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DISTRIBUTIONAL ANALYSIS OF RADIATION CONDITIONS FOR THE $3+1$ WAVE EQUATION

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ABSTRACT. Consider the Cauchy problem for the ordinary 3+1 wave equation. Reduction of the spatial domain to a half-space involves an exact radiation boundary condition enforced on a planar boundary. This boundary condition is most easily formulated in terms of the tangential-Fourier and time-Laplace transform of the solution. Using the Schwartz theory of distributions, we examine two other formulations: (i) the nonlocal spacetime form and (ii) its three-dimensional (tangential/time) Fourier transform. The spacetime form features a convolution between two tempered distributions.

1. Introduction and preliminaries.

1.1. Introduction. Numerical wave simulation on finite computational domains requires the introduction of fictitious boundaries at which one must specify boundary conditions; see, for example, the review article [**6**]. Ideally, such boundary conditions stem from exact reduction of an infinite domain, allowing for radiation flux off the incomplete computational domain. We consider perhaps the simplest nontrivial example: the ordinary 3+1 wave equation on $\mathbb{R}^3$ and reduction to a half-space. This reduction involves specification of a radiation condition on a planar boundary, here taken as $z = \delta > 0$ with $z \leq \delta$ the half-space of interest. We write the 3+1 wave equation as $\Box U = g(t, \mathbf{x}, z)$, with $\Box := -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$ and $U = U(t, \mathbf{x}, z)$. Here, $\mathbf{x} := (x, y)$ are the coordinates tangential to the boundary. The exact radiation condition encodes assumptions (given below) about the supports of the source g and the initial data $U(0, \cdot, \cdot)$,$U_t(0, \cdot, \cdot)$.

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In order to state the goals of this paper and provide an overview of our results, we first describe the radiation condition in its most compact *spacetime* incarnation:

$$W^-(t,\mathbf{x},\delta) = \int_0^t dt' \frac{\partial}{\partial t} \int_{|\mathbf{x}-\mathbf{x}'|\le(t-t')} d\mathbf{x}' \frac{W^+(t',\mathbf{x}',\delta)}{\pi(t-t')^2}. \tag{1.1}$$

This nonlocal expression relates the *characteristic variables*

$$W^\pm := 2^{-1/2}(-U_t \pm U_z) \tag{1.2}$$

on the planar boundary $z=\delta$. These variables are determined by the outward-pointing normal vector to the boundary. Viewing the "computational domain" as $z \le \delta$, equation (1.1) expresses W^- (which propagates in the negative z-direction) in terms of W^+ (which propagates in the positive z-direction). Confirmation that $W^\pm$ propagate in the stated directions follows from the wave equation expressed in the first-order symmetric hyperbolic (FOSH) form

$$\begin{aligned}
\partial_t W^\pm &= \mp\partial_z W^\pm - \frac{1}{\sqrt{2}}(\partial_x\Phi_1 + \partial_y\Phi_2),\\
\partial_t \Phi_a &= -\frac{1}{\sqrt{2}}\partial_a(W^+ + W^-),\\
\partial_t U &= -\frac{1}{\sqrt{2}}(W^+ + W^-),
\end{aligned} \tag{1.3}$$

where $a = 1,2$. This FOSH form features the variables $W^\pm$, $\Phi_1 := U_x$, $\Phi_2 := U_y$, and U. The right-hand side of equation (1.1), an integral over history, is remarkable in two regards. First, it involves a convolution with $P(t,\mathbf{x}) = 2\partial_t[t^{-2}H(t-|\mathbf{x}|)]$, where $H(\cdot)$ is the Heaviside step function. Second, it features the normalized average of W^+ over a disk: the intersection of the boundary $z=\delta$ and the interior of the backward light cone with apex $(t,\mathbf{x},\delta)$. We also find it remarkable that (1.1) is formally equivalent to

(1.4)

$$W^-(t,\mathbf{x},\delta) = \int_0^t dt' \frac{2}{t-t'}\Bigg[\int\limits_{|\mathbf{x}-\mathbf{x}'|=(t-t')} ds_{\mathbf{x}'} \frac{W^+(t',\mathbf{x}',\delta)}{2\pi(t-t')} - \int\limits_{|\mathbf{x}-\mathbf{x}'|\leq(t-t')} d\mathbf{x}' \frac{W^+(t',\mathbf{x}',\delta)}{\pi(t-t')^2}\Bigg].$$

Here $ds_{\mathbf{x}'}$ is the arc-length measure along the "ring" $r := |\mathbf{x}-\mathbf{x}'| = t-t'$. Within the square brackets resides the difference of two normalized averages, the first over the ring $r = t-t'$, and the second over the disk $r \leq t-t'$. For smooth W^+, this difference approaches 0 sufficiently fast as $t' \to t^-$ to ensure that the singularity in (1.4) is integrable. We analyze a third expression (1.11) below, an alternative to (1.1) and (1.4), which is more amenable to analysis.

The condition (1.1) has a corresponding form in the "Fourier-Laplace domain," namely,

$$\begin{aligned} \check{W}^-(s,\boldsymbol{\xi},\delta) &= \check{P}(s,\boldsymbol{\xi})\check{W}^+(s,\boldsymbol{\xi},\delta), \\ \check{P}(s,\boldsymbol{\xi}) &:= \frac{s-\sqrt{s^2+\xi^2}}{s+\sqrt{s^2+\xi^2}}, \end{aligned} \tag{1.5}$$

where s is the Laplace variable dual to t, and $\boldsymbol{\xi} = (\xi_1,\xi_2)$ are the Fourier variables dual to $\mathbf{x} = (x,y)$. Moreover, $\xi \geq 0$ is defined by $\xi^2 = \xi_1^2+\xi_2^2$, and, as described in the appendix, the branch for $\sqrt{s^2+\xi^2}$ is chosen to ensure that the expression has positive real part for $\mathrm{Re}s > 0$. We view (1.5) as more fundamental than (1.1); it is (1.5) that we are able to derive from the inhomogeneous wave equation together with certain assumptions on the source and initial data (see below).

Strategies have been proposed to numerically implement the radiation condition. Typically, the idea is to approximate (1.5) in a fashion which is amenable to inversion, thereby obtaining a tractable spacetime condition. In particular, a strategy due to Hagstrom, Warburton and Givoli **[7, 8]** introduces auxiliary variables on the boundary. These auxiliary variables obey a system of evolution equations in their own right. When coupled to the interior, this system determines W^- in terms of W^+, and in a fashion which approximates the above conditions. In fact, we have leveraged the Hagstrom-Warburton-Givoli approach in our own work on the evaluation of retarded-time integrals **[3]**. That reference assumed a "sheet source" $g(t, \mathbf{x}, z) = -2f(t, \mathbf{x})\delta(z)$ and vanishing initial data; here, we consider more general sources and data.

While the Hagstrom-Warburton-Givoli *approximation* is both fascinating and important for numerical implementation, our focus here lies with the above forms of the *exact* radiation condition. More precisely, rather than focusing on (1.1) and (1.5), we will instead examine equivalent (and similar) expressions which are more amenable to analysis. These equivalent expressions, (1.11) and (1.9), are introduced shortly. Our key objective is to establish the correspondence between the spacetime and Fourier-Laplace (or "frequency domain") forms of the radiation condition. We have considered this correspondence before. Indeed, this correspondence is addressed in **[3,** Appendix] via classical arguments, with the Laplace inversion carried out in terms of a Bromwich contour properly in the right-half s-plane. Our goal here is to establish the correspondence using the Schwartz theory of distributions. The analysis we present involves the three-dimensional Fourier transform (in $t, \mathbf{x}$) associated with the history of the planar boundary. The motivation for a second distributional investigation of the correspondence is the following.

• Our "pure" Fourier transform approach also treats the time dimension via Fourier transformation. This approach is often adopted in investigations by physicists.

• The analysis features structures of theoretical interest, e.g., a convolution in which both factors are tempered distributions. Our results may then be viewed as a nonstandard version of the Fourier convolution theorem; see the concluding section.

• The results constitute a first step towards understanding the issue of domain reduction in the context of rough (distributional) solutions to the wave equation, whence our results may, for example, prove relevant for shocks. Regarding this bullet, note that, as seen in equation (1.12) below, equation (1.1) involves a three-dimensional convolution in $(t, \mathbf{x})$ with a kernel Q. Since equations (1.1) and (1.12) hold for any $z = \delta > 0$, this three-dimensional convolution may be viewed as a four-dimensional convolution in $(t, \mathbf{x}, z)$ involving a kernel which is the tensor product of Q with the Dirac distribution (the convolutional identity). This four-dimensional perspective is likely the natural one from which to investigate whether a particular distributional solution to the 3+1 wave equation satisfies equation (1.1).

