posedness of the Cauchy problem (1.1)–(1.4).

4. Asymptotic of $(V^H(t))_{t\ge 0}$. In the present section, we are interested in the large time behavior ($t \to \infty$) of the solution to the problem (1.1)–(1.4). The main tool in this task is the following result due to Weis [**27**, Theorem 3.1].

Theorem 4.1. *Let (Ω, μ) be a measure space, and let $(S(t))_{t\ge 0}$ be a strongly continuous semigroup on $L_p(\Omega, \mu)$, $1 \le p \le \infty$, with infinitesimal generator Z. Let W be a bounded operator, and denote by $(T(t))_{t\ge 0}$ the semigroup generated by $Z + W$. Assume for some $n \in \mathbf{N}$ all products of the form*

$$WS(s_1)WS(s_2)\cdots WS(s_n)W, \quad s_i > 0$$

are strictly singular. Then

$$r_e(T(t)) = r_e(S(t)) \quad \textit{for all} \quad t,$$

i.e., $(S(t))_{t\ge 0}$ and $(T(t))_{t\ge 0}$ have the same essential type.

Remark 4.1. The class of strictly singular operators was introduced by Kato as a generalization of the notion of compact operators. If X is a Banach space, the set of strictly singular operators is a closed two-sided ideal of $\mathcal{L}(X)$ (the algebra of all bounded operators on X) which contains the set of compact operators, see, for instance, [**2** and the references therein]. It should be noticed that, following Pelczynski [**18**], the classes of strictly singular operators and weakly compact operators coincide on L_1 spaces. Note also that, following a result due to Calkin [**2**, p. 81] which asserts the uniqueness of the ideal compact operators on separable Hilbert spaces, the ideal of strictly singular operators and that of compact ones coincide on L_2 spaces. $\square$

Throughout this section the boundary operator K will be assumed to obey the assumption:

($\mathcal{A}2$) The transition operator $K \in \mathcal{L}(X_p^1; X_p^0)$ is compact or weakly compact following that $p \in]1, \infty[$ or $p = 1$.

Now we are ready to state our main result.

Theorem 4.2. *Assume that the collision operator B and the transition operator K satisfy the assumptions* ($\mathcal{A}1$) *and* ($\mathcal{A}2$), *respectively. Then, for all $t > 0$,*

$$r_e(V^K(t)) = r_e(U^K(t)) \quad \textit{for all} \quad t > 0,$$

i.e., $(U^K(t))_{t\geq 0}$ and $(V^K(t))_{t\geq 0}$ have the same essential type ω_e.

As already noticed in the introduction, the interest of Theorem 4.2 lies in the fact that it implies $\omega_e(V^K) = \omega_e(U^K)$, which is very useful for the description, cf. (1.3), of the large time behavior of $(V^K(t))_{t\geq 0}$. In fact, for any $t > 0$, the part of the spectrum of $V^K(t)$ outside the essential spectral disc (disc of center 0 and radius $e^{t\omega_e(U^K)}$) of $U^K(t)$ can consist only of eigenvalues of finite algebraic multiplicity. Assuming the existence of such eigenvalues, then the semigroup $(V^K(t))_{t\geq 0}$ can be decomposed into two parts, the first containing the time development of finitely many eigenmodes, the second being of faster decay.

To prove Theorem 4.2 we need the following two lemmas.

Lemma 4.1. *Let B_1 and B_2 be two collision operators satisfying the hypothesis* $(\mathcal{A}1)$*. Then, for all $t > 0$, the operator $B_1 U_0(t) B_2$ is compact, respectively weakly compact, on X_p, $1 < p < \infty$, respectively X_1.*

Proof. By Lemma 3.1, it is sufficient to prove the result for a one rank collision operators which we denote again by B_1 and B_2, i.e., for $i \in \{1, 2\}$,

$$X_p \ni \varphi \to B_i\varphi(\mu, v) = \int_0^c \eta_i(\mu)\ \theta_i(v)\ \beta_i(v')\ \varphi(\mu, v')\, dv'$$

where $\eta_i(.) \in L^\infty([0,a], d\mu)$, $\theta_i(.) \in L_p([0,c], dv)$, $\beta_i(.) \in L_q([0,c], dv)$ and $q = p/(p-1)$.

