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ON THE EXISTENCE OF GROUND STATES OF NONLINEAR FRACTIONAL SCHRÖDINGER SYSTEMS WITH CLOSE-TO-PERIODIC POTENTIALS

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ABSTRACT. We are concerned with the nonlinear fractional Schrödinger system

$$\begin{cases} (-\Delta)^s u + V_1(x)u = f(x,u) + \Gamma(x)|u|^{q-2}u|v|^q & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v + V_2(x)v = g(x,v) + \Gamma(x)|v|^{q-2}v|u|^q & \text{in } \mathbb{R}^N, \\ u, v \in H^s(\mathbb{R}^N), \end{cases}$$

where $(-\Delta)^s$ is the fractional Laplacian operator, $s \in (0,1)$, $N > 2s$, $4 \leq 2q < p < 2^*$, $2^* = 2N/(N-2s)$. $V_i(x) = V^i_{\text{per}}(x) + V^i_{\text{loc}}(x)$ is closed-to-periodic for $i = 1,2$, f and g have subcritical growths and $\Gamma(x) \geq 0$ vanishes at infinity. Using the Nehari manifold minimization technique, we first obtain a bounded minimizing sequence, and then we adopt the approach of Jeanjean-Tanaka [**8**] to obtain a decomposition of the bounded Palais-Smale sequence. Finally, we prove the existence of ground state solutions for the nonlinear fractional Schrödinger system.

1. Introduction. Recently, Bieganowski and Mederski [**4**] considered a class of nonlinear Schrödinger equations with a sum of periodic and vanishing potentials and sign-changing nonlinearities. Under the positivity assumption on the spectrum of Schrödinger operators, they investigated the existence of ground state solutions being minimizers on the Nehari manifold. To the authors' knowledge, this is the first paper for such a type of problem. Since we are considering nonlinear Schrödinger equations on the entire space $\mathbb{R}^N$, there is no obvious compact imbedding for Sobolev spaces; furthermore, the concentration-compactness lemma can be used. For the case of autonomous systems (with constant potentials), the symmetry of solutions may be used

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to obtain compactness. However, for general nonconstant potentials, since there is no symmetry of solutions, some restrictions must be used on the potentials, such as boundedness and asymptotically constant or monotonicity at infinity. Hence, it is completely nontrivial and different techniques are required for the case of periodic potentials. For further background and survey on nonlinear Schrödinger equations with periodic potentials, the interested reader may refer to Pankov [**19**], in addition to [**6, 9, 12, 22**]. Pioneering research via variational methods on nonlinear Schrödinger equations with general potentials may be found in [**20**].

In 2006, Maia, Montefusco and Pellacci [**13**], via variational methods, considered a weakly coupled nonlinear Schödinger system. This is one of the earliest papers on the subject. Using techniques of the Mountain Pass theorem, the Nehari manifold method, the concentration-compactness lemma and the Pohozaev identity, the authors proved the existence of the positive solution of the Schrödinger system, i.e., each component of the solution of the system is nontrivial. In 2016, using an approach based on a new linking-type result involving the Nehari-Pankov manifold, Mederski [**14**] found a ground state solution of a system of nonlinear Schrödinger equations with periodic potentials. This may be viewed as an extension, not only from a single equation to a system, but also from a constant potential to a periodic potential. When we consider the nonlinear Schödinger system, although the system may have a similar variational structure to a single equation and we can obtain a solution of the system using similar techniques as for the single equation, usually it is not an easy task to prove that each component of the solution of the system is nontrivial.

Applications in pure and applied mathematics and other natural sciences, as well as fractional Laplacians, have been widely studied via variational methods in the last decade. In 2015, Servadei and Valdinoci [**21**] extended the Brezis-Nirenberg result for the classical Laplacian to the fractional Laplacian. The reader may find a general picture on the development of the fractional Laplacian from the introduction and references in this paper. Nonlinear fractional Schrödinger equations also appear in fractional quantum physics; thus, it is interesting to study such equations for every type of potential. Our aim in this paper is to extend the results on ground state solutions of nonlinear Schrödinger equations with periodic potentials [**4**] to coupled nonlinear fractional Schrödinger systems with periodic potentials.

After this paper was completed, the authors obtained the paper **[3]**. The variational structure in **[3]** is very similar to ours. Nonlinear Schrödinger equations with vanishing potentials are more complicated; the reader is referred to Benci, Crisanti and Michellttti **[2]** and Mederski **[15]**. How to extend the results of **[2, 15]** to nonlinear fractional Schrödinger equations is a problem still to be addressed.

Motivated by the above papers, in this paper, we focus on ground state solutions of the nonlinear fractional Schrödinger system with periodic or close-to-periodic potentials, i.e.,

$$
\begin{cases} (-\Delta)^s u + V_1(x)u = f(x,u) + \Gamma(x)|u|^{q-2}u|v|^q & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v + V_2(x)v = g(x,v) + \Gamma(x)|v|^{q-2}v|u|^q & \text{in } \mathbb{R}^N, \\ u, v \in H^s(\mathbb{R}^N), \end{cases} \tag{1.1}
$$

where $(-\Delta)^s$ is the fractional Laplacian operator,

$$
s \in (0,1), \qquad N > 2s, \qquad 4 \le 2q < p < 2^*, \qquad 2^* = \frac{2N}{N-2s}.
$$

Our aim is to study the existence of least energy solutions for system (1.1).