This paper is organized as follows. We complete this first section with a rough derivation of the above radiation condition (1.5), one meant to fix ideas. Section 2 presents our theoretical results and main theorems. The key theorem involves a certain assumption, an estimate which must be obeyed by the tangential-Fourier/time-Laplace transform of a solution to the wave equation. Section 3 verifies this assumption for a class of solutions to the wave equation. The concluding section, Section 4, describes our results as a version of the Fourier convolution theorem. An appendix describes the key function $s + \sqrt{s^2 + \xi^2}$ as an analytic function of s.

1.2. Derivation of the radiation condition. We sketch the derivation of (1.5), that is, the radiation condition in the Fourier-Laplace domain; for details, see **[3]** and the references therein. Consider then the 3+1 wave equation $\Box U = g$, and assume the following *reduction conditions*:

(R1) For $z > 0$ the inhomogeneity vanishes: $g(t, \mathbf{x}, z) = 0$ for all $t \in [0, \infty)$, $\mathbf{x} \in \mathbb{R}^2$.

(R2) For $z > 0$ the initial data vanishes: $U(0, \mathbf{x}, z) = 0 = U_t(0, \mathbf{x}, z)$ for all $\mathbf{x} \in \mathbb{R}^2$.

To these, we might add a third condition.

(R3) The initial data is of compact support in space, and the source of compact support in spacetime.

These assumptions are depicted schematically in Figure 1. In particular, (R1) and (R2) imply that, for $z > 0$, the solution U is a superposition of Fourier-Laplace "modes"

$$E_{s,\boldsymbol{\xi}}(t, \mathbf{x}, z) = e^{st + \mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x} - z\sqrt{s^2+\xi^2}}, \quad z > 0. \tag{1.6}$$

Note that each mode $E_{s,\boldsymbol{\xi}}$ solves the homogeneous wave equation, and that $E_{s,\boldsymbol{\xi}}(t, \mathbf{x}, z) \to 0$ as $z \to \infty$, provided $\mathrm{Re}s > 0$. More precisely, we have

$$U(t,\mathbf{x},z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} d\boldsymbol{\xi}\, \frac{1}{2\pi \mathrm{i}} \int_{\eta-\mathrm{i}\infty}^{\eta+\mathrm{i}\infty} ds \check{U}(s,\boldsymbol{\xi},\delta) E_{s,\boldsymbol{\xi}}(t,\mathbf{x},z-\delta), \tag{1.7}$$

where $z-\delta$, $\eta > 0$, and $\check{U}(\cdot,\cdot,\delta)$ is the Fourier-Laplace transform of the boundary trace $U(\cdot,\cdot,\delta)$. The variable η defines the Bromwich contour for the inverse Laplace transform.

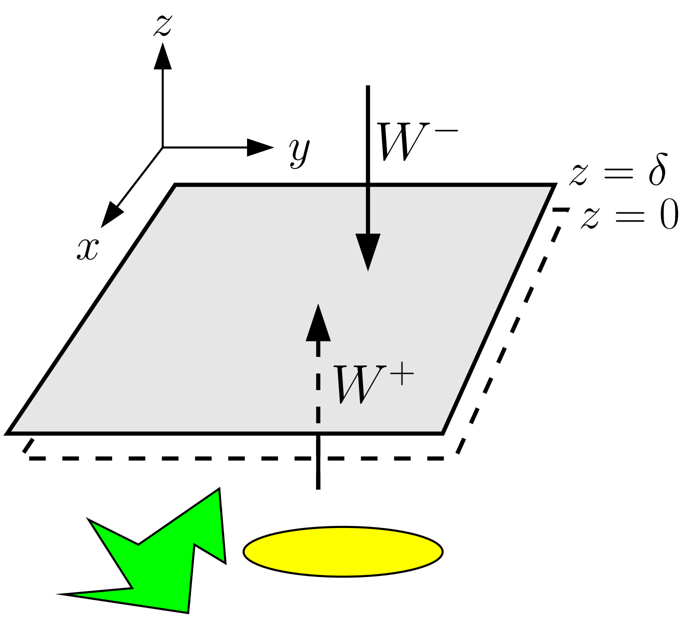

FIGURE 1. Planar boundary $z = \delta$. The "computational domain" corresponds to $z \leq \delta$. As depicted by the green and yellow objects, the source and initial data may be nontrivial (although both are of compact support) on the computational domain; however, both vanish for $z > 0$.

As above, we introduce the characteristic variables $W^{\pm}$ relative to the planar boundary $z = \delta$. If the z-axis points straight up the page, as depicted in Figure 1, then W^+ propagates like $\uparrow$ and W^- as $\downarrow$. The exact radiation condition both expresses W^- in terms of W^+ and encodes assumptions (R1) and (R2) above. This exact condition is easily expressed in the Fourier-Laplace domain. Indeed, due to the form of the mode (1.6), the Fourier-Laplace transforms of the characteristic variables $W^{\pm}(\cdot,\cdot,\delta)$ from (1.2) are

$$\check{W}^{\pm}(s,\boldsymbol{\xi},\delta) = -2^{-1/2}\big(s \pm \sqrt{s^2+\xi^2}\big)\check{U}(s,\boldsymbol{\xi},\delta). \tag{1.8}$$

These formulas then yield the radiation condition (1.5), and (1.5) is algebraically equivalent to

$$\check{W}^{-}(s,\boldsymbol{\xi},\delta) = -\check{W}^{+}(s,\boldsymbol{\xi},\delta) + \check{Q}(s,\boldsymbol{\xi})s\check{W}^{+}(s,\boldsymbol{\xi},\delta), \tag{1.9}$$

where

$$\check{Q}(s,\boldsymbol{\xi}) := \frac{2}{\sqrt{s^2+\xi^2}+s}. \tag{1.10}$$

Reference **[3]** used $\check{P}_1$ and P_1 in place of $\check{Q}$ and Q, where the subscript 1 merely served to distinguish P_1 from P.

As shown below and in **[3]**, the spacetime form of (1.9) is the "classical" (singularity free) relationship

$$\begin{aligned} W^{-}(t,\mathbf{x},\delta) &= -W^{+}(t,\mathbf{x},\delta) \\ &\quad + \frac{1}{\pi}\int_0^t dt' \int_0^1 d\varrho\varrho \int_0^{2\pi} d\phi D_0 W^{+}(t',\mathbf{x}+(t-t')\varrho\boldsymbol{\nu}(\phi),\delta), \end{aligned} \tag{1.11}$$

where D_0 indicates differentiation in the time slot and $\boldsymbol{\nu}(\phi) = (\cos\phi, \sin\phi)$. We also compactly write this expression as

$$W^{-}(t,\mathbf{x},\delta) = -W^{+}(t,\mathbf{x},\delta) + \int_0^t dt' \int_{|\mathbf{x}-\mathbf{x}'|\le(t-t')} d\mathbf{x}' \frac{D_0W^{+}(t',\mathbf{x}',\delta)}{\pi(t-t')^2}, \tag{1.12}$$

where the integral on the right-hand side is a spacetime convolution involving $Q(t,\mathbf{x}) = 2t^{-2}H(t-|\mathbf{x}|)$. We note that we have neither a spacetime derivation nor a direct spacetime verification of (1.11). Rather than (1.1) and (1.5), we shall work instead with (1.11) and (1.9).

The equivalency of (1.1), (1.4) and (1.12) is easily established. Indeed, with mild conditions (A1)–(A3) on $D_0W^{+}(\cdot,\cdot,\delta)$ given shortly, the D_0 can be pulled outside of the ϕ and ρ integrations in (1.11) as the operator $\partial_t + \partial_{t'}$. The t' integration of the $\partial_{t'}$ derivative then gives

$$W^-(t,\mathbf{x},\delta) = \frac{1}{\pi}\int_0^t dt' \frac{\partial}{\partial t}\int_0^1 d\varrho\varrho \int_0^{2\pi} d\phi W^+(t',\mathbf{x}+(t-t')\varrho\boldsymbol{\nu}(\phi),\delta), \tag{1.13}$$

that is, (1.1) written as an iterated integral. With the change of variables $r = (t-t')\varrho$, the last integral becomes

$$W^-(t,\mathbf{x},\delta) = \frac{1}{\pi}\int_0^t dt' \frac{\partial}{\partial t}\int_0^{(t-t')} drr \int_0^{2\pi} d\phi \frac{W^+(t',\mathbf{x}+r\boldsymbol{\nu}(\phi),\delta)}{(t-t')^2}. \tag{1.14}$$

Performance of the t differentiation here yields (1.4) expressed as two iterated integrals.

