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THE NEWTON-KANTOROVICH APPROXIMATIONS FOR NONLINEAR SINGULAR INTEGRAL EQUATIONS WITH SHIFT

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ABSTRACT. We obtain a result on the convergence for the Newton-Kantorovich method applied to a class of singular integral equations with shift (SIES). We study the convergence of the approximations in the case the operator associated to the equation has a local Lipschitz derivative.

1. Introduction. There is a large literature on the classical theory of nonlinear singular integral equations (SIE) (see [**6**], [**7**], [**15**], [**16**]) and on the successful theory of the non linear singular integral equations with shift (SIES) (see for example [**10**], [**11**], [**18**]).

The approximate solutions of nonlinear equations, involving integral operators on closed curves, have been intensively investigated by using many approximation methods, specially the modified Newton-Kantorovich method, the method of reduction, of collocation and of mechanical quadratures. In the theory of the approximate solutions of nonlinear singular integral equations, many results are obtained by applying the modified Newton-Kantorovich method under the classical hypothesis of global Lipschitz continuity of the derivative of the operator associated to the equation (see [**1**], [**2**]).

In this paper, our aim is to apply the Newton-Kantorovich method (faster than the modified method) to a class of singular integral equations under the weaker hypothesis of local Lipschitz continuity of the derivative.

In particular we study, in a generalized Hölder space $\mathcal{H}_\varphi(\Gamma)$, the

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equation

$$\begin{aligned} C(u)(t) :&= F(t,u(t)) + G(\alpha(t),u(\alpha(t))) \\ &\quad - \frac{1}{\pi i}\int_\Gamma \left(\frac{H(\tau,u(\tau))}{\tau - t} + \frac{M(\tau,u(\tau))}{\tau-\alpha(t)}\right) d\tau \ = 0\,, \end{aligned}$$

where Γ is a Lyapunov curve (see for example [**6**], [**15**]) in the complex plane $\mathbf{C}$, α is a Carleman shift ($\alpha(\alpha(t) = t$ for all $t \in \Gamma$) which preserves the orientation on Γ and F, G, H, M are functions from Γx$\mathbf{C}$ in $\mathbf{C}$.

For a fixed function $u_0 \in \mathcal{H}_\varphi(\Gamma)$, we study the problem of existence and unicity of the linearized equation $C'(u_0)h = f$, $f \in \mathcal{H}_\varphi(\Gamma)$, using classical results of linear operator theory and of fixed point theory. We observe that in [**1**] and [**2**] the same problem has been transformed into a Riemann-Hilbert boundary value problem.

2. Generalized Hölder spaces and auxiliary results. In this section, we recall the theorem of convergence for the Newton-Kantorovich method [**20**] which we use and some definitions and results from the theory of singular integral equations which we will need in the sequel.

Let X and Y be Banach spaces, $B(u_0,R)$ the closed ball in X with center u_0 and radius R.

Let $F : B(u_0,R) \subseteq X \to Y$ be a nonlinear operator Fréchet differentiable at every interior point of $B(u_0,R)$. Suppose that $F'(u_0)$ is invertible and F' satisfies the condition

$$\begin{gathered} ||F'(u) - F'(v)||_{\mathcal{L}(X,Y)} \le k(r)||u-v||_X \\ \forall\, u, v \in B^\circ(u_0,r), \quad 0 < r \le R. \end{gathered}$$

Let us define the following constants and numerical functions

$$\begin{gathered} \beta = ||F'(u_0)^{-1}F(u_0)||_X, \qquad \gamma = ||F'(u_0)^{-1}||_{\mathcal{L}(Y,X)}, \\ \omega(r) = \int_0^r k(t)\,dt, \qquad \phi(r) = \gamma + \beta\int_0^r \omega(t)\,dt - r. \end{gathered}$$

Theorem 2.1. [**20**] *Suppose that the function ϕ has a unique positive root r_* in $[0,R]$ and $\phi(R) \le 0$, then the equation $F(u) = 0$ has a unique* *solution* u_* *in* $B(u_0, R)$ *and the Newton-Kantorovich approximations*

$$u_n = u_{n-1} - F'(u_{n-1})^{-1} F(u_{n-1}), \quad n \in N$$

are defined for all $n \in N$*, belong to* $B(u_0, r_*)$ *and converge to* u_**. Moreover, the following estimate holds*

$$(1) \qquad ||u_{n+1} - x_n|| \leq r_{n+1} - r_n; \qquad ||u_* - x_n|| \leq r_* - r_n,$$

where the sequence $(r_n)_{n\in N}$*, increasing and convergent to* r_**, is defined by the recurrence formula*

$$r_0 = 0, \qquad r_n = r_{n-1} - \frac{\phi(r_{n-1})}{\phi'(r_{n-1})}, \quad n \in N\,.$$

In the sequel, Γ denotes a closed Lyapunov curve of lenght $\Lambda(\Gamma)$.