To do so, let $t > 0$ be a real and $\varphi \in X_p$, then

$$\begin{aligned}(U_0(t)(B_2\varphi))(\mu, v) &= \exp(-\sigma(v)t)\ (B_2\varphi)(\mu - vt, v)\ \chi_{[0,(\mu/v)]}(t)\\ &= \exp(-\sigma(v)t)\ \eta_2(\mu - vt)\ \theta_2(v)\\ &\quad \times \int_0^c \beta_2(v')\ \varphi(\mu - vt, v')\ \chi_{[0,(\mu/v)]}(t)\, dv'.\end{aligned}$$

Therefore,

$$\begin{aligned}&(B_1 U_0(t) B_2\varphi)(\mu, v)\\ &\quad = \eta_1(\mu)\theta_1(v) \int_0^c \beta_1(v'')\ (U_0(t)B_2\varphi)(\mu, v'')\, dv''\\ &\quad = \eta_1(\mu)\theta_1(v) \int_0^c dv''\, \beta_1(v'')\exp(-\sigma(v'')t)\ \eta_2(\mu - v''t)\,\theta_2(v'')\\ &\qquad \times \int_0^c dv'\, \beta_2(v')\,\varphi(\mu - v''t, v')\,\chi_{[0,(\mu/v'')]}(t).\end{aligned}$$

Making the change of variables $\mu' = \mu - v''t$, we get

$$\begin{aligned}&(B_1 U_0(t) B_2\varphi)(\mu, v)\\ &\quad = \frac{1}{t}\ \eta_1(\mu)\,\theta_1(v) \int_0^a d\mu'\ h\left(\frac{\mu-\mu'}{t}\right) \exp\left(-\,\sigma\left(\frac{\mu-\mu'}{t}\right)t\right)\eta_2(\mu')\\ &\qquad \times \int_0^c dv'\, \beta_2(v')\,\varphi\,(\mu', v')\,\chi_{[\mu-ct,\mu]}(\mu'),\end{aligned}$$

where $h(.) = \beta_1(.)\,\theta_2(.) \in L_1([0,c])$.

Note that $BU_0(t)B$ is an integral operator having the kernel

$$k(t)(\mu,v,\mu',v') \\ = \frac{1}{t}\,\eta(\mu)\theta(v)\,h\left(\frac{\mu-\mu'}{t}\right)\exp\left(-\,\sigma\left(\frac{\mu-\mu'}{t}\right)t\right)\eta(\mu')\,\beta(v')\,\chi_{[0,\mu]}(\mu').$$

It may be decomposed as

$$BU_0(t)B = \mathcal{O}_t \circ \mathcal{J},$$

where
$$\begin{cases} \mathcal{J} : X_p \longrightarrow L_p([0,a]), \\ \varphi \longrightarrow (\mathcal{J}\varphi)(\mu) := \displaystyle\int_0^c dv\,\beta(v)\,\varphi(\mu,v), \end{cases}$$

while $\mathcal{O}_t$ is given by

$$\begin{cases} \mathcal{O}_t : L_p([0,a]) \longrightarrow X_p \\ \varphi \longrightarrow (\mathcal{O}_t\varphi)(\mu,v) := (1/t)\ \eta(\mu)\theta(v) \displaystyle\int_0^a d\mu'\ h((\mu-\mu')/t) \\ \times \exp\left(-\,\sigma\left(\dfrac{\mu-\mu'}{t}\right)t\right)\eta(\mu')\,\varphi(\mu')\,\chi_{[0,\mu]}(\mu'). \end{cases}$$

It is clear that $\mathcal{J}$ is a bounded operator. So the decomposition above reduces the proof to the compactness of the operator $\mathcal{O}_t$ for $1 < p < \infty$ or its weak compactness for $p = 1$. To this end, let $\varphi \in L_p([0,a])$; then we have

$$\begin{aligned} &\|\mathcal{O}_t\varphi\|_{X_p}^p \\ &= \int_{[0,a]\times[0,c]} d\mu\,dv \left| \frac{1}{t}\,\eta(\mu)\theta(v) \int_0^a d\mu'\,h\left(\frac{\mu-\mu'}{t}\right) \right. \\ &\qquad\qquad \left. \times \exp\left(-\,\sigma\left(\frac{\mu-\mu'}{t}\right)t\right)\ \eta(\mu')\ \varphi(\mu')\ \chi_{[0,\mu]}(\mu') \right|^p \\ &\leq \frac{1}{t}\,(\text{ess-sup}\,|\eta(\mu)|)^2 \int_0^c dv\,|\theta(v)|^p \int_0^a d\mu \left| \int_0^a d\mu'\,h\left(\frac{\mu-\mu'}{t}\right)\ \varphi(\mu') \right|^p \\ &\qquad \eta \in L^\infty([0,a],d\mu) \quad \text{and} \quad \sigma(.) \text{ nonnegative}. \end{aligned}$$