2. Distributional analysis. Let $\check{V}(s,\boldsymbol{\xi}) := s\check{W}^+(s,\boldsymbol{\xi},\delta)$, so that $V(t,\mathbf{x}) = D_0 W^+(t,\mathbf{x},\delta)$. Then, (1.9) and (1.11) become

$$\check{W}^-(s,\boldsymbol{\xi},\delta) = -\check{W}^+(s,\boldsymbol{\xi},\delta) + \check{Q}(s,\boldsymbol{\xi})\check{V}(s,\boldsymbol{\xi}) \tag{2.1}$$

and

$$W^-(t,\mathbf{x},\delta) = -W^+(t,\mathbf{x},\delta) + \mathcal{I}(t,\mathbf{x};V), \tag{2.2}$$

where

$$\mathcal{I}(t,\mathbf{x};V) := \frac{1}{\pi}\int_0^t dt' \int_0^1 d\varrho\varrho \int_0^{2\pi} d\theta V\big(t',\mathbf{x}+(t-t')\varrho\boldsymbol{\nu}(\theta)\big). \tag{2.3}$$

To investigate the equivalency between (2.1) and (2.2), we only need focus on the relationship between $\check{Q}\check{V}$ and $\mathcal{I}$. As shown below, $\check{Q}(s,\boldsymbol{\xi})$ is the Fourier-Laplace transform of

$$Q(t,\mathbf{x}) := 2t^{-2}H(t-|\mathbf{x}|), \tag{2.4}$$

and another expression for (2.3) is the Fourier-Laplace convolution

$$\mathcal{I}(t,\mathbf{x};V) = \frac{1}{2\pi}\int_0^t dt' \int_{\mathbb{R}^2} d\mathbf{x}' Q(t-t',\mathbf{x}-\mathbf{x}')V(t',\mathbf{x}'). \tag{2.5}$$

Now, $V = D_0W^+(\cdot,\cdot,\delta)$ for the specific case of equation (1.11), but this section views V as a generic function, although one subject to further assumptions:

(A1) V is continuous in all its arguments, except possibly in t at $t = 0$. Moreover, $V(t,\mathbf{x}) = 0$ for $t < 0$, $\mathbf{x} \in \mathbb{R}^2$.

(A2) An $R > 0$ exists such that $V(t,\cdot)$ is supported in the ball $B(R+t) = \{\mathbf{x} : |\mathbf{x}| < R+t\}$.

(A3) The function V is bounded in spacetime, i.e., $|V(t,\mathbf{x})| \le V_B$ for all $t \ge 0$, $\mathbf{x} \in \mathbb{R}^2$. Likely, we could consider a uniform bound in space which permits polynomial growth in time.

Assumptions (A1)–(A3) are common to the whole section; by design, they ensure, in particular, that the Fourier-Laplace transform $\check{V}$ of V exists for Re$s > 0$. Below, we consider an extension of $\check{V}$ to Re$s = 0$. The (A.1) condition $V(t,\cdot) = 0$ for $t < 0$ is also described by the statement "V is *causal*." Further assumptions on V are stated later in (A4).

Conditions (A1)–(A2) for V place restrictions on the support of $\mathcal{I}$. First, $\mathcal{I}(t,\cdot)$ is also causal by (A1), i.e., it vanishes for $t < 0$. Moreover, notice that, in (2.3), the spatial argument of V is $\mathbf{x}' = \mathbf{x} + (t-t')\varrho\boldsymbol{\nu}(\theta)$, where $0 \le \varrho \le 1$, $0 \le t - t' \le t$ and $|\boldsymbol{\nu}(\theta)| = 1$. Therefore, since

$$|\mathbf{x}| = |\mathbf{x}' - (t-t')\varrho\boldsymbol{\nu}(\theta)| \le |\mathbf{x}'| + t, \tag{2.6}$$

the inequality $R + 2t < |\mathbf{x}|$ implies that $R + t < |\mathbf{x}'|$ and, in turn, that $V(\cdot,\mathbf{x}') = 0$ by condition (A2). We conclude that $\mathcal{I}(t,\cdot)$ is supported in $B(2t+R)$.

Remark 2.1. For the target application of equation (1.11), the conditions (A1)–(A3) on $V = D_0W^+(\cdot,\cdot,\delta) = 2^{-1/2}\big[-U_{tt}(\cdot,\cdot,\delta)+U_{tz}(\cdot,\cdot,\delta)\big]$ hold, provided there exists a unique, bounded, C^2 solution U to the Cauchy problem for $\Box U = g$, subject to assumptions (R1)–(R3).

We aim to show that the formula (2.3) can be written

$$\mathcal{I}(t,\mathbf{x};V) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} d\boldsymbol{\xi}\,\frac{1}{2\pi\mathrm{i}}\int\limits_{\eta-\mathrm{i}\infty}^{\eta+\mathrm{i}\infty} ds e^{st+\mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}}\check{Q}(s,\boldsymbol{\xi})\check{V}(s,\boldsymbol{\xi}). \tag{2.7}$$

We have two arguments. One is classical, and takes $\eta = \mathrm{Re} s > 0$; it appears in [**3,** Appendix]. The other, relying on distributions and considered here, makes sense of the case $\eta = 0$, in which case the inversion corresponds to a three-dimensional inverse Fourier transform. Both arguments also consider the "forward transform" of $\mathcal{I}(t, \mathbf{x}; V)$.

Before examining (2.3) in terms of distributions, we first study the generalized function (2.4). Throughout this section, $\widehat{\ }$ (a wide hat) and $^\wedge$ (an upside down "vee") refer to different *Fourier* transforms; the first is with respect to the *spatial* boundary variables $\mathbf{x}$, and the latter with respect to the *spacetime* boundary variables $(t, \mathbf{x})$.

Lemma 2.2. *The Fourier-Laplace transform of* Q *in* (2.4) *is the expression* $\overset{\vee}{Q}$ *in* (1.10) *for* $\mathrm{Re} s > 0$.

Proof. The ordinary spatial Fourier transform $\widehat{Q}$ of Q is

$$\widehat{Q}(t, \boldsymbol{\xi}) = \begin{cases} \dfrac{1}{\pi t^2} \displaystyle\int_{|\mathbf{x}| \leq t} d\mathbf{x} e^{-\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}} & t > 0, \\ 1 & t = 0^+, \\ 0 & t < 0. \end{cases} \tag{2.8}$$

Direct calculations and standard identities for Bessel functions then yield

$$\widehat{Q}(t, \boldsymbol{\xi}) = \begin{cases} 2(t\xi)^{-1} J_1(t\xi) & t\xi > 0, \\ 1 & t\xi = 0^+. \end{cases} \tag{2.9}$$

Finally, [**1,** formula 29.3.58] shows that

$$\mathcal{L}(\widehat{Q}(\cdot, \boldsymbol{\xi}))(s) = 2/(s + \sqrt{s^2 + \xi^2}). \qquad \square$$

Note that $\widehat{Q}(t, \boldsymbol{\xi}) = 0$ for $t < 0$, since $Q(t, \mathbf{x}) = 0$ for $t < 0$. Let us attempt to compute the three-dimensional (boundary spacetime) Fourier transform of Q. Respectively, the small and large-z asymptotics of $J_1(z)$ imply that $\lim_{t \to 0^+} \widehat{Q}(t, \boldsymbol{\xi}) = 1$ and $|\widehat{Q}(t, \boldsymbol{\xi})| = O(t^{-3/2})$ as $t \to \infty$, provided $\xi \neq 0$. Therefore, $\widehat{Q}(\cdot, \boldsymbol{\xi})$ is integrable on the half-line when $\xi \neq 0$, and the following Fourier integral exists:

$$\hat{Q}(\xi_0, \boldsymbol{\xi}) := \frac{1}{\sqrt{2\pi}} \int_0^\infty dt e^{-\mathrm{i}\xi_0 t} \widehat{Q}(t, \boldsymbol{\xi}), \quad \xi \neq 0. \tag{2.10}$$

On the other hand, the integral does not exist for $\xi = 0$, in part motivating our distributional analysis below.

Remark 2.3. Note that $\widehat{Q}(t, \mathbf{0}) = H(t)$, and let $h \in \mathcal{S}'(\mathbb{R})$ be the tempered distribution defined by $\langle h, \psi \rangle := \int_{\mathbb{R}} dt H(t)\psi(t)$ on Schwartz-class functions. Denote by $\hat{h} \in \mathcal{S}'(\mathbb{R})$ its one-dimensional distributional Fourier transform. Then, the action of $\hat{h}$ on Schwartz-class functions is **[5]**:

$$\langle \hat{h}, \psi \rangle = \sqrt{\frac{\pi}{2}}\psi(0) - \frac{\mathrm{i}}{\sqrt{2\pi}} \left[\text{p.v.} \int_{\mathbb{R}} d\lambda \frac{\psi(\lambda)}{\lambda} \right], \quad \psi \in \mathcal{S}(\mathbb{R}). \tag{2.11}$$

In this paragraph, the Schwartz space is associated with $\mathbb{R}$. However, we stress that, in order to find a distributional interpretation for the Fourier transform of Q, we must instead consider $\mathcal{S}(\mathbb{R}^3)$; the relevant analysis is presented below.

Remark 2.4. Throughout this remark assume $\xi \neq 0$ is fixed, so that $\widehat{Q}(\cdot, \boldsymbol{\xi}) \in L_2(\mathbb{R}^+)$. It follows that $\overset{\vee}{Q}(\cdot, \boldsymbol{\xi})$ lies in the Hardy space for the right-half s-plane. Moreover, $\lim_{\eta \to 0^+} \overset{\vee}{Q}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi})$ exists as

(i) a limit in $L_2(\mathbb{R})$, and

(ii) a limit for almost every ξ_0.

