

APPROXIMATION AND COMMUTATOR PROPERTIES  
OF PROJECTIONS ONTO SHIFT-INVARIANT  
SUBSPACES AND APPLICATIONS TO  
BOUNDARY INTEGRAL EQUATIONS

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*Dedicated to Professor Phil Anselone*

**ABSTRACT.** The main purpose of the present paper is to prove approximation and commutator properties for projections mapping periodic Sobolev spaces onto shift-invariant spaces generated by a finite number of compactly supported functions. With these prerequisites at hand, and using certain localization techniques, we then characterize the stability of generalized Galerkin-Petrov schemes for solving periodic pseudodifferential equations in terms of elliptic type estimates of the numerical symbol. Moreover, we establish optimal convergence rates for the approximate solutions with respect to the Sobolev norms.

**1. Introduction.** It is well known that one of the central problems of the numerical analysis for pseudodifferential equations is to find conditions ensuring the stability of the numerical scheme in consideration. One possible approach to stability analysis for variable symbols is a reduction to the case of constant symbols by means of certain localization techniques which could be viewed as a numerical counterpart to the well-known principle of freezing coefficients in the theory of partial differential equations. The main ingredients for applying such techniques are certain *superapproximation* results for the projections defining the numerical schemes.

Of course, the basic idea of localizing techniques has a long history in theory as well as in the numerical analysis of partial differential

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equations. The first papers addressing this particular aspect seem to be [30], [31], where classical Galerkin schemes with trigonometric trial functions for singular integral equations are investigated. The analysis of piecewise linear spline collocation for Cauchy singular integral equations in [22] already involved implicitly certain superapproximation properties and localization arguments as well which also then played a crucial role in various subsequent papers treating one- and multi-dimensional problems, see, e.g., [2, 19, 20, 21, 28, 5, 23, 24, 7, 16, 17]. An explicit abstract formulation of these principles was given in [18, 19]. For an overview of the various univariate results, see also [26].

The main purpose of the present paper is to prove superapproximation properties for projections  $(P_m)_{m \in \mathbf{N}_0}$  mapping periodic Sobolev spaces onto shift-invariant spaces generated by a finite number of compactly supported functions and defining the numerical methods in consideration. (Notice that such spaces are frequently used as trial spaces in numerical procedures for engineering applications, see [16].) These results are sometimes referred to as *discrete commutator properties* and have the form

$$(\text{CP I}) \quad \|(1 - P_m)fP_mu\|_s \leq c2^{-m(t-s+\delta)}\|P_mu\|_t$$

as well as

$$(\text{CP II}) \quad \|P_mf(1 - P_m)u\|_s \leq c\delta_m2^{-m(t-s)}\|u\|_t$$

where  $c$  and  $\delta$  are positive constants independent of  $u \in H^t(\mathbf{T})$  (but depending on  $s$  and  $t$ , in general). As usually  $\|\cdot\|_t$  denotes the norm of the periodic Sobolev space  $H^t(\mathbf{T})$  of order  $t \in \mathbf{R}$  on the torus  $\mathbf{T} := \mathbf{R}/\mathbf{Z}$ . The orders  $t$  and  $s$  satisfy  $t \geq s$  and are restricted by the choice of the projections. Here  $f$  is a smooth periodic function,  $(\delta_m)_{m \in \mathbf{N}_0}$  tends to zero as  $m \rightarrow \infty$  and  $P_m$  is the projection onto a finite dimensional space defined over a uniform mesh with mesh size  $h = 2^{-m}$ . In view of multiresolution analysis, as it typically arises in connection with the construction of wavelets and multiwavelets, we have chosen the step size to be a power of two. In addition, we say that the projections  $(P_m)_{m \in \mathbf{N}_0}$  have the *approximation property* for  $s \leq t$  if and only if

$$(\text{AP}) \quad \|(1 - P_m)u\|_s \leq c2^{-m(t-s)}\|u\|_t$$

holds for any  $u \in H^t(\mathbf{T})$ . Another important property is the *inverse property* for  $s \leq t$ , which means

$$(IP) \quad \|P_m u\|_t \leq c 2^{m(t-s)} \|P_m u\|_s.$$

Notice that the periodic pseudodifferential operators with constant symbols are convolution operators. This property in combination with using shift-invariant subspaces as trial and test spaces in the corresponding Galerkin-Petrov methods imply a block circulant structure of the resulting stiffness matrices which allows to characterize the stability in the case at hand in terms of eigenvalues, see [7, 25]. On the other hand, the stability and error estimates for the aforementioned methods is independent of the choice of the bases of the trial and test spaces. Hence, our results can be used in a corresponding wavelet or multi-wavelet setting, in particular, for developing compression techniques and fast solvers, see [8, 29].

To our knowledge, our results, see Theorems 2.1 through 2.4, are new in the present form and generality and should be of some independent interest. In particular, they unify and generalize all results in the univariate case and for uniform grids known from the literature, e.g., for spline or classical wavelet spaces. Moreover, in some cases, see Theorems 2.1 and 2.4, we do not need any refinability of the generators, i.e., the spaces  $S_m(\phi)$ , see (1), are not required to be nested.

The paper is organized as follows. The precise formulation of the properties (CP I), (CP II) and (AP) is given in Section 2. The proofs of the main results are presented in Section 3. With these prerequisites at hand, we characterize in Section 4 the stability of the generalized Galerkin-Petrov schemes defined by the aforementioned projections in terms of elliptic type estimates of the numerical symbol. Further, we establish optimal convergence rates for the approximate solutions with respect to the Sobolev norms.

Note that our results can be extended to the corresponding spaces of functions defined on  $\mathbf{R}^n$  or  $\mathbf{T}^n$ , respectively. Moreover, the estimates obtained in the present paper are important in the case of function spaces over subdomains of  $\mathbf{R}^n$ , since the crucial ingredients of our analysis are local in nature.

**2. Notations and the main approximation results.** The main purpose of this paper is to prove (CP I) and (CP II) for appropriate

families of projections. For  $M \in \mathbf{N}$ , let

$$\Lambda_M := \mathbf{Z} \cap \left[ -\frac{M}{2}, \frac{M}{2} \right).$$

We choose a sequence  $\phi := (\phi^j)_{j \in \Lambda_M}$  of generators for the spaces of approximating functions satisfying

$$\phi^j \in \mathcal{L}_2 := \left\{ f \in L^2(\mathbf{R}) : \sum_{k \in \mathbf{Z}} |f(\cdot + k)| \in L^2([0, 1]) \right\}$$

for  $j \in \Lambda_M$ . It is easy to see that  $\mathcal{L}_2 \subseteq L^1(\mathbf{R})$  and therefore  $\hat{f} \in \mathcal{C}(\mathbf{R})$  for  $f \in \mathcal{L}_2$ , where

$$\hat{f}(x) := \int_{\mathbf{R}} e^{-2\pi ixy} f(y) dy$$

is the Fourier transform on  $L^2(\mathbf{R})$ . For the step size  $h := (1/N)$ ,  $N := 2^m$  with  $m \in \mathbf{N}_0$ , we define the approximation spaces

$$(1) \quad S_m(\phi) := \text{Lin} \{ \phi_{k,m}^j := 2^{m/2} [\phi^j(2^m \cdot -k)] : k \in \Lambda_N, j \in \Lambda_M \}.$$

Hereby the periodization operator  $[\cdot]$  is given by  $[f] := \sum_{l \in \mathbf{Z}} f(\cdot + l)$  for  $f \in \mathcal{L}_2$ .