We recall the definition of a generalized Hölder space (see for example [**4**], [**7**]). Let $\varphi : (0, +\infty) \to (0, +\infty)$ be a continuous increasing function such that

(i) $\lim\limits_{s\to 0^+} \varphi(s) = 0$;

(ii) $\varphi(s) > 0 \quad \forall\, s > 0$;

(iii) $\dfrac{\varphi(s)}{s}$ is almost decreasing (i.e., there exists a constant $C > 0$ such that for all s, t with $0 < s \leq t$ we have $\varphi(s) \leq C\varphi(t)$);

$$(iv) \qquad \sup_{0\leq\delta\leq\Lambda} \frac{1}{\varphi(\delta)} \int_0^\delta \frac{\varphi(s)}{s}\, ds < +\infty;$$

$$(v) \qquad \sup_{0\leq\delta\leq\Lambda} \frac{\delta}{\varphi(\delta)} \int_\delta^\Lambda \frac{\varphi(s)}{s^2}\, ds < +\infty.$$

It is simple to verify that, under these conditions, φ is a subadditive function, i.e. $\varphi(s + t) \leq \varphi(s) + \varphi(t)$ for all $t, s > 0$.

We denote by $\mathcal{H}_\varphi := \mathcal{H}_\varphi(\Gamma)$ the collection of all functions u defined on Γ satisfying the condition

$$\omega_u(\delta) \leq l\varphi(\delta), \quad \forall\, 0 < \delta \leq \Lambda(\Gamma),$$

for some $l > 0$, where ω_u is the classical modulus of continuity of u. We endow $\mathcal{H}_\varphi$ by the norm

$$||u|| := ||u||_\infty + ||u||_\varphi,$$

where

$$||u||_\varphi := \sup_{0<\delta\leq\Lambda(\Gamma)} \frac{\omega_u(\delta)}{\varphi(\delta)} = \sup_{0<|t-s|\leq\Lambda(\Gamma)} \frac{|u(t)-u(s)|}{\varphi(|t-s|)}.$$

and $||.||_\infty$ is the usual norm in the space $\mathcal{C}(\Gamma)$ of the continuous functions defined on Γ.

It is well known that $\mathcal{H}_\varphi$ forms a commutative Banach algebra.

Let $u : \Gamma \to \mathbf{C}$ be an integrable function on Γ. We consider the operator

$$Su(t) := \frac{1}{\pi i}\int_\Gamma \frac{u(\tau)}{\tau - t}\,d\tau = \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \frac{u(\tau)}{\tau - t}\,d\tau, \tag{2}$$

where the integral is understood in the sense of the Cauchy principal value.

The function u is usually referred to as the *density* of the singular integral and the expression $\frac{d\tau}{\tau-t}$ as the *Cauchy kernel*.

The operator S, which associates Su to u, is the classical *singular integral operator*.

It is well known that S is a bounded operator in the classical Hölder space $\mathcal{H}_\theta$ $(0<\theta<1)$ and in the space $\mathcal{L}^p(\Gamma)$ $(1<p<+\infty)$ (see [**11**], [**14**]); we show that, under some assumptions on φ, S is a bounded operator in the generalized Hölder space as well.

We begin by observing that, if

$$\int_0^\Lambda \frac{\varphi(s)}{s}\,ds < +\infty,$$

then Su is well defined for every $u \in \mathcal{H}_\varphi$ (see [**7**]).

We recall the following result:

Theorem 2.2. **[7]** *If φ satisfies the hypotheses $(i)-(v)$, then the operator S maps $\mathcal{H}_\varphi$ into $\mathcal{H}_\varphi$ and is bounded.*

Throughout this paper, φ is a fixed function satisfying conditions (i), (ii), (iii) (iv) and (v). Moreover $S^2 = I$ and the commutator $[aI, S] : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$ defined by

$$[aI, S] := aS - SaI$$

is a compact operator for every $a \in \mathcal{H}_\varphi$ (see [**10**]).