These statements then follow from the treatment in **[10,** Chapter 5], with the observation that $\hat{Q}(\cdot, \boldsymbol{\xi})$ in (2.10) lies in the Hardy space for the lower-half ξ_0-plane.

2.1. Tempered distribution q**.** Stemming from (2.4) is a distribution $q \in \mathcal{D}'(\mathbb{R}^3)$ induced by the following test-function pairing:

$$\langle q, \varphi \rangle := \int_{\mathbb{R}^3} dt\, d\mathbf{x} Q(t, \mathbf{x})\varphi(t, \mathbf{x}), \quad \varphi \in C_0(\mathbb{R}^3). \tag{2.12}$$

Note that $Q\varphi \in L^1(\mathbb{R}^3)$, so the integration is well defined. Here, and in what follows, if a function induces a distribution via integration against test functions, then we denote the distribution by the corresponding lowercase letter.

Lemma 2.5. $q \in \mathcal{S}'(\mathbb{R}^3)$ *is a tempered distribution.*

Proof. Let $\psi \in \mathcal{S}(\mathbb{R}^3)$ be a Schwartz-class function. First, $Q\psi$ is also in $L^1(\mathbb{R}^3)$. Moreover, by Fubini's theorem,

$$\langle q, \psi \rangle = \int_0^\infty dt t^{-2} \int_{|\mathbf{x}|\leq t} d\mathbf{x} \psi(t, \mathbf{x}). \tag{2.13}$$

Since $|\langle q, \psi \rangle| \leq 2\pi\big(\sup|\psi| + \sup|t^2\psi|\big)$ follows from this expression, q indeed defines a continuous linear form on $\mathcal{S}(\mathbb{R}^3)$. $\square$

Every $q \in \mathcal{S}'(\mathbb{R}^3)$ has a Fourier transform $\hat{q} \in \mathcal{S}'(\mathbb{R}^3)$ defined by $\langle \hat{q}, \psi \rangle = \langle q, \hat{\psi} \rangle$ for all Schwartz-class functions ψ. We will show that the action of our $\hat{q}$ on Schwartz-class functions is given by

$$\langle \hat{q}, \psi \rangle = \int_{\mathbb{R}^3} d\xi_0 d\boldsymbol{\xi} \hat{Q}(\xi_0, \boldsymbol{\xi}) \psi(\xi_0, \boldsymbol{\xi}), \tag{2.14}$$

where

$$\hat{Q}(\xi_0, \boldsymbol{\xi}) := \lim_{\eta\to 0^+} \frac{1}{\sqrt{2\pi}} \check{Q}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi}). \tag{2.15}$$

This expression for $\hat{Q}(\xi_0, \boldsymbol{\xi})$ agrees with (2.10) when $\xi \neq 0$; it is singular when $(\xi_0, \boldsymbol{\xi}) = (0, \mathbf{0})$. We prove this in Theorem 2.7, which relies on the next lemma.

Lemma 2.6. *For* $\eta > 0$*, define the approximation* q^η *to* q *by the following action on test functions:*

$$\langle q^\eta, \varphi \rangle := \int_{\mathbb{R}^3} dt\, d\mathbf{x} e^{-\eta t} Q(t, \mathbf{x}) \varphi(t, \mathbf{x}). \tag{2.16}$$

Then, q^η is a tempered distribution, and its Fourier transform $\hat{q}^\eta$ is induced by the function

(2.17)
$$\hat{Q}^\eta(\xi_0, \boldsymbol{\xi}) := \frac{1}{\sqrt{2\pi}} \check{Q}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi}) = \sqrt{\frac{2}{\pi}} \frac{1}{(s^2 + \xi^2)^{1/2} + s}, \quad s = \eta + \mathrm{i}\xi_0.$$

Proof. The proof that $q^\eta \in \mathcal{S}'(\mathbb{R}^3)$ is similar to that for Lemma 2.5. To show that the action of $\hat{q}^\eta$ on Schwartz-class functions is induced by integration against expression (2.17), note that the action of q^η is induced by integration against $Q^\eta(t, \mathbf{x}) := 2t^{-2} H(t - |\mathbf{x}|) e^{-\eta t}$. It can easily be shown that $Q^\eta \in L_1(\mathbb{R}^3)$; its Fourier transform (as an integral) therefore exists:

$$\hat{Q}^\eta(\xi_0, \boldsymbol{\xi}) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} dt\, d\mathbf{x} e^{-(\eta + \mathrm{i}\xi_0)t + \mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}} Q(t, \mathbf{x}), \tag{2.18}$$

and, clearly, the stated relationship between $\hat{Q}^\eta$ and $\check{Q}$ holds. Moreover, by the definition of the distributional Fourier transform,

$$\begin{aligned} \langle \hat{q}^\eta, \psi \rangle &= \langle q^\eta, \hat{\psi} \rangle \\ &= \int\limits_{\mathbb{R}^3} dt\, d\mathbf{x} Q^\eta(t, \mathbf{x}) \hat{\psi}(t, \mathbf{x}) \\ &= \int\limits_{\mathbb{R}^3} d\xi_0\, d\boldsymbol{\xi} \hat{Q}^\eta(\xi_0, \boldsymbol{\xi}) \psi(\xi_0, \boldsymbol{\xi}). \end{aligned} \tag{2.19}$$

The last equality is established in [**5,** Theorem 8.1.3]. □

Theorem 2.7. *The tempered distribution $\hat{q}$ is induced by the function*

$$\hat{Q}(\xi_0, \boldsymbol{\xi}) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{-\xi_0^2 + \xi^2} + \mathrm{i}\xi_0}, \tag{2.20}$$

where the square-root factor is defined by

$$\sqrt{-\xi_0^2 + \xi^2} = \begin{cases} (\xi^2 - \xi_0^2)^{1/2} & \textit{for } |\xi_0| < \xi, \\ \operatorname{sign}(\xi_0) \cdot \mathrm{i}(\xi_0^2 - \xi^2)^{1/2} & \textit{for } \xi < |\xi_0|. \end{cases} \tag{2.21}$$

Therefore,

$$|\hat{Q}(\xi_0, \boldsymbol{\xi})| = \begin{cases} (2/\pi)^{1/2}\xi^{-1} & \text{for } |\xi_0| < \xi, \\ (2/\pi)^{1/2}\big[(\xi_0^2 - \xi^2)^{1/2} + |\xi_0|\big]^{-1} & \text{for } \xi < |\xi_0|, \end{cases} \tag{2.22}$$

and the singularity at $(\xi_0, \boldsymbol{\xi}) = (0, \mathbf{0})$ *is integrable.*

Proof. First, the expression (2.20) is the $\eta \to 0^+$ limit of $\hat{Q}^\eta(\xi_0, \boldsymbol{\xi})$ from (2.17). The cases in (2.21) stem from the definition of $\sqrt{s^2 + \xi^2}$; see the appendix. Equation (2.22) then easily follows. To establish the integrability of the origin singularity, note that, if $\xi < |\xi_0|$, then $\big[(\xi_0^2 - \xi^2)^{1/2} + |\xi_0|\big]^{-1} \leq \xi^{-1}$. Therefore, a combination of the cases in (2.22) yields $|\hat{Q}(\xi_0, \boldsymbol{\xi})| \leq (2/\pi)^{1/2}\xi^{-1}$ and

$$\lim_{\gamma \to 0^+} \int_{-1}^{1} d\xi_0 \int_{\gamma \leq |\boldsymbol{\xi}| \leq 1} d\boldsymbol{\xi} |\hat{Q}(\xi_0, \boldsymbol{\xi})| \leq \lim_{\gamma \to 0^+} 4\sqrt{2\pi}(1 - \gamma) = 4\sqrt{2\pi}. \tag{2.23}$$

It remains to show that (2.20) induces $\hat{q}$. For q^η defined in the last lemma, we have $q = \lim_{\eta \to 0^+} q^\eta$ holding in $\mathcal{S}'(\mathbb{R}^3)$. By the sequential continuity of Fourier transformation as a map on $\mathcal{S}'(\mathbb{R}^3)$,

$$q = \lim_{\eta \to 0^+} q^\eta \iff \hat{q} = \lim_{\eta \to 0^+} \hat{q}^\eta, \tag{2.24}$$

with the second limit also holding in $\mathcal{S}'(\mathbb{R}^3)$. Therefore, the previous lemma yields

$$\begin{aligned} \langle \hat{q}, \psi \rangle &= \lim_{\eta \to 0^+} \langle \hat{q}^\eta, \psi \rangle \\ &= \lim_{\eta \to 0^+} \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^3} d\xi_0 \, d\boldsymbol{\xi} \frac{\psi(\xi_0, \boldsymbol{\xi})}{(\eta + \mathrm{i}\xi_0) + \sqrt{(\eta + \mathrm{i}\xi_0)^2 + \xi^2}}. \end{aligned} \tag{2.25}$$