Next we consider a family of distributions  $\eta := (\eta^j)_{j \in \Lambda_M} \in (H^{-s'}(\mathbf{R}))^M$ ,  $s' \geq 0$ , with compact support to define the functionals

$$(2) \quad \eta_{k,m}^l(f) := 2^{-m/2} \eta^l(f(2^{-m}(\cdot + k))), \quad l \in \Lambda_M, k \in \Lambda_N,$$

for  $f \in H^{s'}(\mathbf{T})$ . We require that  $S_m(\phi) \subseteq H^{s'}(\mathbf{T})$  and

$$(3) \quad \eta^r(\phi^j(\cdot - k)) = \delta_{r,j} \delta_{0,k}$$

for  $r, j \in \Lambda_M, k \in \mathbf{Z}$ . Therefore, the operators

$$(4) \quad Q_m(f) := \sum_{\substack{k \in \Lambda_N \\ l \in \Lambda_M}} \eta_{k,m}^l(f) \phi_{k,m}^l$$

are projections defined for sufficiently smooth functions  $f$ . From now on, we denote by  $P_m$  the orthogonal projections onto  $S_m(\phi)$  in  $H^0(\mathbf{T}) =$

$L^2(\mathbf{T})$ . We remark that these projections have a representation like (4), too, provided the functions  $\phi := (\phi^j)_{j \in \Lambda_M}$  have compact support and their integer translates form a Riesz bases. Indeed, we have

$$P_m(u) = \sum \langle u, \phi_{k,m}^l \rangle \psi_{k,m}^l$$

for  $u \in L^2(\mathbf{T})$  with  $(\psi^j)_{j \in \Lambda_M}$  defined by

$$(5) \quad (\hat{\psi}^j(pM + l + x))_{l \in \Lambda_M, j \in \Lambda_M} \\ := (\hat{\phi}^j(pM + l + x))_{l \in \Lambda_M, j \in \Lambda_M} \left( \sum_{m \in \mathbf{Z}} \overline{\hat{\phi}^r(m + x)} \hat{\phi}^j(m + x) \right)_{r \in \Lambda_M, j \in \Lambda_M}^{-1}$$

for  $p \in \mathbf{Z}$ . We will require the linear independence of the generators which is stronger than Riesz stability. A family of compactly supported functions  $\phi := (\phi^j)_{j \in \Lambda_M}$  is called *linearly independent* if the mapping  $((c_k^j)_{j \in \Lambda_M})_{k \in \mathbf{Z}} \mapsto \sum_{j \in \Lambda_M, k \in \mathbf{Z}} c_k^j \phi^j(\cdot - k)$  is injective on the space of sequences of  $M$ -dimensional complex vectors. It is known, cf. [9], that in the univariate case there always exist linearly independent generators if the spaces  $S_m(\phi)$  are generated by compactly supported functions. By definition the functions  $\phi := (\phi^j)_{j \in \Lambda_M}$  are said to satisfy the *Strang-Fix* condition of order  $d \in \mathbf{N}$  if there exists a finite linear combination  $\tilde{\phi}$  of integer translates of the  $(\phi^j)_{j \in \Lambda_M}$  which fulfills the usual Strang-Fix condition of order  $d$ , i.e.,  $\tilde{\phi}^{(k)}(l) = 0$  for any  $l \in \mathbf{Z} \setminus \{0\}$ ,  $k = 0, \dots, d-1$  and  $\tilde{\phi}(0) \neq 0$ . The integer translates of  $\tilde{\phi}$  reproduce algebraic polynomials up to degree  $d-1$  if  $\tilde{\phi}$  is a compactly supported continuous function of bounded variation satisfying the Strang-Fix condition of order  $d$ , cf. [11].

For the approximating functions, we require

**Hypothesis.** *There exist compactly supported functions  $\gamma := (\gamma^j)_{j \in \Lambda_M}$  and  $M \times M$  matrices  $\omega := (\omega_l)_{l \in \mathbf{Z}}$  with exponential decay such that both  $\Pi(x) := \sum_{l \in \mathbf{Z}} \omega_l e^{2\pi i l x}$  is invertible on  $[0, 1]$  and*

$$(6) \quad (\phi^j)_{j \in \Lambda_M} = \omega *_M' (\gamma^j)_{j \in \Lambda_M} := \sum_{l \in \mathbf{Z}} \omega_l (\gamma^j(\cdot - l))_{j \in \Lambda_M}.$$

*The integer translates of  $\gamma$  have to be linearly independent. Further, there exist numbers  $0 < \rho < 1$ ,  $d, d' \in \mathbf{N}_0$ , with  $d' \geq d$  satisfying*

1.  $S_m(\phi) \subseteq H^s(\mathbf{T}) \cap H^{s'}(\mathbf{T})$  for any  $s < d + \rho$ ,  $m \in \mathbf{N}_0$ , where  $s'$  is the Sobolev index of the functionals  $\eta$ ;
2. the integer translates of  $\phi$  reproduce algebraic polynomials up to degree  $d' \in \mathbf{N}_0$ ;
3.  $S_m(\phi)$  fulfills the inverse property, i.e.,  $\|u_m\|_t \leq c2^{m(t-s)}\|u_m\|_s$  for any  $u_m \in S_m(\phi)$ ,  $s \leq t < d + \rho$ .

The third condition implies the inverse property of the projections and is fulfilled for generators satisfying Definition 4.4, cf. also Theorem 3.1 in [25]. In particular, splines of order  $r$  and defect  $M$ , i.e., piecewise polynomials of degree less than or equal to  $r - 1$  which are  $r - M - 1$  times continuously differentiable, fulfill the inverse property for  $s \leq t < r - M + (1/2)$ , cf., [25, Example 3.3]. Further, if  $\phi$  in (5) satisfies the hypothesis, then so does  $\psi$ .

Now we are in the position to formulate the main theorems.

**Theorem 2.1.** *Let  $\phi$  fulfill the hypothesis. Then the projections  $(Q_m)_{m \in \mathbf{N}_0}$  defined by (4) satisfy (AP) and (CP II) for  $s = 0$  and any  $s' \leq t \leq d' + 1$ .*

In contrast to [7], to prove Theorem 2.1 we do not need any refinability of the generators.

Next we give a generalization under the additional assumptions that the projections are orthogonal in  $H^0(\mathbf{T})$  and the spaces  $S_m(\phi)$  are nested, i.e., in the case when  $\phi$  is refinable.

**Theorem 2.2.** *Let  $\phi$  satisfy the hypothesis and let  $S_m(\phi) \subseteq S_{m+1}(\phi)$  for  $m \in \mathbf{N}$ . Then the orthogonal projections  $(P_m)_{m \in \mathbf{N}_0}$  onto  $S_m(\phi)$  fulfill the approximation property (AP) for  $-d' - 1 \leq s < d + \rho$ ,  $-d - \rho < t \leq d' + 1$  and  $s \leq t$ .*

Note that Theorem 2.2 and the subsequent Corollaries 2.3, 2.5, 2.8 and Theorem 2.4 generalize the results obtained in [7, Section 5] for  $M = 1$ , at least in the univariate case. Moreover, here the range of Sobolev indices is larger than in the aforementioned paper.

If the generators are splines of order  $r$  with knots of multiplicity  $M$ , then we are in the situation of [16]. In this case Theorem 2.2 is valid with  $d' = r - 1$ ,  $d = r - M$  and  $\rho = 1/2$ . Then the results can be obtained from Theorem 3.4 in [16].

From now on we use the letter  $c$  to denote a generic constant, the value of which varies from instance to instance.

**Corollary 2.3.** *Let  $\phi$  satisfy the hypothesis and let  $S_m(\phi) \subseteq S_{m+1}(\phi)$  for  $m \in \mathbf{N}$ . Then the projections  $(Q_m)_{m \in \mathbf{N}_0}$  defined by (4) satisfy (AP) for any  $s \leq t$  with  $s' \leq t \leq d' + 1$ ,  $0 \leq s < d + \rho$ .*

*Proof.* Using the inverse estimate, we conclude that

$$(7) \quad \|(1 - Q_m)u\|_s \leq \|(1 - P_m)u\|_s + c2^{ms}\|(1 - Q_m)u\|_0 \leq c2^{-m(t-s)}\|u\|_t$$

for  $u \in H^t(\mathbf{T})$ .  $\square$

Now we turn to the commutator property (CP I) for the orthogonal projections.

**Theorem 2.4.** *Let  $\phi$  satisfy the hypothesis and let the orthogonal projections  $(P_m)_{m \in \mathbf{N}_0}$  onto  $S_m(\phi)$  fulfill the approximation property for  $s \leq t$  with  $0 \leq t \leq d' + 1$  and  $0 \leq s < d + \rho$ . Then the orthogonal projections  $(P_m)_{m \in \mathbf{N}_0}$  have the following properties:*

- i) (CP I) is valid for  $-d' - 1 \leq s \leq t < d + \rho$ ;
- ii) (CP II) is valid for  $s \leq t \leq d' + 1$  and  $-d - \rho < s < d + \rho$ .