The shift operator plays an important role in the theory of the singular integral equations with shift. We begin by recalling the definition of a shift function.

A diffeomorphism $\alpha : \Gamma \to \Gamma$ whose derivative belongs to some classical Hölder space and does not vanish on Γ, is called a *shift function* or briefly *a shift*.

A shift α which preserves the orientation on Γ and such that all the points of the curve are periodic of multiplicity 2 (i.e., $\alpha^2(t) = \alpha(\alpha(t)) = t$ for all $t \in \Gamma$), is called a *Carleman shift*.

For the classification of the shift and its properties, see for example [**10**].

Let α be a shift. The operator $W_\alpha : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$ defined by

$$W_\alpha u(t) := u(\alpha(t))$$

is called a *shift operator*.

Theorem 2.3. *W_α is a bounded invertible operator from $\mathcal{H}_\varphi$ into $\mathcal{H}_\varphi$.*

Proof. It is trivial to verify that W_α is invertible with inverse $W_\alpha^{-1}(u)(t) = W_{\alpha^{-1}}$. Now we prove that W_α is a bounded operator. Since

$$|\alpha(t) - \alpha(s)| = \left| \int_s^t \alpha'(\tau) \right| \leq C|t-s| \tag{3}$$

with $M := \sup_{t\in\Gamma}|\alpha'(t)|$, we have

$$\varphi(|\alpha(t)-\alpha(s)|) \leq \varphi(M|t-s|) \leq \varphi((1+[M])|t-s|)$$

where [.] is the entire part of a real number.

From the subadditivity of φ, it follows that, for all $t, s \in \Gamma$

$$|W_\alpha u(t) - W_\alpha u(s)| \leq (1+[M])\varphi(|t-s|)||u||_\varphi$$

and

$$||W_\alpha u||_\varphi \leq B||u||_\varphi, \quad B := 1+[M].$$

Finally, the assertion follows from

$$||W_\alpha u||_\infty = ||u||_\infty \quad \forall\, u \in \mathcal{H}_\varphi. \qquad \square$$

It is evident that, if $\alpha^2(t) = t$ for all $t \in \Gamma$, we have $W^2 = I$.

The following proposition is a classical result of the theory of the singular integral operators (see for example [**10**]).

Proposition 2.1. *If we set*

$$\delta(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ preserves the orientation on } \Gamma \\ -1 & \text{if } \alpha \text{ changes the orientation on } \Gamma \end{cases};$$

we have

$$WS \cong \delta(\alpha) SW \quad \text{in} \quad \mathcal{H}_\varphi$$

where $\cong$ denotes the following relation of equivalence

$$A \cong B \quad \Leftrightarrow \quad A - B \text{ is a compact operator.}$$

We conclude this section with the superposition operator (see for example [**4**]). Given a function $F : \Gamma \mathrm{x} \mathbf{C} \to \mathbf{C}$, the operator $\mathcal{F}$ generated by F

$$\mathcal{F}(u)(t) := F(t, u(t)), \quad u \in \mathcal{H}_\varphi,\ t \in \Gamma$$

is the classical *superposition operator*.

In the next proposition (for the proof see [**4**]), we give some conditions to ensure that the operator $\mathcal{F}$ maps $\mathcal{H}_\varphi$ into $\mathcal{H}_\varphi$, is Fréchet differentiable and $\mathcal{F}'$ satisfies a local Lipschitz condition.

Proposition 2.2. *Let u_0 be a fixed function of $\mathcal{H}_\varphi$.*

Assume that the first and second partial derivatives F_u and F_{uu} of F with respect to u exist at every point of $\Gamma \mathrm{x} \mathbf{C}$, and that there are positive and increasing functions $c_i : [0, R] \to [0, +\infty)$ $(1 \le i \le 6, R > 0)$ such that

$$|F(t, u) - F(s, v)| \le c_1(r)\varphi(|s - t|) + c_2(r)|u - v|, \tag{4}$$

$$|F_u(t, u) - F_u(s, v)| \le c_3(r)\varphi(|s - t|) + c_4(r)|u - v|, \tag{5}$$

$$|F_{uu}(t, u) - F_{uu}(s, v)| \le c_5(r)\varphi(|s - t|) + c_6(r)|u - v|, \tag{6}$$