The integrand here is controlled by the integrable function $|\psi(\xi_0, \boldsymbol{\xi})|\xi^{-1}$, as seen by

$$\left| \frac{1}{s + \sqrt{s^2 + \xi^2}} \right| \leq \left| \frac{1}{\eta + \sqrt{\eta^2 + \xi^2}} \right| \leq \frac{1}{\xi}, \quad s = \eta + \mathrm{i}\xi_0, \tag{2.26}$$

where the second inequality is trivial, and the first is proven in the appendix. The limit/integration exchange relies on the Lebesgue dominated convergence theorem. □

Remark 2.8. The integrand in (2.25) is $\hat{Q}^\eta(\xi_0, \boldsymbol{\xi})$ from (2.17). Define

$$K^\eta(\xi_0) := \hat{Q}^\eta(\xi_0, \mathbf{0}) = \sqrt{\frac{1}{2\pi}}\frac{\eta - \mathrm{i}\xi_0}{\eta^2 + \xi_0^2}, \tag{2.27}$$

and the associated tempered distribution $k^\eta \in \mathcal{S}'(\mathbb{R})$, where $\langle k^\eta, \psi\rangle = \int_{\mathbb{R}} d\xi_0 K^\eta(\xi_0)\psi(\xi_0)$ for all $\psi \in \mathcal{S}(\mathbb{R})$. Standard computations then show

$$\langle k^\eta, \psi\rangle \to \sqrt{\frac{\pi}{2}}\psi(0) - \frac{\mathrm{i}}{\sqrt{2\pi}}\Big[\text{p.v.} \int_{\mathbb{R}} d\lambda \frac{\psi(\lambda)}{\lambda}\Big], \quad \psi \in \mathcal{S}(\mathbb{R}). \tag{2.28}$$

This observation is consistent with that in Remark 2.3; in the $\eta \to 0^+$ limit $k^\eta \to \hat{h}$ in $\mathcal{S}'(\mathbb{R})$. Again, we stress that the observations made here and in Remark 2.3 pertain to distributions in $\mathcal{S}'(\mathbb{R})$, whereas, in the main text, we are concerned with distributions in $\mathcal{S}'(\mathbb{R}^3)$.

2.2. Main result. Our distributional statement of (2.7) stems from Theorem 2.10 below. We now set the stage for its statement. Throughout this subsection, we suppress the V in $\mathcal{I}(t, \mathbf{x}; V)$. The function $\mathcal{I}(t, \mathbf{x})$ defined by (2.3) is continuous in $(t, \mathbf{x})$; whence, it induces a distribution $\imath \in \mathcal{D}'(\mathbb{R}^3)$ via the test-function pairing

$$\langle \imath, \varphi\rangle = \int_{\mathbb{R}^3} dt\, d\mathbf{x} \mathcal{I}(t, \mathbf{x})\varphi(t, \mathbf{x}), \quad \varphi \in C_0(\mathbb{R}^3). \tag{2.29}$$

Lemma 2.9. *With V obeying conditions* (A1)–(A3)*, we have $\imath \in \mathcal{S}'(\mathbb{R}^3)$.*

Proof. By assumption (A3), $|\mathcal{I}(t, \mathbf{x})| \le V_B t$; therefore, $\mathcal{I}\psi \in L_1(\mathbb{R}^3)$ for $\psi \in \mathcal{S}(\mathbb{R}^3)$. Consider the pairing

$$\langle \imath, \psi\rangle = \int_{\mathbb{R}^3} dt\, d\mathbf{x} \mathcal{I}(t, \mathbf{x})\psi(t, \mathbf{x}) = \int_0^\infty dt \int_{|\mathbf{x}|\le R+2t} d\mathbf{x} \mathcal{I}(t, \mathbf{x})\psi(t, \mathbf{x}), \tag{2.30}$$

where the last equality stems from Fubini's theorem and the causality and support properties of $\mathcal{I}$; see the discussion after assumptions (A1)–

(A3). The last formula shows that

$$\big|\langle \imath,\psi\rangle\big| \le \sup|\psi|\cdot\int_0^1 dt \int_{|\mathbf{x}|\le R+2t} d\mathbf{x}\big|\mathcal{I}(t,\mathbf{x})\big| \\ + \sup|t^5\psi|\cdot\int_1^\infty dt\int_{|\mathbf{x}|\le R+2t} d\mathbf{x}\, t^{-5}\big|\mathcal{I}(t,\mathbf{x})\big|, \tag{2.31}$$

where the last integral on the right-hand side is convergent since $|\mathcal{I}(t,\mathbf{x})| \le V_B t$. □

We now consider the spacetime Fourier transform $\hat{\imath} \in \mathcal{S}'(\mathbb{R}^3)$ of $\imath$. To state the main result, we first note that, by assumptions (A1)–(A3), the function V also defines a tempered distribution v. Moreover, with $V^\eta(t,\cdot) := e^{-\eta t}V(t,\cdot)$ for $\eta > 0$, by (A1)–(A3), we also have $V^\eta \in L_1(\mathbb{R}^3)$ and $\hat{V}^\eta(\xi_0,\boldsymbol{\xi}) = (2\pi)^{-1/2}\check{V}(\eta + \mathrm{i}\xi_0,\boldsymbol{\xi})$. In order to prove our main result, we need three more conditions on V, which are combined into assumption (A4).

(A4) (a) As $\eta \to 0^+$, the function $\hat{V}^\eta$ converges almost everywhere on $\mathbb{R}^3$. We denote by $\hat{V}$ the function which equals the $\eta \to 0^+$ limit of $\hat{V}^\eta$, where it exists, and 0 otherwise.

(A4) (b) Through pairing by integration, $\hat{V}$ induces a tempered distribution $\hat{v}$.

(A4) (c) The Fourier-Laplace transform $\check{V}$ of V obeys

$$|\check{Q}(\eta+\mathrm{i}\xi_0,\boldsymbol{\xi})\check{V}(\eta+\mathrm{i}\xi_0,\boldsymbol{\xi})\psi(\xi_0,\boldsymbol{\xi})| \le h_\psi(\xi_0,\boldsymbol{\xi}),$$

where h_ψ is integrable and the bound is uniform in $\eta > 0$.

We view $\hat{V}(\cdot,\boldsymbol{\xi})$ as the continuation of $(2\pi)^{-1/2}\check{V}(\cdot,\boldsymbol{\xi})$ from the open right-half s-plane to the imaginary axis.

Theorem 2.10. *Assume conditions* (A1)–(A3) *so that, in particular, by the last lemma, $\imath$ is a tempered distribution. Assume conditions* (A1)–(A4) *on V. Then, the pairing of $\hat{\imath}$ with Schwartz-class functions* *is given by*

$$\langle \hat{\imath}, \psi \rangle = \int_{\mathbb{R}^3} d\xi_0 d\boldsymbol{\xi} \hat{\mathcal{I}}(\xi_0, \boldsymbol{\xi}) \psi(\xi_0, \boldsymbol{\xi}), \tag{2.32}$$

where we have defined

$$\hat{\mathcal{I}}(\xi_0, \boldsymbol{\xi}) := \sqrt{2\pi} \hat{Q}(\xi_0, \boldsymbol{\xi}) \hat{V}(\xi_0, \boldsymbol{\xi}). \tag{2.33}$$

Proof. To compute the Fourier transform of $\imath$ and prove Theorem 2.10, we again rely on exponential damping. First, with $V^\eta(t, \mathbf{x}) = e^{-\eta t} V(t, \mathbf{x})$ and $Q^\eta(t, \mathbf{x}) = 2t^{-2} H(t - |\mathbf{x}|) e^{-\eta t}$, both V^η and Q^η are in $L_1(\mathbb{R}^3)$. In the ordinary sense, we have $\mathcal{I}^\eta = \sqrt{2\pi} Q^\eta * V^\eta$, where

$$\mathcal{I}^\eta(t, \mathbf{x}) := \frac{e^{-\eta t}}{2\pi} \int_{\mathbb{R}} dt' \int_{\mathbb{R}^2} d\mathbf{x}' Q(t - t', \mathbf{x} - \mathbf{x}') V(t', \mathbf{x}') = e^{-\eta t} \mathcal{I}(t, \mathbf{x}). \tag{2.34}$$

Moreover, by a standard version of the Fourier convolution theorem,

$$\hat{\mathcal{I}}^\eta(\xi_0, \boldsymbol{\xi}) = \sqrt{2\pi} \hat{Q}^\eta(\xi_0, \boldsymbol{\xi}) \hat{V}^\eta(\xi_0, \boldsymbol{\xi}) = \frac{1}{\sqrt{2\pi}} \check{Q}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi}) \check{V}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi}). \tag{2.35}$$