Using the boundedness of the projections  $(Q_m)_{m \in \mathbf{N}_0}$  in  $\|\cdot\|_s$  for  $s' \leq s < d + \rho$ , we infer from Theorem 2.4

**Corollary 2.5.** *Let  $\phi$  fulfill the hypothesis and let  $S_m(\phi) \subseteq S_{m+1}(\phi)$  for  $m \in \mathbf{N}$ . Then the projections  $(Q_m)_{m \in \mathbf{N}_0}$  satisfy (CP I) for  $s' \leq s \leq t < d + \rho$ .*

*Proof.* The assertion follows directly from the identity

$$(8) \quad 1 - Q_m = (1 - P_m) + Q_m(P_m - 1). \quad \square$$

The proofs of Theorems 2.1, 2.2 and 2.4 are deferred until the next section.

The interpolation property of Sobolev spaces allows to reduce the proofs of (CP I) and (CP II) from different orders of Sobolev spaces to the order  $s = 0$ .

**Proposition 2.6.** *Let  $(R_m)_{m \in \mathbf{N}_0}$  be a family of projections and let  $0 \leq a \leq b \leq c$  be real numbers such that, for any*

$$(9) \quad 0 \leq s < b, \quad a \leq t \leq c \quad \text{with} \quad s \leq t$$

*the approximation property (AP) and for  $0 \leq s \leq t < b$  the inverse property (IP) are satisfied. Then we infer from (CP I) (respectively, (CP II)) for  $s = 0$  and  $a \leq t \leq c$  that (CP I) (respectively, (CP II)) is valid for any  $s$  and  $t$  restricted by (9).*

*Proof.* Let (CP I) be satisfied with  $s = 0$  and  $a \leq t \leq c$ . Assume  $0 < s < b$  and  $s \leq t$ . For  $s < t$  choose  $\varepsilon > 0$  such that  $s + \varepsilon < t$  and  $s + \varepsilon < b$ . We obtain

$$(10) \quad \|(1 - R_m)fR_mu\|_{s+\varepsilon} \leq c2^{-m(t-s-\varepsilon)}\|R_mu\|_t$$

by the approximation property. In the case  $s = t$ , we conclude (10) from the uniform boundedness of the projections in  $\|\cdot\|_{s+\varepsilon}$  with  $s + \varepsilon < b$  and from (IP). Using the interpolation inequality

$$(IE) \quad \|\cdot\|_s \leq c\|\cdot\|_0^{1-(s/(s+\varepsilon))}\|\cdot\|_{s+\varepsilon}^{s/(s+\varepsilon)}$$

we conclude from (CP I) for  $s = 0$  and (10) that

$$\|(1 - R_m)fR_mu\|_s \leq c2^{-m(t-s)}2^{-m\delta(\varepsilon/(s+\varepsilon))}\|R_mu\|_t.$$

For (CP II) the assertion follows directly from the inverse property.  $\square$

Let  $(\tilde{R}_m)_{m \in \mathbf{N}_0}$  be another family of projections with the same ranges as  $(R_m)_{m \in \mathbf{N}_0}$ , i.e.,  $\mathcal{R}(\tilde{R}_m) = \mathcal{R}(R_m)$  for any  $m \in \mathbf{N}_0$ , and such that  $\|\tilde{R}_m\|_0 \leq c$ . Then all properties excepting (CP II) remain valid with



$(R_m)_{m \in \mathbf{N}_0}$  replaced by  $(\tilde{R}_m)_{m \in \mathbf{N}_0}$ . More precisely, because of (8), the following holds.

**Corollary 2.7.** *If the  $(R_m)_{m \in \mathbf{N}_0}$  fulfill the assumptions of Proposition 2.6, then so do the  $(\tilde{R}_m)_{m \in \mathbf{N}_0}$ . Further, we infer from (CP I) for  $(R_m)_{m \in \mathbf{N}_0}$  and  $s = 0$ ,  $a \leq t \leq c$ , that (CP I) is valid for  $(\tilde{R}_m)_{m \in \mathbf{N}_0}$  and any  $s$  and  $t$  restricted by (9).*

Further, we infer from Proposition 2.6, Theorem 2.1 and Corollary 2.3

**Corollary 2.8.** *Let  $\phi$  satisfy the hypothesis and let  $S_m(\phi) \subseteq S_{m+1}(\phi)$  for  $m \in \mathbf{N}$ . Then the projections  $(Q_m)_{m \in \mathbf{N}_0}$  defined by (4) satisfy (CP II) for any  $s \leq t$  with  $s' \leq t \leq d' + 1$ ,  $0 \leq s < d + \rho$ .*

*Remark 2.9.* Let  $(R_m)_{m \in \mathbf{N}_0}$  be a family of projections satisfying (CP I) and (CP II). Let  $L : H^t(\mathbf{T}) \rightarrow H^{t-r}(\mathbf{T})$  be a bounded operator with  $r \in \mathbf{R}$  given. Using the identity

$$R_m f L - f R_m L = R_m f (1 - R_m) L - (1 - R_m) f R_m L,$$

we obtain from (CP I) and (CP II) that

$$(11) \quad \|R_m f L u - f R_m L u\|_{s-r} \leq c 2^{-m(t-s)} \delta_m \|u\|_t$$

is valid for the corresponding Sobolev indices  $s \leq t$ , where the constant does not depend on  $u \in H^t(\mathbf{T})$ .

Property (11) has been proved in [32] for the case of qualocation projections  $R_m$ , periodic pseudodifferential operators  $L$  and functions  $u$  belonging to subspaces of smoothest splines. In this case, property (CP I) follows from Theorem 2.1 in [32]. On the other hand, the qualocation projections satisfy our hypothesis. Hence, for the qualocation method studied in [32], results similar to Theorems 4.8 and 4.10 of the present paper can be derived.

**3. Proof of the theorems.** The proof of Theorem 2.1 is based on techniques of [7], but uses no refinability. It will be divided into two

steps. We show first (AP) and then (CP II). To this end we need the  $l$ th forward differences of  $u$  defined by

$$(\Delta_h^l u)(x) := \sum_{j=0}^l \binom{l}{j} (-1)^{l-j} u(x + jh)$$

for  $h \in \mathbf{R}$ . As usual,  $\|\cdot\|_0(\Omega)$  denotes the  $L^2$ -norm relative to some domain  $\Omega \subseteq \mathbf{R}$ . The corresponding  $l$ th order modulus of continuity is given by

$$\omega_l(u, t, \Omega) := \sup_{|h| \leq t} \|\Delta_h^l u\|_0(\Omega_{h,l}),$$

where  $\Omega_{h,l} := \{x \in \Omega : x + jh \in \Omega, j = 0, \dots, l\}$ . For  $\Omega = \mathbf{T}$  we write  $\omega_l(u, t)$  instead of  $\omega_l(u, t, \mathbf{T})$ . Now we are ready to introduce the Besov norms

$$\|u\|_{B_{2,2}^t}(\Omega) := \|u\|_0(\Omega) + |u|_t(\Omega)$$

where, for any fixed  $l \in \mathbf{N}$ ,  $l > t$ ,

$$|u|_t(\Omega)^2 := \sum_{j=0}^{\infty} 2^{2jt} \omega_l(u, 2^{-j}, \Omega)^2.$$

In the following we need the norm equivalence, see, e.g., [12, 35]

$$(12) \quad \|\cdot\|_t(\Omega) \sim \|\cdot\|_{B_{2,2}^t}(\Omega)$$

for any  $0 < t < l$  where  $\Omega$  is an interval. We remind that  $\eta \in H^{-s'}(\mathbf{R})$  with support in  $\Gamma := (-a, a)$  for some fixed  $a \in \mathbf{N}$ .

*Proof of Theorem 2.1.* Step 1. For any  $t \geq s'$  we obtain from (12),

$$(13) \quad \begin{aligned} |\eta_{k,m}^l(u)|^2 &= |2^{-m/2} \eta^l(u(2^{-m}(\cdot + k)))|^2 \\ &\leq c 2^{-m} \|u(2^{-m}(\cdot + k))\|_t^2(\Gamma) \\ &\leq c \left( \|u\|_0(\Gamma_k^m)^2 + \sum_{j=0}^{\infty} 2^{2tj} \omega_{d'+1}(u, 2^{-m-j}, \Gamma_k^m)^2 \right) \\ &\leq c (\|u\|_0(\Gamma_k^m)^2 + 2^{-2tm} \|u\|_{B_{2,2}^t}(\Gamma_k^m)^2) \end{aligned}$$

with  $\Gamma_k^m := 2^{-m}(k + \Gamma)$ . By our hypothesis

$$(14) \quad (\phi_{k,m}^j)_{j \in \Lambda_M} = \sum_{i \in \Lambda_N} \omega_{i-k}^m (\gamma_{i,m}^j)_{j \in \Lambda_M}$$

is valid with

$$\omega_k^m := \sum_{p \in \mathbf{Z}} \omega_{pN+k}$$

where  $N = 2^{-m}$ . Using the exponential decay of the coefficients, one concludes that

$$(15) \quad \sum_{k \in \Lambda_N} \sum_{i \in \square_{l,\gamma}^m} \|\omega_{i-k}^m\| \leq c, \quad \sum_{l \in \Lambda_N} \sum_{i \in \square_{l,\gamma}^m} \|\omega_{i-k}^m\| \leq c$$

with  $\square_{l,\gamma}^m := \{i \in \Lambda_N : \cup_{j \in \Lambda_M} \text{supp } \gamma_{i,m}^j \cap \square_l^m \neq \emptyset\}$ ,  $\square := [-1/2, 1/2]$  and  $\square_l^m := 2^{-m}(l + \square)$ .