$\forall\, t, s \in \Gamma, |u - u_0(t)| \le r$, $|v - u_0(s)| \le r$, $0 < r \le R$. *Then $\mathcal{F}$ maps the ball $B(u_0, R)$ of $\mathcal{H}_\varphi$ into $\mathcal{H}_\varphi$, $\mathcal{F}$ is Fréchet differentiable and*

$$\mathcal{F}'(u)h(t) = F_u(t, u(t))h(t)$$

is the Fréchet derivative of F at u. Moreover, there exists a positive and increasing function $k_{\mathcal{F}}(r)$ such that

$$||\mathcal{F}'(u) - \mathcal{F}'(v)||_{\mathcal{L}(\mathcal{H}_\varphi)} \le k_{\mathcal{F}}(r)||u - v||, \tag{7}$$

for all $u, v \in B(u_0, r)$ and $0 < r \le R$.

3. The singular integral equation with shift (SIES). Let α be a Carleman shift, we consider the singular integral equation with shift (SIES)

$$\begin{aligned} C(u)(t) := {} & F(t, u(t)) + G(\alpha(t), u(\alpha(t))) \\ & - \frac{1}{\pi i} \int_\Gamma \left(\frac{H(\tau, u(\tau))}{\tau - t} + \frac{M(\tau, u(\tau))}{\tau - \alpha(t)} \right) d\tau \ = 0, \end{aligned}$$

where F, G, H, M are functions from $\Gamma \mathrm{x} \mathbf{C}$ in $\mathbf{C}$. In this section and in the next one, we are going to establish some sufficient conditions which ensure that the operator C verifies the hypotheses of Theorem 2.1.

Theorem 3.1. *Assume that the functions F, G, H and M satisfy the hypotheses of Proposition* 2.2 *and let*

$$c_i, d_i, e_i : [0, R] \to [0, +\infty), \quad 1 \le i \le 6, \ R > 0$$

be the functions of Proposition 2.2 *associated respectively to F, G, H and M.*

Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and $\mathcal{M}$ be the superposition operators generated respectively by F, G, H and M.

Then $C : B(u_0, R) \subseteq \mathcal{H}_\varphi \to \mathcal{H}_\varphi$ is Fréchet differentiable in $B(u_0, R)$ and

$$\begin{aligned} C'(u)v(t) &= F_u(t, u(t))v(t) + G_u(\alpha(t), u(\alpha(t))v(\alpha(t)) \\ &\quad - \frac{1}{\pi i} \int_\Gamma \left(\frac{H_u(\tau, u(\tau))}{\tau - t}\, v(\tau) + \frac{M_u(\tau, u(\tau))}{\tau - \alpha(t)}\, v(\tau) \right) d\tau \end{aligned}$$

is the Fréchet derivative of C at u. Moreover

$$||C'(u) - C'(v)||_{\mathcal{L}(\mathcal{H}_\varphi)} \le k(r)||u - v|| \quad \forall\, u, v \in B(u_0, r), \quad 0 < r \le R,$$

with

$$k(r) := k_{\mathcal{F}}(r) + k_{\mathcal{G}}(r) + ||S||k_{\mathcal{H}}(r) + ||S||\, ||W_\alpha||k_{\mathcal{M}}(r).$$

Proof. Let

$$\begin{aligned} Z(u)h : &= F_u(t, u(t))v(t) + G_u(\alpha(t), u(\alpha(t))v(\alpha(t)) \\ &\quad - \frac{1}{\pi i} \int_\Gamma \left(\frac{H_u(\tau, u(\tau))}{\tau - t}\, v(\tau) + \frac{M_u(\tau, u(\tau))}{\tau - \alpha(t)} v(\tau) \right) d\tau, \end{aligned}$$

then we have

$$\begin{aligned} ||C(u{+}h) - C(u) - Z(u)h|| \le{} & ||\mathcal{F}(u{+}h) - \mathcal{F}(u) - \mathcal{F}'(u)h|| \\ & + ||W_\alpha\big(\mathcal{G}(u{+}h) - \mathcal{G}(u) - \mathcal{G}'(u)h\big)|| \\ & + ||S\big(\mathcal{H}(u{+}h) - \mathcal{H}(u)) - \mathcal{H}'(u)h\big)|| \\ & + ||W_\alpha S\big(\mathcal{M}(u{+}h) - \mathcal{M}(u)) - \mathcal{M}'(u)h\big)||. \end{aligned}$$