Finally, as is easily shown, $\imath^\eta \to \imath$ in $\mathcal{S}'(\mathbb{R}^3)$. Therefore, the action of $\hat{\imath}$ on Schwartz-class functions is

$$\begin{aligned} \langle \hat{\imath}, \psi \rangle &= \lim_{\eta \to 0^+} \langle \hat{\imath}^\eta, \psi \rangle \\ &= \lim_{\eta \to 0^+} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi_0 \int_{\mathbb{R}^2} d\boldsymbol{\xi} \check{Q}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi}) \check{V}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi}) \psi(\xi_0, \boldsymbol{\xi}). \end{aligned} \tag{2.36}$$

The result now follows from (A4), (2.26), and the Lebesgue dominated convergence theorem. □

Remark 2.11. Evidently, $\hat{\mathcal{I}}(\xi_0, \boldsymbol{\xi})$ in (2.33) stems from the ordinary Fourier transform in the following sense:

$$\hat{\mathcal{I}}(\xi_0, \boldsymbol{\xi}) = \lim_{\eta \to 0^+} \hat{\mathcal{I}}^\eta(\xi_0, \boldsymbol{\xi}) = \lim_{\eta \to 0^+} (2\pi)^{-1/2} \check{\mathcal{I}}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi}), \tag{2.37}$$

although the limit need not be finite for certain arguments. Since both $\hat{Q}(\xi_0, \boldsymbol{\xi})$ and $\hat{V}(\xi_0, \boldsymbol{\xi})$ are, up to $(2\pi)^{-1/2}$ factors, also extensions to the imaginary axis of Fourier-Laplace transforms, we may formally write (2.33) as

$$\hat{\mathcal{I}}(\xi_0, \boldsymbol{\xi}) = \frac{(2\pi)^{3/2}}{4\pi^2} \check{Q}(\mathrm{i}\xi_0, \boldsymbol{\xi}) \check{V}(\mathrm{i}\xi_0, \boldsymbol{\xi}). \tag{2.38}$$

Of course, this is a statement regarding the function which induces the action of $\hat{\imath}$ on Schwartz-class functions. However, performing a formal, spacetime, inverse Fourier transform of the last expression, we obtain equation (2.7). Therefore, Theorem 2.10 indicates in what sense equation (2.7) holds for $\eta = 0$.

3. Boundary Fourier-Laplace transform of monopole solutions. This section considers a class of solutions to the homogeneous 3+1 wave equation for which conditions (A4) hold, thereby confirming that the analysis of Section 2 is relevant for at least some solutions. For the solutions considered, verification that (A1)–(A3) hold is trivial.

3.1. Monopole solutions. Define the *retarded time*

$$u = t - \sqrt{\rho^2 + (z + \ell)^2}, \quad \rho^2 = x^2 + y^2, \tag{3.1}$$

where $\ell > 0$ is a positive constant. As shown in Figure 2, choose a smooth *profile function* f supported in $[u_1, u_2] := [-\ell\beta_1, -\ell\beta_2]$, where $0 \le \beta_2 < \beta_1 \le 1$. It follows that

$$-\sqrt{\rho^2 + (\delta + \ell)^2} \le -(\delta + \ell) < u_1 < u_2 < 0. \tag{3.2}$$

Then,

$$U(t, \mathbf{x}, z) = \frac{f\big(t - \sqrt{\rho^2 + (z + \ell)^2}\big)}{\sqrt{\rho^2 + (z + \ell)^2}}, \quad t > 0 \tag{3.3}$$

is a solution of the 3+1 wave equation, namely, a monopole solution relative to the point $(x_0, y_0, z_0) = (0, 0, -\ell)$; see **[2]** for this terminology.

We have arranged for the support of the initial data, $U(0, \mathbf{x}, z)$ and $U_t(0, \mathbf{x}, z)$, to look as in Figure 3 since, for $x = 0 = y$, its support is such that $\ell\beta_2 \le z + \ell \le \ell\beta_1$. Therefore, the initial data vanishes for $z > 0$. For simplicity, we have placed the (x_0, y_0) location of the source at the origin of the planar-boundary $z = \delta$. Note that the case

$(x_0, y_0) \neq (0,0)$ then corresponds to the replacement

$$\overset{\vee}{U}(s, \boldsymbol{\xi}, \delta) \longrightarrow e^{-\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}_0}\overset{\vee}{U}(s, \boldsymbol{\xi}, \delta), \tag{3.4}$$

in terms of the Fourier-Laplace transform $\overset{\vee}{U}(\cdot,\cdot,\delta)$ of the boundary trace $U(\cdot,\cdot,\delta)$. Therefore, our placement of the source results in no loss of generality.

3.2. Fourier-Laplace transform of the monopole. The key result is the following.

Claim 3.1. *For the monopole solution* (3.3)*, the Fourier-Laplace transform* $\overset{\vee}{U}(\cdot,\cdot,\delta)$ *of the boundary trace* $U(\cdot,\cdot,\delta)$ *is given by*

$$\overset{\vee}{U}(s, \boldsymbol{\xi}, \delta) = a(s)\frac{e^{-(\ell+\delta)\sqrt{s^2+\xi^2}}}{\sqrt{s^2+\xi^2}}, \tag{3.5}$$

where $a(s) := \int_{u_1}^{u_2} du e^{-su} f(u)$ *is analytic in* s*.*

Proof. The same result is found whether we perform the spatial Fourier transform first and then the Laplace transform or vice versa. We take the Laplace transform first; the computation proceeds as follows:

(3.6)

$$\mathcal{L}\big(U(\cdot,\mathbf{x},\delta)\big)(s) = \frac{1}{\sqrt{\rho^2+(\delta+\ell)^2}}\int_0^\infty e^{-st}dt f\big(t-\sqrt{\rho^2+(\delta+\ell)^2}\big)$$

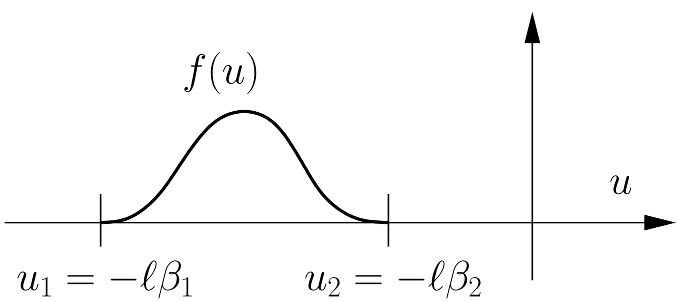

FIGURE 2. Profile function.

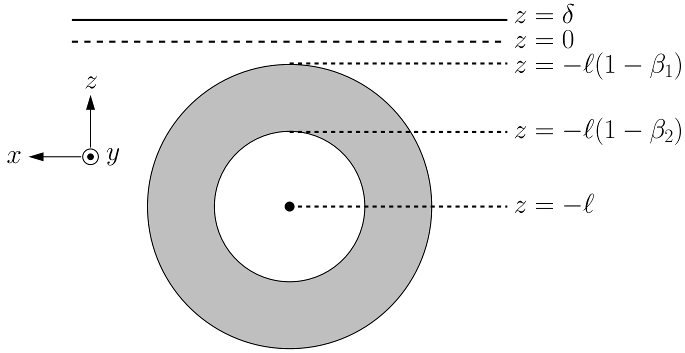

FIGURE 3. Initial support of the monopole solution.

$$= \frac{e^{-s\sqrt{\rho^2+(\delta+\ell)^2}}}{\sqrt{\rho^2+(\delta+\ell)^2}} \int\limits_{-\sqrt{\rho^2+(\delta+\ell)^2}}^{\infty} due^{-su} f(u)$$
$$= \frac{e^{-s\sqrt{\rho^2+(\delta+\ell)^2}}}{\sqrt{\rho^2+(\delta+\ell)^2}} \int\limits_{u_1}^{u_2} due^{-su} f(u),$$

where we have used (3.2) and the assumed support of f to obtain the last equality. Now, perform the boundary spatial Fourier transform:

(3.7)
$$\widehat{U}(s,\boldsymbol{\xi},\delta) = \frac{a(s)}{2\pi} \int\limits_0^{2\pi} d\phi \int\limits_0^{\infty} d\rho\rho \exp\big(-\mathrm{i}\rho\boldsymbol{\nu}(\phi)\cdot\boldsymbol{\xi}\big) \frac{e^{-s\sqrt{\rho^2+(\delta+\ell)^2}}}{\sqrt{\rho^2+(\delta+\ell)^2}},$$

where $\boldsymbol{\nu}(\phi) = (\cos\phi, \sin\phi)$ is a unit vector. Therefore,

$$\check{U}(s,\boldsymbol{\xi},\delta) = a(s) \int\limits_0^{\infty} d\rho\rho J_0(\rho\xi) \frac{e^{-s\sqrt{\rho^2+(\delta+\ell)^2}}}{\sqrt{\rho^2+(\delta+\ell)^2}}, \tag{3.8}$$

which we recognize as a Hankel transform $\mathcal{H}\big(g(\cdot)\big)(\xi)$ of order 0, where

$$g(\rho) = \frac{e^{-s\sqrt{\rho^2+(\delta+\ell)^2}}}{\sqrt{\rho^2+(\delta+\ell)^2}}. \tag{3.9}$$

According to [**4,** page 9, formula (24)], if

(3.10)
$$f(x) = x^{1/2}(x^2+\beta^2)^{-1/2}\exp[-\alpha(x^2+\beta^2)^{1/2}]; \quad \mathrm{Re}\alpha > 0,\ \mathrm{Re}\beta > 0,$$

then, for $y > 0$,

(3.11)
$$\int_0^\infty f(x)J_0(xy)(xy)^{1/2}dx = y^{1/2}(y^2+\alpha^2)^{-1/2}\exp[-\beta(y^2+\alpha^2)^{1/2}].$$

A combination of these formulas yields

$$\int_0^\infty J_0(xy)\frac{e^{-\alpha(x^2+\beta^2)^{1/2}}}{(x^2+\beta^2)^{1/2}}\,dx = \frac{e^{-\beta(y^2+\alpha^2)^{1/2}}}{(y^2+\alpha^2)^{1/2}}. \tag{3.12}$$

This formula yields the result. □

3.3. Verification of assumption (A4). The following claim establishes that assumption (A4) is indeed satisfied by some solutions to the wave equation, namely, those of the form (3.3).