**Lemma 3.1.** *For  $0 \leq s \leq t$ ,  $s' \leq t$ ,  $s < d + \rho$ ,  $u \in H^t(\mathbf{T})$ , we have*

$$\begin{aligned} & \|Q_m u\|_s(\square_l^m) \\ & \leq c 2^{ms} \sum_{k \in \Lambda_N} \left( \sum_{i \in \square_{l,\gamma}^m} \|\omega_{i-k}^m\| \right) (\|u\|_0(\Gamma_k^m) + 2^{-mt} \|u\|_{B_{2,2}^t}(\Gamma_k^m)). \end{aligned}$$

*Proof.* From our hypothesis and (14), we deduce that

$$\|\phi_{k,m}^j\|_s(\square_l^m) \leq c 2^{ms} \sum_{i \in \square_{l,\gamma}^m} \|\omega_{i-k}^m\|.$$

Since

$$\|Q_m u\|_s(\square_l^m) \leq \sum_{\substack{k \in \Lambda_N \\ j \in \Lambda_M}} |\eta_{k,m}^j(u)| \|\phi_{k,m}^j\|_s(\square_l^m),$$

the assertion follows from (13).  $\square$

Because of the linear independence of the integer translates of  $\gamma$  there exist functionals  $(F^j)_{j \in \Lambda_M}$  with compact support  $\mathcal{F} := \cup_{j \in \Lambda_M} \text{supp } F^j$  such that

$$|F^j(g)| \leq \int_{\mathcal{F}} |g(x)|^2 dx$$

for  $g \in L^2(\mathbf{R})$  and that

$$G_m u : \sum_{\substack{k \in \Lambda_N \\ j \in \Lambda_M}} F_{k,m}^j(u) \gamma_{k,m}^j$$

are projections, cf. [3].

**Lemma 3.2.** *Let  $\Omega \subseteq \mathbf{R}$  be a fixed bounded interval. Then we have, for  $0 \leq s \leq t \leq d' + 1$  with  $s < d + \rho$ ,*

$$\|G_m u - u\|_{B_{2,2}^s(\Omega_k^m)} \leq c 2^{-m(t-s)} \|u\|_{B_{2,2}^t(\tilde{\Omega}_k^m)}$$

for any  $u \in H^t(\mathbf{T})$  where  $\tilde{\Omega}_k^m := \cup_{l \in \Lambda_N} \{\mathcal{F}_l^m : \cup_{j \in \Lambda_M} \text{supp } \gamma_{l,m}^j \cap \Omega_k^m \neq \emptyset\} \cup \Omega_k^m$ .

*Proof.* Without loss of generality, we may assume that  $m$  is sufficiently large such that  $|\Omega_k^m| < 1$ . Because of ii) in the hypothesis, the  $G_m$  reproduce all polynomials up to degree  $d'$  on  $\Omega_k^m$ , and hence,

$$(16) \quad \|G_m u - u\|_{B_{2,2}^s(\Omega_k^m)} \leq \|G_m(u - p)\|_{B_{2,2}^s(\Omega_k^m)} + \|u - p\|_{B_{2,2}^s(\Omega_k^m)}.$$

Thus, when  $s = 0$ , we conclude that

$$(17) \quad \|G_m u - u\|_0(\Omega_k^m) \leq c \inf_{p \in \Pi_{d'}} \|u - p\|_0(\tilde{\Omega}_k^m)$$

where  $\Pi_{d'}$  is the space of all polynomials of degree less than or equal to  $d'$ . A Whitney type estimate, cf. [12] ensures the existence of a polynomial  $p_0 \in \Pi_{d'}$  such that

$$(18) \quad \begin{aligned} \|u - p_0\|_0(\tilde{\Omega}_k^m) &\leq c \omega_{d'+1}(u, 2^{-m}, \tilde{\Omega}_k^m) \\ &\leq c 2^{-mt} \left( \sum_{j=0}^{\infty} 2^{2jt} \omega_{d'+1}(u, 2^{-j}, \tilde{\Omega}_k^m)^2 \right)^{1/2} \end{aligned}$$

$$(19) \quad \leq c 2^{-mt} \|u\|_{B_{2,2}^t(\tilde{\Omega}_k^m)}.$$

Using the inverse property we obtain, by (19),

$$(20) \quad \begin{aligned} \|G_m(u - p_0)\|_{B_{2,2}^s(\Omega_k^m)} &\leq \left\| \sum_{\substack{j \in \Lambda_M \\ l \in \Omega_{k,\gamma}^m}} F_{l,m}^j(u - p_0) \gamma_{l,m}^j \right\|_s \\ &\leq c 2^{ms} \|u - p_0\|_0(\tilde{\Omega}_k^m) \\ &\leq c 2^{-m(t-s)} \|u\|_{B_{2,2}^t(\tilde{\Omega}_k^m)}. \end{aligned}$$

The estimate  $\omega_{d'+2}(u, t, \Omega) \leq c\|u\|_0(\Omega)$  reveals

$$\begin{aligned}
& \sum_{l=0}^{\infty} 2^{2sl} \omega_{d'+2}(u - p_0, 2^{-l}, \Omega_k^m)^2 \\
&= \sum_{l=0}^m 2^{2sl} \omega_{d'+2}(u - p_0, 2^{-l}, \Omega_k^m)^2 \\
&\quad + \sum_{l=m+1}^{\infty} 2^{2sl} \omega_{d'+2}(u - p_0, 2^{-l}, \Omega_k^m)^2 \\
&\leq c \sum_{l=0}^m 2^{2sl} \|u - p_0\|_0 (\Omega_k^m)^2 \\
&\quad + 2^{-2m(t-s)} \sum_{l=m+1}^{\infty} 2^{2sl} 2^{2l(t-s)} \omega_{d'+2}(u - p_0, 2^{-l}, \Omega_k^m)^2 \\
&\leq c 2^{-2m(t-s)} \|u\|_{B_{2,2}^t} (\Omega_k^m)^2 \\
&\quad + 2^{-2m(t-s)} \sum_{l=m+1}^{\infty} 2^{2lt} \omega_{d'+2}(u, 2^{-l}, \Omega_k^m)^2 \\
(21) \quad &\leq c 2^{-2m(t-s)} \|u\|_{B_{2,2}^t} (\Omega_k^m)^2
\end{aligned}$$

where we have used  $\Delta_h^{d'+2} p_0 = 0$ . Thus, the assertion follows from (16), (20) and (21).  $\square$

Now we are ready to prove the approximation property. Using  $Q_m u - u = -Q_m(G_m u - u) + (G_m u - u)$ , we infer from the Lemmas 3.1 and 3.2 that

$$\begin{aligned}
& \|Q_m u - u\|_0^2 \\
&\leq c \sum_{l \in \Lambda_N} \|Q_m u - u\|_0 (\square_l^m)^2 \\
&< c 2^{-2mt} \sum_{l \in \Lambda_N} \left( \sum_{k \in \Lambda_N} \left( \sum_{i \in \square_{l,\gamma}^m} \|\omega_i^m k\| \right) \|u\|_{B_{2,2}^t}(\tilde{\Gamma}_k^m) + \|u\|_{B_{2,2}^t}(\tilde{\square}_l^m) \right)^2 \\
&\leq c 2^{-2mt} \sum_{l \in \Lambda_N} \left( \sum_{k \in \Lambda_N} \sum_{i \in \square_{l,\gamma}^m} \|\omega_{i-k}^m\| \right)
\end{aligned}$$

$$\begin{aligned}
& \cdot \left( \sum_{k \in \Lambda_N} \sum_{i \in \square_{i,\gamma}^m} \|\omega_{i-k}^m\| \|u\|_{B_{2,2}^t} (\tilde{\Gamma}_k^m)^2 \right) \\
& + c2^{-2mt} \sum_{l \in \Lambda_N} \|u\|_{B_{2,2}^t} (\tilde{\square}_l^m)^2 \\
& \leq c2^{-2mt} \sum_{k \in \Lambda_N} \|u\|_{B_{2,2}^t} (\tilde{\Gamma}_k^m)^2 \\
& \leq c2^{-2mt} \|u\|_{B_{2,2}^t}^2 \\
& \leq c2^{-2mt} \|u\|_t^2
\end{aligned}$$

where we have used  $\text{diam } \tilde{\Gamma}_k^m = 2^{-m}$ .