The properties of the superposition operator and the boundness of the operators S and W_α imply that

$$||C(u+h) - C(u) - Z(u)h|| = o(||h||), \quad ||h|| \to 0.$$

Hence the assertion on the differentiability follows. Moreover we have

$$\begin{aligned} ||C'(u) - C'(v)||_{\mathcal{L}(\mathcal{H}_\varphi)} &\le ||\mathcal{F}'(u) - \mathcal{F}'(v)||_{\mathcal{L}(\mathcal{H}_\varphi)} + ||\mathcal{G}'(u) - \mathcal{G}'(v)||_{\mathcal{L}(\mathcal{H}_\varphi)} \\ &\quad + ||S||\, ||\mathcal{H}'(u) - \mathcal{H}'(v)||_{\mathcal{L}(\mathcal{H}_\varphi)} \\ &\quad + ||S||\, ||W_\alpha||\, ||\mathcal{M}'(u) - \mathcal{M}'(v)||_{\mathcal{L}(\mathcal{H}_\varphi)} \\ &\le (k_{\mathcal{F}}(r) + k_{\mathcal{G}}(r) + ||S||\, k_{\mathcal{H}}(r) \\ &\quad + ||S||\, ||W_\alpha||\, k_{\mathcal{M}}(r))||u - v|| \end{aligned}$$

which completes the proof. □

4. The invertibility of $C'(u_0)$. Next we want to show that the linear singular integral equation

$$C'(u_0)h(t) = f(t) \tag{8}$$

has a unique solution for every $f \in \mathcal{H}_\varphi$. Rewriting the above equation, we obtain

$$\begin{aligned} F_u(t, u_0(t))h(t) &+ G_u(\alpha(t), u_0(\alpha(t))(W_\alpha h)(t) \\ &- H_u(t, u_0(t))(Sh)(t) - M_u(\alpha(t), u_0(\alpha(t)))(W_\alpha S h)(t) \\ &- \frac{1}{\pi i} \int_\Gamma \frac{H_u(\tau, u_0(\tau)) - H_u(t, u_0(t))}{\tau - t}\, h(\tau)\, d\tau \\ &- \frac{1}{\pi i} \int_\Gamma \frac{M_u(\tau, u_0(\tau)) - M_u(\alpha(t), u_0(\alpha(t)))}{\tau - \alpha(t)}\, h(\tau)\, d\tau \\ &= f(t). \end{aligned}$$

If we set

$$\begin{aligned} a(t) :&= F_u(t, u_0(t)), \qquad b(t) := G_u(\alpha(t), u_0(\alpha(t)), \\ c(t) :&= -H_u(t, u_0(t)), \qquad d(t) := -M_u(\alpha(t), u_0(\alpha(t))), \\ k(t,\tau) :&= \frac{H_u(t, u_0(t)) - H_u(\tau, u_0(\tau))}{\tau - t} \\ &\quad + \frac{M_u(\alpha(t), u_0(\alpha(t))) - M_u(\tau, u_0(\tau))}{\tau - \alpha(t)}, \\ Kh(t) :&= \frac{1}{\pi i} \int_\Gamma k(t,\tau)\, h(\tau)\, d\tau, \end{aligned}$$

then (8) becomes

$$(aI + bW_\alpha + cS + dW_\alpha S + K)h = f. \tag{9}$$

In order to prove the next result, which establishes some sufficient conditions for the invertibility of $C'(u_0)$, we introduce the space of matrices of elements of $\mathcal{H}_\varphi$ and we define a norm in this space.

Let A be a matrix with n lines and m columns, whose elements belong to $\mathcal{H}_\varphi$.

Then, if we set $A = (a_{ij})\,(1 \le i \le n;\ 1 \le j \le m)$, $a_{ij} \in \mathcal{H}_\varphi$, we define the norm of A as follows:

$$\begin{aligned} ||A|| &= \max\{||a_{ij}||,\ i = 1, 2, \dots, n;\ j = 1, 2, \dots, m\} \\ &= \max\{||a_{ij}||_\infty + ||a_{ij}||_\varphi,\ i = 1, 2, \dots, n;\ j = 1, 2, \dots, m\}. \end{aligned}$$

Straightforward computations prove that if A and B are two matrices of elements of $\mathcal{H}_\varphi$ with nxm and mxh elements respectively, then

$$||AB|| \le ||A||\,||B||\,.$$

We can now establish our main result.