Claim 3.2. *Using the monopole* (3.5), *define* $V = D_0W^+(\cdot,\cdot,\delta) = 2^{-1/2}[-U_{tt}(\cdot,\cdot,\delta) + U_{zt}(\cdot,\cdot,\delta)]$. *Then* V *satisfies assumption* (A4) *above.*

Proof. In the Fourier-Laplace domain,

$$\check{V}(s,\boldsymbol{\xi}) = -2^{-1/2}s\big(s+\sqrt{s^2+\xi^2}\big)\check{U}(s,\boldsymbol{\xi},\delta). \tag{3.13}$$

This result relies on the fact that the initial data vanishes for $z=\delta$. To establish (A4) (a) and (A4) (b) for $\hat{V}^\eta(\xi_0,\boldsymbol{\xi}) = (2\pi)^{-1/2}\check{V}(\eta+\mathrm{i}\xi_0,\boldsymbol{\xi})$, we first examine $\check{U}(\eta+\mathrm{i}\xi_0,\boldsymbol{\xi},\delta)$ using the expression in equation (3.5). With the definition of $\sqrt{s^2+\xi^2}$ given in the appendix, we have

(3.14)
$$\big|\sqrt{s^2+\xi^2}\big| = \sqrt{\sqrt{\eta^4+2\eta^2(\xi_0^2+\xi^2)+(\xi_0^2-\xi^2)^2}} \geq \sqrt{|\xi_0^2-\xi^2|}.$$

Moreover,

$$\mathrm{Re}\sqrt{(\eta+\mathrm{i}\xi_0)^2+\xi^2} > 0 \quad \text{for } \eta > 0. \tag{3.15}$$

Finally, from the definition of $a(s)$ and $u_1 < u_2 < 0$, we get

$$\big|a(\eta + \mathrm{i}\xi_0)\big| \leq \int_{u_1}^{u_2} du e^{-\eta u}|f(u)| \leq e^{\eta|u_1|}\|f\|_{L_1(u_1,u_2)}. \tag{3.16}$$

Taken together, the last three estimates yield

$$\big|\check{U}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi}, \delta)\psi(\xi_0, \boldsymbol{\xi})\big| \leq e^{H|u_1|}\|f\|_{L_1(u_1,u_2)} \frac{\big|\psi(\xi_0,\boldsymbol{\xi})\big|}{\sqrt{|\xi_0^2 - \xi^2|}}, \quad \eta \in (0, H], \tag{3.17}$$

where $\psi \in \mathcal{S}(\mathbb{R}^3)$. Now, the $\eta \to 0^+$ limit of $(2\pi)^{-1/2}\check{U}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi}, \delta)$ exists and is finite, provided $\xi_0^2 \neq \xi^2$. Furthermore, the limiting function $(2\pi)^{-1/2}\check{U}(\mathrm{i}\xi_0, \boldsymbol{\xi}, \delta)$ induces a tempered distribution $\hat{u}$. Indeed, with (3.17) and essentially the same argument used to compute $\hat{q}$ in the last section, we see that $\hat{u}$ is the Fourier transform of the tempered distribution u induced by $U(\cdot,\cdot,\delta)$. Up until now, we have established that (A4) (a) and (A4) (b) hold for $\hat{U}^\eta(\xi_0, \boldsymbol{\xi}) = (2\pi)^{-1/2}\check{U}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi})$, rather than $\hat{V}^\eta(\xi_0, \boldsymbol{\xi})$. That (A4) (a) and (A4) (b) hold for $\hat{V}^\eta(\xi_0, \boldsymbol{\xi})$ then follows from equation (3.13).

Finally, to establish (A4) (c), consider

$$\check{Q}(s, \boldsymbol{\xi})\check{V}(s, \boldsymbol{\xi}) = -2^{1/2}s\check{U}(s, \boldsymbol{\xi}, \delta), \tag{3.18}$$

where we have used the explicit expressions for $\check{Q}$ and $\check{V}$. Therefore, the results from the last paragraph also yield

$$\begin{aligned} &\big|\check{Q}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi})\check{V}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi})\psi(\xi_0, \boldsymbol{\xi})\big| \\ &\quad\leq 2^{1/2}(\eta^2 + \xi_0^2)^{1/2}|\check{U}(\eta + \mathrm{i}\xi_0, \boldsymbol{\xi}, \delta)\psi(\xi_0, \boldsymbol{\xi})| \\ &\quad\leq 2^{1/2}(H^2 + \xi_0^2)^{1/2}e^{H|u_1|}\|f\|_{L_1(u_1,u_2)} \frac{\big|\psi(\xi_0,\boldsymbol{\xi})\big|}{\sqrt{|\xi_0^2 - \xi^2|}}, \end{aligned} \tag{3.19}$$

for $\eta \in (0, H]$. The last expression is integrable, establishing (A4) (c). □

Remark 3.3. The singularities in (3.19) correspond to glancing waves. Indeed, a plane wave with the dispersion relation $\xi_0^2 = \xi_1^2 + \xi_2^2$ has a propagation direction which is parallel to the boundary.

4. Conclusions. Here we discuss in what sense our results are an incarnation of the Fourier convolution theorem. As shown, the functions Q, V, and $\mathcal{I}$ define tempered distributions q, v, $\imath$. In fact, $\imath = \sqrt{2\pi} q * v$, that is, $\imath$ is a convolution in the distributional sense. To establish this statement, first choose $\varphi \in C_0^\infty(\mathbb{R}^3)$, and define

$$w_\varphi(t, \mathbf{x}) := \int_{\mathbb{R}} d\lambda \int_{\mathbb{R}^2} d\mathbf{y} V(\lambda, \mathbf{y}) \varphi(t + \lambda, \mathbf{x} + \mathbf{y}) = \int_0^\infty d\lambda \int_{|\mathbf{y}| \le R+\lambda} d\mathbf{y} V(\lambda, \mathbf{y}) \varphi(t + \lambda, \mathbf{x} + \mathbf{y}), \tag{4.1}$$

where the second equality stems from the assumed support properties for V. Since V is not of compact support, $w_\varphi \notin C_0^\infty(\mathbb{R}^3)$ in general. However, we define the pairing $\langle q*v, \varphi\rangle$ as $(2\pi)^{-3/2}\langle q, w_\varphi\rangle$, provided the latter exists. Theorem 5.1.1 of [**5**] motivates this definition; however, the $(2\pi)^{-3/2}$ factor here stems from our "democratic" convention for the Fourier transform (with 2π factors appearing symmetrically in the transform and its inverse). This convention is different from that in [**5**]. Since q is only supported for non-negative times, the w_φ in $\langle q, w_\varphi\rangle$ is then effectively of compact support. These observations show $q*v \in \mathcal{D}'(\mathbb{R}^3)$. Further calculation, relying on exchange of integrations, shows that $(2\pi)^{-3/2}\langle q, w_\varphi\rangle = (2\pi)^{-1/2}\langle \imath, \varphi\rangle$, whence, $\imath = \sqrt{2\pi} q * v$ in $\mathcal{D}'(\mathbb{R}^3)$, and so also in $\mathcal{S}'(\mathbb{R}^3)$.

Since, as shown in the last paragraph, $\imath = \sqrt{2\pi} q*v$, a naive statement of the Fourier convolution theorem would be

$$\hat{\imath} = \sqrt{2\pi}\hat{q}\hat{v}, \tag{4.2}$$

a formal statement only! A literal interpretation of this statement is problematic, and we examine in what sense it does hold. Theorem 2.10 above shows that the distributional Fourier transform $\hat{\imath} = \sqrt{2\pi}(q * v)^\wedge$ is induced through integration by the function

$$\hat{\mathcal{I}}(\xi_0, \boldsymbol{\xi}) = \sqrt{2\pi}\hat{Q}(\xi_0, \boldsymbol{\xi})\hat{V}(\xi_0, \boldsymbol{\xi}). \tag{4.3}$$

This result is a form of the Fourier convolution theorem; however, $\hat{\imath}$ cannot be directly expressed in terms of $\hat{q}$ and $\hat{v}$. Indeed, a standard interpretation of $\hat{q}\hat{v}$ is problematic, since $\hat{v}$ need not be a multiplier on $\mathcal{S}'(\mathbb{R}^3)$; see [**5**] for a discussion of multipliers. The function $\hat{V}$ which induces $\hat{\upsilon}$ does not have the requisite smoothness to define a map from $\mathcal{S}(\mathbb{R}^3)$ to itself.