*Step 2.* The same arguments as in the proof of Theorem 5.4 in [7] with  $r_{l,k} := \sum_{i \in \square_{k,\gamma}^m} \|\omega_{i-l}^m\|$ ,  $N_m : 2^{m/2}$ ,  $K_{k,m} := \{l \in \Lambda_N : |l - k| < 2^{m/2}\}$ ,  $d$  replaced by  $d'$ , imply (CP II) for  $s = 0$ .  $\square$

*Proof of Theorem 2.2.* First we show the assertion for  $0 \leq s \leq t$ . A crucial point is to prove the equivalence of the norm

$$(22) \quad \|u\|_{\phi,t}^2 := \|u\|_0^2 + \sum_{m \in \mathbf{N}} 2^{2mt} \|(P_m - P_{m-1})u\|_0^2$$

and the Sobolev norm of order  $t$  on  $\cup_{m \in \mathbf{N}_0} S_m(\phi)$ . By Theorem 2.1 the orthogonal projections  $P_m$  onto  $S_m(\phi)$  satisfy, for  $0 \leq t \leq d' + 1$ , the estimate

$$(23) \quad \|(1 - P_m)f\|_0 \leq c2^{-mt} \|f\|_t, \quad f \in H^t(\mathbf{T}).$$

Now we introduce the norm  $|||\cdot|||_{d'+1}$  defined by  $|||f|||_{d'+1} := \|f^{(d'+1)}\|_0 + |f(0)|$  for  $f \in H^{d'+1}(\mathbf{T})$ . Using the norm equivalence of  $\|\cdot\|_{d'+1}$  and  $|||\cdot|||_{d'+1}$  and the fact that the constant functions are contained in  $S_m(\phi)$ , we obtain from (23) the relation

$$(24) \quad \|(1 - P_m)f\|_0 \leq c2^{-m(d'+1)} \|f^{(d'+1)}\|_0, \quad f \in H^{d'+1}(\mathbf{T}).$$

From (24) we get, similarly to the proof of Proposition 4.1 in [6], that

$$(25) \quad \|(1 - P_m)f\|_0 \leq c\omega_{d'+1}(f, 2^{-m})$$

for  $f \in H^{d'+1}(\mathbf{T})$ .

**Lemma 3.3.** *For any  $u \in H^{d+s}(\mathbf{T})$ ,  $0 \leq s \leq 1$ , we have*

$$\omega_{d+1}(u, t) \leq ct^{d+s} \|u\|_{d+s}.$$

*Proof.* For  $u \in H^s(\mathbf{T})$ ,  $0 \leq s \leq 1$ , it is known that

$$\begin{aligned} \omega_1(u, t)^2 &\leq c \frac{1}{t} \int_{-t}^t \int_0^{1-h} |f(x+h) - f(x)|^2 dx dh \\ &\leq ct^{2s} \int_0^1 \int_{-1}^2 \frac{|f(x) - f(y)|^2}{|\sin(\pi(x-y))|^{1+2s}} dx dy \\ &\leq ct^{2s} \|u\|_s^2. \end{aligned}$$

Therefore, we obtain for any  $u \in H^{d+s}(\mathbf{T})$  that

$$\omega_{d+1}(u, t) \leq ct^d \omega_1(u^{(d)}, t) \leq ct^{d+s} \|u^{(d)}\|_s \leq ct^{d+s} \|u\|_{d+s}. \quad \square$$

Let  $0 \leq s < \rho$ . From the inverse property and Lemma 3.3, we conclude that

$$\begin{aligned} (26) \quad \omega_{d+1}(u_m, t) &\leq c \min\{\|u_m\|_0, t^{d+s} \|u_m\|_{d+s}\} \\ &\leq c \min\{1, (2^m t)\}^{d+s} \|u_m\|_0 \end{aligned}$$

for  $u_m \in S_m(\phi)$ . In view of (25) and (26), Theorem 4.1 of [6] applies and yields

$$\|u\|_t \doteq \|u\|_{\phi, t}, \quad u \in \bigcup_{m \in \mathbf{N}_0} S_m(\phi)$$

for any fixed  $0 \leq t < d + \rho$ . By (23), the smooth functions are contained in  $\text{clos}_{\|\cdot\|_{\phi, t}} \bigcup_{m \in \mathbf{N}_0} S_m(\phi)$ . Hence, the norm equivalence is valid even on  $H^t(\mathbf{T})$ . Arguing as in the proof to Theorem 5.1 of [7], we obtain for

$0 \leq s \leq t \leq d' + 1$ ,  $s < d + \rho$ , that

$$\begin{aligned}
\|(1 - P_m)u\|_s^2 &\leq c \left( \|(1 - P_m)u\|_0^2 + \sum_{j=1}^{\infty} 2^{2js} \|(P_j - P_{j-1})(1 - P_m)u\|_0^2 \right) \\
&= c \left( \|(1 - P_m)u\|_0^2 + \sum_{j=m+1}^{\infty} 2^{2js} \|(P_j - P_{j-1})(1 - P_m)u\|_0^2 \right) \\
&\leq c \sum_{j=m+1}^{\infty} 2^{2j(s-t)} \|u\|_t^2 \\
&\leq c 2^{-2m(t-s)} \|u\|_t^2
\end{aligned}$$

where  $u \in H^t(\mathbf{T})$ . In the last step we have used (23). It remains to prove the case  $s < 0$ . If  $s \leq t < 0$ , then

$$\begin{aligned}
\|u - P_m u\|_s &= \sup_{0 \neq v \in H^{-s}(\mathbf{T})} \frac{\langle u - P_m u, v \rangle}{\|v\|_{-s}} \\
&= \sup_{0 \neq v \in H^{-s}(\mathbf{T})} \frac{\langle u, v - P_m v \rangle}{\|v\|_{-s}} \\
&\leq \|u\|_t \sup_{0 \neq v \in H^{-s}(\mathbf{T})} \frac{\|v - P_m v\|_{-t}}{\|v\|_{-s}} \\
&\leq c 2^{m(s-t)} \|u\|_t.
\end{aligned}$$

In the case  $s < 0 \leq t \leq d' + 1$  we obtain

$$\begin{aligned}
\|u - P_m u\|_s &= \sup_{0 \neq v \in H^{-s}(\mathbf{T})} \frac{\langle u - P_m u, v \rangle}{\|v\|_{-s}} \\
&= \sup_{0 \neq v \in H^{-s}(\mathbf{T})} \frac{\langle u - P_m u, v - P_m v \rangle}{\|v\|_{-s}} \\
&\leq \|u - P_m u\|_0 \sup_{0 \neq v \in H^{-s}(\mathbf{T})} \frac{\|v - P_m v\|_0}{\|v\|_{-s}} \\
&\leq c 2^{m(s-t)} \|u\|_t. \quad \square
\end{aligned}$$

*Proof of Theorem 2.4. Step 1.* For any  $h > 0$ , we have

$$\Delta_h^{d+1} f u_m(x) = \sum_{k=0}^{d+1} \binom{d+1}{k} \Delta_h^k f(x) \Delta_h^{d+1-k} u_m(x + kh).$$