Theorem 4.1. *Suppose that the hypotheses of Theorem* 3.1 *are satisfied. Let A be a matrix of functions defined by*

$$A := \begin{pmatrix} a & b & c & d \\ W_\alpha(b) & W_\alpha(a) & W_\alpha(d) & W(c) \\ c & d & a & b \\ W_\alpha(d) & W_\alpha(c) & W_\alpha(b) & W_\alpha(a) \end{pmatrix}.$$

Let $\mathcal{D}$ the closed subspace of $\mathcal{H}_\varphi^4 := \mathcal{H}_\varphi \mathrm{x} \mathcal{H}_\varphi \mathrm{x} \mathcal{H}_\varphi \mathrm{x}\ \mathcal{H}_\varphi$ defined by

$$\mathcal{D} := \{(h, W_\alpha h, Sh, W_\alpha Sh),\ h \in \mathcal{H}_\varphi\},$$

and let $\mathcal{A}$ be the linear operator from $\mathcal{D}$ into H_φ^4 defined by

$$\mathcal{A}H(t) = A(t)H(t)^T.$$

Set:

$$
\begin{aligned}
K_1 : &= S(aI + bW_\alpha + cS + dW_\alpha S + K) - (cI + dW_\alpha + aS + bW_\alpha S),\\
K_2 : &= W_\alpha S(aI + bW_\alpha + cS + dW_\alpha S + K) - (W_\alpha(d)I + W_\alpha(c)W_\alpha\\
&\quad + W_\alpha(b)S + W_\alpha(a)W_\alpha S),
\end{aligned}
$$

$$
\mathcal{K} := \begin{pmatrix} K \\ W_\alpha K W_\alpha \\ K_1 S \\ K_2 S W_\alpha \end{pmatrix}.
$$

Suppose that the two conditions are satisfied

$$
\det A(t) \neq 0 \quad \forall\, t \in \Gamma, \tag{10}
$$

$$
||\mathcal{A}^{-1}\mathcal{K}||_{\mathcal{L}(\mathcal{D})} < 1\,. \tag{11}
$$

Then $C'(u_0)$ is invertible; moreover

$$
||C'(u_0)^{-1}||_{\mathcal{L}(\mathcal{H}_\varphi)} \leq \frac{||Ad(A)||}{1 - ||\mathcal{A}^{-1}\mathcal{K}||_{\mathcal{L}(\mathcal{D})}} \left(\frac{1}{m} + \frac{||\det A||_\varphi}{m^2} \right), \tag{12}
$$

where

$$
m := \min_{t \in \Gamma} |\det A(t)|.
$$

Proof. A function $u \in \mathcal{H}_\varphi$ is a solution of the equation (9), if, and only if, it is a solution of the system

$$
\begin{cases}
(aI + bW_\alpha + cS + dW_\alpha S + K)h = f \\
W_\alpha((aI + bW_\alpha + cS + dW_\alpha S + K)h) = W_\alpha f \\
S((aI + bW_\alpha + cS + dW_\alpha S + K)h) = Sf \\
W_\alpha S((aI + bW_\alpha + cS + dW_\alpha S + K)h) = (W_\alpha S)f
\end{cases}. \tag{13}
$$

Let $\cong$ be the relation of equivalence defined in the Section 2, then we have

$$
\begin{gathered}
S^2 = W_\alpha^2 = I, \qquad S(uv) \cong uS(v),\\
W_\alpha(uv) = W_\alpha(u)W_\alpha(v), \qquad W_\alpha S \cong SW_\alpha\,.
\end{gathered}
$$

Therefore, the above system (13) is equivalent to the following one

$$(14)\quad \begin{cases} (aI + bW_\alpha + cS + dW_\alpha S)h \cong f \\ (W_\alpha(b)I + W_\alpha(a)W_\alpha + W_\alpha(d)S + W_\alpha(c)W_\alpha S)h \cong W_\alpha f \\ (cI + dW_\alpha + aS + bW_\alpha S)h \cong Sf \\ (W_\alpha(d)I + W_\alpha(c)W_\alpha + W_\alpha(b)S + W_\alpha(a)W_\alpha S)h \cong (W_\alpha S)f \end{cases}.$$