To achieve a "pure distributional" statement of our results in the form of a Fourier convolution theorem, we follow a different route. Define V_T to be V such that $V_T(t,\cdot) = V(t,\cdot)$ for $t \leq T$, but $V_T(t,\cdot) = 0$ for $t > 2T$. Moreover, we may arrange for V_T to be continuous in its arguments. The cut-off function V_T then defines a distribution υ_T of compact support. In this case, $\hat{\upsilon}_T$ is a multiplier on $\mathcal{S}'(\mathbb{R}^3)$. Also, $\imath_T := \sqrt{2\pi} q * \upsilon_T \to \imath$ in $\mathcal{S}'(\mathbb{R}^3)$. These observations and a standard distributional form [**5**] of the Fourier convolution theorem show that

$$\hat{\imath} = \sqrt{2\pi}(q * \upsilon)^\wedge = \lim_{T\to\infty} \sqrt{2\pi}\hat{q}\hat{\upsilon}_T, \tag{4.4}$$

with the limit holding in $\mathcal{S}'(\mathbb{R}^3)$. We view this result as a nonstandard instance of the Fourier convolution theorem.

We close with remarks on an alternative to (1.5), namely, the $(-)$ choice in (1.8),

$$\check{W}^-(s,\boldsymbol{\xi},\delta) = -2^{-1/2}\big(s - \sqrt{s^2+\xi^2}\big)\check{U}(s,\boldsymbol{\xi},\delta), \tag{4.5}$$

which relates the incoming characteristic variable directly to the fundamental wave variable. In [**9,** Appendix], this relationship was formally inverted to achieve the spacetime relationship

$$W^-(t,\mathbf{x},\delta) = -2^{-1/2}\int_0^t dt' \frac{1}{(t-t')}\frac{\partial}{\partial t}\int_{|\mathbf{x}-\mathbf{x}'|=t-t'} ds_{\mathbf{x}'}\frac{U(t',\mathbf{x}',\delta)}{2\pi(t-t')}, \tag{4.6}$$

where, as in (1.4), the formula features arc-length measure along a ring. This formula was apparently first discovered by Weston [**11**]. Remarkably, (4.6) can also be derived in spacetime directly from the wave equation via integration by parts [**9**]. We have been unable to find a corresponding "spacetime derivation" of either (1.1), (1.11), or (1.12).

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A. APPENDIX

A.1. Analytic function $s+\sqrt{s^2+\xi^2}$. We assume throughout that η, $\xi>0$ and $s=\eta+\mathrm{i}\xi_0$ for $\xi_0\in\mathbb{R}$.

Definition A.1. With $-\pi/2<\theta_1$, $\theta_2<\pi/2$, we write

$$s=\eta+\mathrm{i}\xi_0=-\mathrm{i}\xi+\rho_1e^{\mathrm{i}\theta_1}=\mathrm{i}\xi+\rho_2e^{\mathrm{i}\theta_2}, \tag{A.1}$$

and, using this expression, define

$$\sqrt{s^2+\xi^2}:=\rho_1^{1/2}\rho_2^{1/2}e^{\mathrm{i}(1/2)(\theta_1+\theta_2)}. \tag{A.2}$$

The restrictions on θ_1 and θ_2 ensure that $\sqrt{s^2+\xi^2}$ has positive real part for $\eta=\mathrm{Re}s>0$. From (A.1) and (A.2),

$$\begin{aligned} s+\sqrt{s^2+\xi^2}&=(1/2)\rho_1e^{\mathrm{i}\theta_1}+(1/2)\rho_2e^{\mathrm{i}\theta_2}+\rho_1^{1/2}\rho_2^{1/2}e^{\mathrm{i}(1/2)(\theta_1+\theta_2)}\\ &=(1/2)\big(\rho_1^{1/2}e^{\mathrm{i}(1/2)\theta_1}+\rho_2^{1/2}e^{\mathrm{i}(1/2)\theta_2}\big)^2. \end{aligned} \tag{A.3}$$

A.2. Argument, modulus and inverse of $s+\sqrt{s^2+\xi^2}$. Inspecting the first right-hand expression in (A.3), we see from the angle restrictions that $s+\sqrt{s^2+\xi^2}$ has strictly positive real part (assuming, as we are, that $\mathrm{Re}s>0$). Moreover, the second right-hand expression yields

$$\big|s+\sqrt{s^2+\xi^2}\big|=\big(\rho_1+\rho_2+2\rho_1^{1/2}\rho_2^{1/2}\cos\vartheta\big)/2, \tag{A.4}$$

where $\vartheta:=(\theta_1-\theta_2)/2$. Since $-\pi<\theta_1-\theta_2<\pi$, we have $-\pi/2<\vartheta<\pi/2$ and $\cos\vartheta$ is positive. It follows that

$$\big|s+\sqrt{s^2+\xi^2}\big|>\big(\rho_1+\rho_2\big)/2>\big(|\xi+\xi_0|+|\xi-\xi_0|\big)/2>\xi. \tag{A.5}$$

Formula (A.3) also establishes the following.

Claim A.2. *Let $w=s+\sqrt{s^2+\xi^2}$. Then, the inverse function is*

$$s=\big(w-\xi^2w^{-1}\big)/2. \tag{A.6}$$

Proof. The claim is trivial to formally establish, but we confirm it using the definition (A.3). From (A.1), we have

$$4\xi^2=-\big(\rho_1e^{\mathrm{i}\theta_1}-\rho_2e^{\mathrm{i}\theta_2}\big)^2, \tag{A.7}$$

where the left-hand side is manifestly positive and real since $\rho_1 e^{i\theta_1}$ and $\rho_2 e^{i\theta_2}$ have the same real parts, as is geometrically seen. Substitution of $w = (1/2)\big(\rho_1^{1/2} e^{i(1/2)\theta_1} + \rho_2^{1/2} e^{i(1/2)\theta_2}\big)^2$ into the right-hand side of (A.6) followed by use of (A.7) consistently yields

$$s = \big(\rho_1 e^{i\theta_1} + \rho_2 e^{i\theta_2}\big)/2, \tag{A.8}$$

a formula which also follows from (A.1). □

A.3. Key inequality. Here, we establish the result (2.26).

Claim A.3. $\big|s + \sqrt{s^2+\xi^2}\big| \geq \eta + \sqrt{\eta^2+\xi^2}$.

Proof. Again, set $w = s + \sqrt{s^2+\xi^2}$, and consider the polar form $w = re^{i\psi}$, where $r > \xi$, as seen above. From $s = \eta + i\xi_0$ and (A.6),

$$\eta = (1/2)(r - \xi^2 r^{-1})\cos\psi, \qquad \xi_0 = (1/2)(r + \xi^2 r^{-1})\sin\psi. \tag{A.9}$$

Therefore, $|w| = r$ corresponds to an ellipse in the s-plane:

$$\frac{\eta^2}{a^2} + \frac{\xi_0^2}{b^2} = 1, \quad a(r) := (r - \xi^2 r^{-1})/2, \quad b(r) := (r + \xi^2 r^{-1})/2. \tag{A.10}$$

Respectively, a and b are the semi-minor and semi-major axes. These parameters are increasing with increasing $r > \xi$. Whence concentric semi-circles $w = re^{i\psi}$ for $-\pi/2 < \psi < \pi/2$ in the w-plane correspond to concentric semi-ellipses in the right-half s-plane. The value $r = \xi$ corresponds to the degenerate semi-ellipse (segment) between $-i\xi$ and $i\xi$, and $r = \infty$ to the infinite semi-circle ($a \sim b$ as $r \to \infty$).

Choose a fixed $r = R > \xi$ and corresponding semi-minor $A := a(R)$ and semi-major $B := b(R)$ axes. Notice A and R (both real positive) by (A.10) are related in the same fashion as s and w (both generally complex) in (A.6). Therefore, $R = A + \sqrt{A^2+\xi^2}$. Consider the vertical line $s = A + i\mathbb{R}$, which intersects the ellipse $(\eta/A)^2 + (\xi_0/B)^2 = 1$ only at $s = A$. At all other points besides $s = A$ the line intersects other ellipses whose corresponding r-values are larger than the fixed one R. For any point on the line, we then have $|s + \sqrt{s^2+\xi^2}| = r \geq R = A + \sqrt{A^2+\xi^2}$, namely, the claimed inequality. □

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