Hence,

$$\begin{aligned} \|\Delta_h^{d+1} f u_m\|_0(\Omega)^2 &\leq c \left( \sum_{k=1}^{d+1} \|\Delta_h^k f\|_{0,\infty}(\Omega)^2 \|\Delta_h^{d+1-k} u_m\|_0(\Omega_{h,k})^2 \right. \\ &\quad \left. + \|f\|_{0,\infty}(\Omega)^2 \|\Delta_h^{d+1} u_m\|_0(\Omega_{h,d+1})^2 \right) \end{aligned}$$

where  $\|f\|_{l,\infty}(\Omega) := \sup_{\nu \leq l} \sup_{x \in \Omega} |f^{(\nu)}(x)|$ . Now we conclude from (17) and (18), with  $d'$  replaced by  $d$ , that

$$\begin{aligned} \|G_m f u_m - f u_m\|_0(\square_k^m)^2 &= \|G_m(f - f(kh))u_m - (f - f(kh))u_m\|_0(\square_k^m)^2 \\ &\leq c \omega_{d+1}((f - f(kh))u_m, 2^{-m}, \tilde{\square}_k^m)^2 \\ &\leq c \left[ 2^{-2m} \|f\|_{1,\infty}^2 \omega_{d+1}(u_m, 2^{-m}, \tilde{\square}_k^m)^2 \right. \\ &\quad \left. + \sum_{q=1}^{d+1} 2^{-2mq} \|f\|_{q,\infty}^2 \omega_{d+1-q}(u_m, 2^{-m}, \tilde{\square}_k^m)^2 \right]. \end{aligned}$$

Since  $u_m \in H^d(\mathbf{T})$  we have  $\omega_l(u_m, \tau, \Omega) \leq c\tau^l \|u_m\|_d(\Omega)$  for  $0 \leq l \leq d$ . So we infer

$$\begin{aligned} \|G_m f u_m - f u_m\|_0(\square_k^m)^2 &\leq c \|f\|_{d+1,\infty}^2 \{ 2^{-2m} \omega_{d+1}(u_m, 2^{-m}, \tilde{\square}_k^m)^2 \\ &\quad + 2^{-2m(d+1)} \|u_m\|_d(\tilde{\square}_k^m)^2 \} \end{aligned}$$

and therefore

$$\begin{aligned} \|G_m f u_m - f u_m\|_0^2 &\leq c \|f\|_{d+1,\infty}^2 \{ 2^{-2m} \omega_{d+1}(u_m, 2^{-m})^2 \\ &\quad + 2^{-2m(d+1)} \|u_m\|_d^2 \}. \end{aligned}$$

For fixed  $s''$  with  $d < s'' < d + \rho$ , we obtain

$$\begin{aligned} \|G_m f u_m - f u_m\|_0^2 &\leq c \|f\|_{d+1,\infty}^2 \{ 2^{-2m(s''+1)} 2^{-2ms''} \omega_{d+1}(u_m, 2^{-m})^2 \\ &\quad + 2^{-2m(d+1)} \|u_m\|_d^2 \} \\ &\leq c \|f\|_{d+1,\infty}^2 2^{-2m(d+1)} \\ &\quad \cdot \left( \sum_{l=0}^{\infty} 2^{-2ls''} \omega_{d+1}(u_m, 2^{-l})^2 + \|u_m\|_d^2 \right) \\ &\leq c \|f\|_{d+1,\infty}^2 2^{-2m(d+1)} \|u_m\|_{s''}^2 \end{aligned}$$

where we have used (12). For  $\delta := 1 - \rho < d + 1 - s''$ , we conclude from the inverse property that

$$\begin{aligned} \|P_m f u_m - f u_m\|_0 &\leq \|G_m f u_m - f u_m\|_0 \\ &\leq c \|f\|_{d+1, \infty} 2^{-m\delta} 2^{-mt} \|u_m\|_t \end{aligned}$$

for any  $t \leq s''$ . Hence, by Proposition 2.6, the assertion i) is shown for  $0 \leq s \leq t < d + \rho$ . Thus, it remains to consider the case  $-d' - 1 \leq s < 0$ . Applying the approximation property, we obtain

$$\|(1 - P_m) f P_m u\|_s \leq c 2^{ms} \|(1 - P_m) f P_m u\|_0.$$

If  $t \geq 0$ , the assertion follows from (CP I) for  $s = 0$ . Otherwise we apply (CP I) with  $\tilde{t} := 0$  and employ the inverse property.

*Step 2.* Next we prove the second part of Theorem 2.4. Because of part i) there exists  $\delta > 0$  such that

$$\begin{aligned} \|P_m f(1 - P_m)u\|_0^2 &\leq \|(1 - P_m) \bar{f} P_m f(1 - P_m)u\|_0 \|u\|_0 \\ &\leq c 2^{-m\delta} \|P_m f(1 - P_m)u\|_0 \|u\|_0 \\ &\leq c 2^{-m\delta} \|u\|_0^2 \end{aligned}$$

for any  $u \in L^2(\mathbf{T})$ . Let  $0 \leq s \leq t \leq d' + 1$  with  $s < d + \rho$  be fixed. Then we conclude from (IP) and (AP) that

$$\begin{aligned} \|P_m f(1 - P_m)u\|_s &\leq c 2^{ms} \|P_m f(1 - P_m)u\|_0 \\ &\leq c 2^{-m(\delta/2-s)} \|(1 - P_m)u\|_0 \\ &\leq c 2^{-m(\delta/2+t-s)} \|u\|_t \end{aligned}$$

provided  $u \in H^t(\mathbf{T})$ .

In the case  $s < 0$  we proceed as in the proof of Corollary 5.2 in [7] to obtain

$$\begin{aligned} \|P_m f(1 - P_m)u\|_s &= \sup_{\|v\|_{-s}=1} \langle (1 - P_m)u, (1 - P_m) \bar{f} P_m v \rangle \\ &\leq \|(1 - P_m)u\|_s \sup_{\|v\|_{-s}=1} \|(1 - P_m) \bar{f} P_m v\|_{-s} \\ &\leq c 2^{-m(t-s+\delta)} \|u\|_t \sup_{\|v\|_{-s}=1} \|P_m v\|_{-s} \\ &\leq c 2^{-m(t-s+\delta)} \|u\|_t \end{aligned}$$

where we have applied (AP) to the first factor and Theorem 2.4 i) to the second factor on the righthand side of the second inequality.  $\square$

**4. Applications to pseudodifferential equations.** In the present section we establish some convergence and stability results for numerical methods to solve periodic pseudodifferential equations. First we introduce the class of pseudodifferential operators which will be studied throughout the remainder of this paper. For  $r \in \mathbf{R}$  we denote by  $S^r(\mathbf{T})$  the class of symbols which consists of functions  $\sigma \in \mathcal{C}^\infty(\mathbf{T} \times \mathbf{Z})$  satisfying

$$|\partial_x^\beta \Delta_{1\xi}^\alpha \sigma(x, \xi)| \leq c_{\alpha, \beta} (1 + |\xi|)^{r-\alpha} \quad \text{for all } x \in \mathbf{T}, \xi \in \mathbf{Z},$$

where  $\Delta_{1\xi}$  is the forward difference operator with respect to  $\xi$  with step size 1 defined in Section 3 and  $\alpha, \beta$  are nonnegative integers. The corresponding pseudodifferential operator with the symbol  $\sigma$  is given by

$$\begin{aligned} \sigma(x, D)u(x) &:= \sum_{\xi \in \mathbf{Z}} e^{2\pi i x \xi} \sigma(x, \xi) \tilde{u}(\xi), \\ \tilde{u}(\xi) &:= \int_0^1 e^{-2\pi i \xi x} u(x) dx, \end{aligned}$$

for  $u \in \mathcal{C}^\infty(\mathbf{T})$ .

It is well known that the symbol of a pseudodifferential operator is uniquely determined up to a function belonging to  $\cap_{r \in \mathbf{R}} S^r(\mathbf{T})$ .

In what follows we restrict ourselves to the subclass  $\Sigma^\mu(\mathbf{T}) \subset S^r(\mathbf{T})$  of all symbols  $\sigma \in S^r(\mathbf{T})$  which admit a decomposition  $\sigma = \sigma_0 + \sigma_1$ , where  $\sigma_1 \in S^{r_1}(\mathbf{T})$  with  $r_1 < r := \operatorname{Re} \mu$ ,  $\mu \in \mathbf{C}$  and  $\sigma_0 \in \mathcal{C}^\infty(\mathbf{T} \times \mathbf{R} \setminus \{0\})$ . The function  $\sigma_0$  is required to be positively homogeneous of degree  $\mu$ , i.e.,

$$\sigma_0(x, \lambda \xi) = \lambda^\mu \sigma_0(x, \xi)$$

for  $\lambda > 0$  and  $\xi \neq 0$ . Without loss of generality, we assume  $\sigma_0(x, 0) = 1$ . We will denote by  $\Psi^r(\mathbf{T})$  and  $\Phi^\mu(\mathbf{T})$  the class of pseudodifferential operators ( $\Psi$ DO's) which admit a decomposition  $L = \sigma(x, D) + K$  where  $\sigma \in S^r(\mathbf{T})$ , and  $\Sigma^\mu(\mathbf{T})$ , respectively, and  $K$  is a smoothing operator given by  $Ku(x) = \int u(y)k(x, y) dy$  with  $k \in \mathcal{C}^\infty(\mathbf{T} \times \mathbf{T})$ . Note that if  $L \in \Phi^\mu(\mathbf{T})$ , then  $L : H^s(\mathbf{T}) \rightarrow H^{s-r}(\mathbf{T})$  is a bounded operator. We

remark that the class  $\Phi^\mu(\mathbf{T})$  contains all classical operators occurring in boundary element methods, cf. [1, 7].