Extending to $\mathbf{R}$ the function $h(.)$, respectively $\varphi(.)$, by zero outside $[0,c]$, respectively $[0,a]$, and applying the Young inequality we get

$$\|\mathcal{O}_t\varphi\|_{X_p}^p \leq \frac{1}{t}\,(\text{ess-sup}\,|\eta(\mu)|)^2\,\|\theta\|_{L_p([0,c])}^p \int_0^a d\mu\,(\|h\|_{L_1(\mathbf{R})}\,\|\varphi\|_{L_1(\mathbf{R})})^p.$$

But the use of the Hölder inequality gives $\|\varphi\|_{L_1(\mathbf{R})} \leq a^{1/q}\,\|\varphi\|_{L_p([0,a])}$ which implies

$$\|\mathcal{O}_t\|_{X_p} \leq C\,\|\theta\|_{L_p([0,c])}\,\|h\|_{L_1(\mathbf{R})},$$

where C is a constant depending on t, a and η. This proves that $\mathcal{O}_t$ depends continuously, in the uniform operators topology, on $\theta(.)$ in $L_p([0,c])$ and on $h(.)$ in $L_1(\mathbf{R})$. Hence, by approximating $\theta(.)$, respectively $h(.)$, in the L_p, respectively L_1, norm by continuous functions with compact support, one sees that $\mathcal{O}_t$ is a limit, in the uniform operator topology, of integral operators with bounded kernels. So $\mathcal{O}_t$ is compact on X_p for $1 < p < \infty$, use Theorem 11.6 in [**12**, p. 275], and weakly compact on X_1, use Corollary 11 in [**6**, p. 294]. This ends the proof. $\square$

Lemma 4.2. *Assume that B and K satisfy the assumptions $(\mathcal{A}1)$ and $(\mathcal{A}2)$, respectively. Then, for all $s > 0$, the operator $BU^K(s)B$ is strictly singular on X_p, $1 \leq p < \infty$.*

Proof. Let $s > 0$ be a real. The operator $BU^K(s)B$ reads

$$BU^K(s)B = BU_0(s)B + \sum_{n\geq 0} BJ_n(s)B.$$

Let us remark that, for each integer $n \geq 0$, in the expression of $J_n(s)$, cf. Theorem 3.1, the operator K appears $(n+1)$ times. Thus, it follows from $(\mathcal{A}2)$ that $J_n(s)$ is compact on X_p for $p \in\,]1,\infty[$ and weakly compact on X_1. So, it is strictly singular on X_p for each p belonging to $[1,\infty)$, use Remark 4.1. Next, since the ideal of strictly singular operators on X_p is a closed two-sided ideal of $\mathcal{L}(X_p)$, we infer that $\sum_{n\geq 0} BJ_n(s)B$ is also strictly singular. Now the use of Lemma 4.1 together with Remark 4.1 ends the proof. $\square$

Proof of Theorem 4.2. It follows from Lemma 4.2 and Theorem 4.1. $\square$

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Faculté des Sciences de Sfax, Département de Mathématiques, Route de Soukra km 3.5, BP 802, Tunisie

Département de Mathématiques, Université de Corse, Quartier Grossetti, BP 52, 20250 Corte, France
E-mail address: `latrach@univ-corse.fr`

Current address: Université Blaise Pascal (Clermont II), Laboratoire de Mathématiques (CNRS UMR 6620), 24 avenue des Landais, 63117 Aubière, France
E-mail address: `Khalid.Latrach@math.univ-bpclermont.fr`

Faculté des Sciences de Sfax, Département de Mathématiques, Route de Soukra km 3.5, BP 802, Tunisie and Dépt. de Math., Université de Corse, Quartier Grossetti, BP 52, 20250 Corte, France