Moreover, if we set

$$F := \begin{pmatrix} f \\ W_\alpha f \\ Sf \\ (W_\alpha S)f \end{pmatrix},$$

the system (14) can be rewritten in the following form

$$(15)\qquad \mathcal{A}H + \mathcal{K}H = F, \quad H \in \mathcal{D}.$$

It is well known that (10) is a necessary and sufficient condition for the invertibility of the operator $\mathcal{A}$ on $\mathcal{D}$ and, if we denote with $Ad(A)$ the adjoint matrix of A, we have

$$\mathcal{A}^{-1}F = \frac{Ad(A)F}{det\, A}.$$

Moreover, by (11) it follows that also the operator $I + \mathcal{A}^{-1}\mathcal{K}$ is invertible. Then the operator $\mathcal{A} + \mathcal{K}$ (and, therefore, $C'(u_0)$) is invertible and

$$(\mathcal{A} + \mathcal{K})^{-1} = (\mathcal{A}(I + \mathcal{A}^{-1}\mathcal{K}))^{-1} = (I + \mathcal{A}^{-1}\mathcal{K})^{-1}\mathcal{A}^{-1}.$$

Thus we have

$$||C'(u_0)^{-1}||_{\mathcal{L}(\mathcal{H}_\varphi)} \le ||(\mathcal{A} + \mathcal{K})^{-1}||_{\mathcal{L}(\mathcal{D})} \le \frac{||\mathcal{A}^{-1}||_{\mathcal{L}(\mathcal{D})}}{1 - ||\mathcal{A}^{-1}\mathcal{K}||_{\mathcal{L}(\mathcal{D})}};$$

moreover

$$\begin{aligned} ||\mathcal{A}^{-1}||_{\mathcal{L}(\mathcal{D})} &= \max_{||F|| \le 1} \left|\left| \frac{Ad(A)F}{det\, A} \right|\right| \\ &\le \max_{||F|| \le 1} ||Ad(A)|| \left|\left| \frac{1}{det\, A} \right|\right| ||F|| \\ &\le ||Ad(A)|| \left(\frac{1}{m} + \frac{||det\, A||_\varphi}{m^2} \right) \end{aligned}$$

from which (12) follows. □

The procedure of reducing a singular integral equation with Carleman shift to a canonical system was stated in [**12**] (see also [**9**]).

Finally, applying Theorem 2.1, we obtain the following result.

Theorem 4.2. *Suppose that the hypotheses of Theorem* 4.1 *are satisfied. We set*

$$\beta := \frac{||Ad(A)||}{1 - ||\mathcal{A}^{-1}\mathcal{K}||_{\mathcal{L}(\mathbf{D})}} \left(\frac{1}{m} + \frac{||det\, A||_{\varphi}}{m^2} \right),$$
$$\gamma := \beta(||\mathcal{F}(u_0)|| + ||\mathcal{G}(u_0)|| + ||S||\, ||\mathcal{H}(u_0)|| + ||S||\, ||W_\alpha||\, ||\mathcal{M}(u_0)||).$$

Suppose that the function

$$\phi(r) = \gamma + \beta \int_0^r \omega(t)\, dt - r, \qquad \omega(r) = \int_0^r k(t)\, dt,$$

has a unique positive root r_* *in* $[0, R]$ *and* $\phi(R) \leq 0$. *Then the equation*

$$F(t, u(t)) + G(\alpha(t), u(\alpha(t))) - \frac{1}{\pi i} \int_\Gamma \left(\frac{H(\tau, u(\tau))}{\tau - t} + \frac{M(\tau, u(\tau))}{\tau - \alpha(t)} \right) d\tau \; = 0,$$

has a unique solution u_* *in* $B(u_0, R)$ *and the Newton-Kantorovich approximations*

$$u_n = u_{n-1} - C'(u_{n-1})^{-1} C(u_{n-1})$$

are defined for all $n \in N$, *belong to* $B(u_0, r_*)$ *and converge to* u_*. *Moreover, the following estimate holds*

$$(16) \qquad ||u_{n+1} - x_n|| \leq r_{n+1} - r_n; \qquad ||u_* - x_n|| \leq r_* - r_n.$$

where the sequence $(r_n)_{n \in N}$, *increasing and convergent to* r_*, *is defined by the recurrence formula*

$$r_0 = 0, \qquad r_n = r_{n-1} - \frac{\phi(r_{n-1})}{\phi'(r_{n-1})}, \quad n \in N.$$

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