Our central objective is to solve the pseudodifferential equation

$$(27) \quad Lu = f$$

where  $L \in \Phi^\mu(\mathbf{T})$  and  $f \in H^{s-r}(\mathbf{T})$  are given. For the solution of (27), we examine the numerical methods considered in [25]. To this end we have to introduce finite-dimensional trial spaces of approximating functions and sets of test functionals. We choose a sequence  $\phi := (\phi^j)_{j \in \Lambda_M} \in \mathcal{L}_2^M$  of generators for the spaces of approximating functions. Then the trial spaces are defined by (1).

Now we turn to the test functionals. Choose a family of distributions  $\eta := (\eta^j)_{j \in \Lambda_M} \in (H^{-s'}(\mathbf{R}))^M$ ,  $s' \geq 0$ , with compact support to define the test functionals

$$(28) \quad \eta_{k,m}^l(f) := 2^{-m/2} \eta^l(f(2^{-m}(\cdot + k))), \quad l \in \Lambda_M, \quad k \in \Lambda_N,$$

for  $f \in H^{s'}(\mathbf{T})$ . The numerical method which we are going to investigate is the Galerkin-Petrov method corresponding to the aforementioned trial spaces and test functionals. This method reads as follows.

Find an approximate solution  $u_m \in S_m(\phi)$  such that

$$(29) \quad \eta_{k,m}^l(Lu_m) = \eta_{k,m}^l(f), \quad l \in \Lambda_M, \quad k \in \Lambda_N$$

for a fixed and all sufficiently large  $m \in \mathbf{N}_0$ . The scheme (29) corresponding to the trial and test spaces generated by  $\phi$  and  $\eta$ , respectively, is called *numerical method*  $\{\eta, \phi\}$  for the operator  $L$ . The following two examples are particular realizations of the scheme (29).

**Example 4.1** Collocation method. Choose a strictly increasing sequence  $(\varepsilon_j)_{j \in \Lambda_M} \in [0, 1)^M$  and define the test functionals by

$$\eta^j(f) := f(\varepsilon_j)$$

for  $j \in \Lambda_M$ . So we have to find a solution  $u_m \in S_m(\phi)$  satisfying

$$Lu_m(2^{-m}(\varepsilon_j + k)) = f(2^{-m}(\varepsilon_j + k)), \quad j \in \Lambda_M, \quad k \in \Lambda_N.$$

**Example 4.2.** Galerkin method. Let  $(\phi^j)_{j \in \Lambda_M} \in \mathcal{L}_2^M$  be a family of compactly supported functions. Then the test functionals in (29) are defined by (28) and

$$\eta^j(f) := \langle f, \phi^j \rangle_{L^2(\mathbf{R})}, \quad j \in \Lambda_M,$$

for  $f \in L^2(\mathbf{T})$ .

**Example 4.3.** Biorthogonal Galerkin method. Let  $(\tilde{\eta}^j)_{j \in \Lambda_M} \in \mathcal{L}_2^M$  be a family of compactly supported functions biorthogonal to  $(\phi^j)_{j \in \Lambda_M}$ , i.e.,

$$\langle \phi^r, \tilde{\eta}^s(\cdot - k) \rangle_{L^2(\mathbf{R})} = \delta_{r,s} \delta_{0,k}$$

for  $r, s \in \Lambda_M$  and  $k \in \mathbf{Z}$ . Then the test functionals in (29) are defined by (28) and

$$\eta^j(f) := \langle f, \tilde{\eta}^j \rangle_{L^2(\mathbf{R})}, \quad j \in \Lambda_M,$$

for  $f \in L^2(\mathbf{T})$ .

It turns out that the convergence analysis of the numerical method (29) essentially depends on the behavior of the matrix valued function  $[\eta\sigma_0\phi]$  defined by

$$[\eta\sigma_0\phi](y, x) := \sum_{l \in \mathbf{Z}} (\eta^r(e^{2\pi i(l+x)\cdot})\sigma_0(y, l+x)\hat{\phi}^s(l+x))_{(r,s) \in \Lambda_M^2},$$

$x \in [-1/2, 1/2]$ . This function  $[\eta\sigma_0\phi]$  will be called *numerical symbol* of the numerical method  $\{\eta, \phi\}$  for the operator  $L$  with principal symbol  $\sigma_0$ . Using the notation

$$\begin{aligned} \hat{\eta}^r(x) &:= \overline{\eta^r(e^{2\pi i x \cdot})}, \\ \hat{\eta}_p(x) &:= (\hat{\eta}^r(pM + l + x))_{(l,r) \in \Lambda_M^2}, \\ \hat{\phi}_p(x) &:= (\hat{\phi}^r(pM + l + x))_{(l,r) \in \Lambda_M^2}, \\ f_p(x) &:= \text{diag}(f(pM + l + x))_{l \in \Lambda_M}, \end{aligned} \tag{30}$$

for  $f : \mathbf{R} \rightarrow \mathbf{C}$ , the numerical symbol takes the simple form

$$[\eta\sigma_0\phi](y, x) = \sum_{p \in \mathbf{Z}} \hat{\eta}_p(x)^* (\sigma_0(y, \cdot))_p(x) \hat{\phi}_p(x). \tag{31}$$

To define a class of admissible numerical methods, cf. [25], we need the notation

$$\langle x \rangle := \begin{cases} |x| & \text{if } x \neq 0, \\ 1 & \text{else.} \end{cases}$$

We recall the following definition from [25].

**Definition 4.4.** The numerical method  $\{\eta, \phi\}$  is called  $s$ -admissible for  $\Psi$ DO's in  $\Phi^\mu(\mathbf{T})$ ,  $s \in \mathbf{R}$ , if the following is satisfied:

- i) the matrices  $\hat{\phi}_0$  and  $\hat{\eta}_0$  are invertible on  $[-1/2, 1/2]$ ;
- ii)  $\sum_{p \neq 0} \|\langle x \rangle_p^s \hat{\phi}_p(x) \hat{\phi}_0(x)^{-1} \langle x \rangle_0^{-s}\|^2$  is uniformly bounded on  $[-1/2, 1/2]$ ;
- iii)  $\sum_{p \neq 0} \|\hat{\eta}_p(x)^* |x|_p^\mu \hat{\phi}_p(x)\|$  is convergent on  $[-1/2, 1/2]$ .

Here the matrices  $\langle \cdot \rangle_p$ ,  $|\cdot|_p^\mu$  arising in ii) and iii) are defined by (30) and  $\|\cdot\|$  means any matrix norm. The letter  $s$  denotes the Sobolev index of the space  $H^s(\mathbf{T})$ .

*Remark 4.5.* Properties i) and ii) are sufficient conditions for a certain discrete Sobolev norm to be equivalent to the continuous Sobolev norm, see [25, Section 3]. Condition i) is stronger than the Riesz stability. Property ii) is a uniform Strang-Fix condition combined with a growth condition for the  $(\hat{\phi}^j)_{j \in \Lambda_M}$ , see [25, Section 4]. The last condition ensures that the numerical symbol is well defined.

To interpret the numerical scheme (29) as a projection method we have to assume

**Hypothesis H.** *There exist functions  $\psi := (\psi^j)_{j \in \Lambda_M} \in \mathcal{L}_2^M$  such that*

- $\hat{\psi}_0$  is invertible on  $[-1/2, 1/2]$  and condition ii) of Definition 4.4 is fulfilled with  $s$  and  $\phi$  replaced by  $s - r$  and  $\psi$ , respectively;
- $\psi$  satisfies the duality conditions  $\eta^l(\psi^j(\cdot - k)) = \delta_{l,j} \delta_{0,k}$  for  $l, j \in \Lambda_M$ ,  $k \in \mathbf{Z}$ ;
- $\| |x|_0^{s-r} \hat{\psi}_0(x) \hat{\eta}_0(x)^* |x|_0^{r-s} \| \leq c$  and  $\| |x|_0^{s-r} (\hat{\psi}_0(x) \hat{\eta}_0(x)^*)^{-1} |x|_0^{r-s} \| \leq c$  for  $x \in [-1/2, 1/2] \setminus \{0\}$ .

Sufficient conditions for the above hypothesis are formulated in [25, Section 3]. There are also given some hints how to construct such functions  $\psi$  in a general situation. Note that the last condition of Hypothesis H is a uniform Strang-Fix condition. Moreover, the second property implies that the operators

$$(32) \quad Q_m f := \sum_{\substack{k \in \Lambda_N \\ l \in \Lambda_M}} \eta_{k,m}^l(f) \psi_{k,m}^l$$

are projections defined for sufficiently smooth functions  $f$ . In view of the notation (32) and the representation of  $u_m \in S_m(\phi)$  as

$$(33) \quad u_m = u *' \phi := \sum_{j \in \Lambda_M} \sum_{k \in \Lambda_N} u_k^j \phi_{k,m}^j$$

with the coefficient vector  $u := ((u_k^j)_{j \in \Lambda_M})_{k \in \Lambda_N} \in \mathbf{C}^{MN}$ , the numerical scheme (29) is equivalent to the projection equation

$$Q_m L(u *' \phi) = Q_m f.$$

**Definition 4.6.** The numerical method  $\{\eta, \phi\}$  is called stable for  $L : H^s(\mathbf{T}) \rightarrow H^{s-r}(\mathbf{T})$  if

$$\|Q_m L u_m\|_{s-r} \geq c \|u_m\|_s$$

for any  $u_m \in S_m(\phi)$  and sufficiently large  $m \in \mathbf{N}_0$ .

Our strategy is to reduce the problem of stability of the numerical method for  $L$  to that of the  $\Psi$ DO of convolution type defined by the principal symbol  $\sigma_0(y, \cdot)$  for fixed  $y$ . The stability and convergence analysis of such operators has been developed in [25]. The aforementioned reduction is based on localization techniques introduced in [19, 7] and essentially uses the properties of the projections proved in Section 2. To this end we need the concept of *local stability*, cf., [19, 7]. Following [19], we denote for fixed  $y \in \mathbf{T}$  by  $\mathcal{M}_y \subset C^\infty(\mathbf{T})$  the *localizing classes* consisting of functions which are equal to 1 in a neighborhood of  $y$ . From now on,  $P_m$  are the orthogonal projections onto  $S_m(\phi)$  relative to the scalar product in  $H^0(\mathbf{T})$ .

**Definition 4.7.** The numerical method  $\{\eta, \phi\}$  is locally stable for  $L : H^s(\mathbf{T}) \rightarrow H^{s-r}(\mathbf{T})$  if, for each  $y \in \mathbf{T}$ , there exist  $g_y \in \mathcal{M}_y$  and operators  $T_y, T'_y \in \Psi^{r'}(\mathbf{T})$ ,  $r' < r$ , and linear operators  $C_{y,m}, D_{y,m} : S_m(\phi) \rightarrow S_m(\phi)$  with  $\sup_{m \in \mathbf{N}} \|C_{y,m}\|_{H^{s-r}(\mathbf{T}), H^s(\mathbf{T})} < \infty$ ,  $\sup_{m \in \mathbf{N}} \|D_{y,m}\|_{H^{s-r}(\mathbf{T}), H^s(\mathbf{T})} < \infty$  such that

$$\begin{aligned} Q_m g_y (\sigma_y(D) + T_y) C_{y,m} &\dot{=}_{s-r} Q_m g_y P_m, \\ D_{y,m} Q_m (\sigma_y(D) + T'_y) g_y P_m &\dot{=}_s P_m g_y P_m. \end{aligned}$$

Here, for any two sequences of operators  $B_m, C_m$ , the notation  $B_m \dot{=}_s C_m$  stands for  $\lim_{m \rightarrow \infty} \|B_m - C_m\|_s = 0$ , and  $\sigma_y(D)$  is the operator with the symbol  $\sigma_0(y, \cdot)$ .

Applying the localization principle proved in [19] and using the results of Section 2, we can now deduce the equivalence of local and global stability. To this end, we assume

$$(34) \quad L : H^s(\mathbf{T}) \rightarrow H^{s-r}(\mathbf{T})$$

to be an invertible  $\Psi$ DO belonging to  $\Phi^\mu(\mathbf{T})$ ,  $r = \operatorname{Re} \mu$  and the functions  $\phi$  to fulfill the hypothesis of Section 2. Further, suppose  $\psi$  satisfies the hypothesis of Section 2 with  $\phi, d, d', \rho, s$  replaced by  $\psi, \tilde{d}, \tilde{d}', \tilde{\rho}, s-r$ , respectively. We assume that  $\psi$  and  $\phi$  are refinable. The number  $s'$  is determined by the choice of the functionals in (28). The following restrictions on the parameter  $s$  in (34) are necessary to ensure the approximation and commutator properties by using the results of Section 2. We require that the Sobolev index  $s$  fulfills the inequalities

$$(35) \quad \begin{aligned} -d' - 1 &\leq s < d + \rho \quad \text{and} \\ 0 &\leq s - r < \tilde{d} + \tilde{\rho} \quad \text{or} \\ -d' - 1 &\leq s - r < d + \rho \quad \text{if } P_m = Q_m. \end{aligned}$$

Using the results of Section 2 and proceeding as in the proofs of Propositions 6.4 and 6.5 of [7], we obtain

**Theorem 4.8.** *Let  $\{\eta, \phi\}$  be  $s$ -admissible for  $\Psi$ DO's in  $\Phi^\mu(\mathbf{T})$  with  $s$  restricted by (35), and let  $\eta$  fulfill Hypothesis H. Assume that  $L : H^s(\mathbf{T}) \rightarrow H^{s-r}(\mathbf{T})$  is invertible and  $L \in \Phi^\mu(\mathbf{T})$ ,  $r = \operatorname{Re} \mu$ . Then*



the numerical method  $\{\eta, \phi\}$  is stable for  $L$  if and only if one of the following conditions is satisfied:

- i) The method is locally stable.
- ii) The method is stable for the operator  $\sigma_y(D)$  with symbol  $\sigma_0(y, \cdot)$  for all  $y \in \mathbf{T}$ .

Now combining Theorem 4.8 and Theorem 2.6 of [25] establishes the main result of this section.

**Theorem 4.9.** *Suppose that the assumption of Theorem 4.8 is fulfilled. Then the numerical method  $\{\eta, \phi\}$  is stable for the operator  $L$  if and only if*

$$(36) \quad \|(\langle x \rangle_0^{s-r} (\hat{\eta}_0(x)^{-1})^* [\eta \sigma_0 \phi](y, x) \hat{\phi}_0(x)^{-1} \langle x \rangle_0^{-s})^{-1}\| \leq c$$

for any  $x, y \in [-1/2, 1/2]$ .  $\square$

The stability combined with approximation properties of the projections guarantees error estimates via standard arguments. (For details, we refer the reader to the proof of Theorem 6.3 in [7] or to the proof of Theorem 13.14 in [26].)

**Theorem 4.10.** *Let the assumption of Theorem 4.8 be satisfied. Further, suppose that the method  $\{\eta, \phi\}$  is stable for  $L : H^s(\mathbf{T}) \rightarrow H^{s-r}(\mathbf{T})$  and, in addition,  $s$  is restricted by  $-d' - 1 \leq s < \tilde{d} + \tilde{\rho}$ . Suppose that  $f \in H^{t-r}(\mathbf{T})$  for some  $t \geq s$  with  $\tilde{d}' + 1 \geq t - r \geq s'$  where  $s'$  is defined by the test functionals, cf. (28). Let  $u$  denote the exact solution of (27), and let  $u_m$  denote the unique solution of (29) whose existence is guaranteed by Theorem 4.9. Then*

$$(37) \quad \|u - u_m\|_s \leq c 2^{-m(t-s)} \|u\|_t$$

is valid. If, in addition,  $s' \leq s - r$ , then

$$(38) \quad \begin{aligned} \|u - u_m\|_{t'} &\leq c 2^{-m(t-t')} \|u\|_t, \\ \max\{-d' - 1, r\} &\leq t' \leq s. \end{aligned}$$

In the case of the Galerkin method, (38) holds for  $\max\{-d' - 1, -d' - 1 - r\} \leq t' \leq s$ .

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