We assume that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ verifies the following hypotheses:

(F_1) f is continuous in $u \in \mathbb{R}$ for almost every $x \in \mathbb{R}^N$ and $\mathbb{Z}^N$-periodic, measurable in $x \in \mathbb{R}^N$, and there is a $c > 0$ such that

$$
|f(x,u)| \le c(1 + |u|^{p-1}) \quad \text{for all } u \in \mathbb{R},\ x \in \mathbb{R}^N.
$$

(F_2) $\lim_{|u|\to 0}(f(x,u)/|u|) = 0$ uniformly in $x \in \mathbb{R}^N$.

(F_3) $\lim_{|u|\to\infty}(F(x,u)/|u|^{2q}) \to \infty$ uniformly in $x \in \mathbb{R}^N$, where $F(x,u) = \int_0^u f(x,s)\,ds \ge 0$.

(F_4) $u \mapsto f(x,u)/|u|^{2q-1}$ is strictly increasing on $(-\infty, 0)$ and $(0,\infty)$ for any $x \in \mathbb{R}^N$.

In view of (F_1)–(F_4), we assume that $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ verifies hypotheses (G_1)–(G_4), and (G_1)–(G_4) are the same as (F_1)–(F_4).

Let $\Gamma : \mathbb{R}^N \to \mathbb{R}$ verify the following hypotheses:

(Γ) $\Gamma \in L^\infty(\mathbb{R}^N)$, $\lim_{|x|\to\infty}\Gamma(x) = 0$, $\Gamma(x) \ge 0$ for almost every $x \in \mathbb{R}^N$, $\Gamma(x) \not\equiv 0$.

We assume that $V_1(x)$ and $V_2(x)$ are close-to-periodic, i.e.,

$$V_i(x) = V_{\mathrm{per}}^i(x) + V_{\mathrm{loc}}^i(x), \quad i = 1, 2,$$

where

(V_1) $V_{\mathrm{per}}^i(x) \in L^\infty(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic, $i = 1, 2$.

(V_2) $0 > V_{\mathrm{loc}}^i(x) \in L^\infty(\mathbb{R}^N)$, $\lim_{|x|\to\infty} V_{\mathrm{loc}}^i(x) = 0$, $i = 1, 2$.

(V_3) $0 < a_i \le V_i(x)$, a_i is a constant, $i = 1, 2$.

Let $\omega = (u, v)$. We obtain solutions of (1.1) from critical points of the functional $\mathcal{J}(\omega) = \mathcal{J}(u, v) : H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \to \mathbb{R}$, where

$$\begin{aligned}
\mathcal{J}(\omega) = \mathcal{J}(u, v) &= \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\, dx\, dy + \frac{1}{2}\int_{\mathbb{R}^N} V_1(x) u^2 dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}}\, dx\, dy + \frac{1}{2}\int_{\mathbb{R}^N} V_2(x) v^2 dx \\
&\quad - \int_{\mathbb{R}^N} F(x, u)\, dx - \int_{\mathbb{R}^N} G(x, v)\, dx - \frac{1}{q}\int_{\mathbb{R}^N} \Gamma(x)|u|^q|v|^q dx.
\end{aligned} \tag{1.2}$$

Denote $|\cdot|_p$ the norm in $L^p(\mathbb{R}^N)$ and $|(\cdot,\cdot)|_p = (|\cdot|_p^p + |\cdot|_p^p)^{1/p}$ the norm of a vector-valued function in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$. In order to look for ground state solutions of problem (1.1), we consider Hilbert space

$$E = E_{V_1} \times E_{V_2}, \tag{1.3}$$

with norm

$$\|\omega\|^2 = \|(u, v)\|^2 = \|u\|_{V_1}^2 + \|v\|_{V_2}^2$$

for $\omega = (u, v) \in E$, where E_{V_i} is the Hilbert space with inner product

$$\begin{aligned}
(u_i, v_i)_{V_i} &= \int_{\mathbb{R}^{2N}} \frac{(u_i(x) - u_i(y))(v_i(x) - v_i(y))}{|x - y|^{N+2s}}\, dx\, dy \\
&\quad + \int_{\mathbb{R}^N} V_i(x) u_i(x) v_i(x)\, dx,
\end{aligned}$$

and the norm is

$$||u_i||_{V_i} = \int_{\mathbb{R}^{2N}} \frac{|u_i(x) - u_i(y)|^2}{|x - y|^{N+2s}}\, dx\, dy + \int_{\mathbb{R}^N} V_i(x)|u_i(x)|^2 dx$$

for $u_i, v_i \in E_{V_i}$ and $i = 1, 2$.

We define the Nehari manifold

$$\begin{aligned}\mathcal{N} &:= \{\omega \in E \setminus \{0\} : \langle \mathcal{J}'(\omega), \omega\rangle = 0\}\\ &= \{\omega \in E \setminus \{0\} : \|\omega\|^2 = \mathcal{I}'(\omega)(\omega)\},\end{aligned}$$

where $\mathcal{I} : E \to \mathbb{R}$ is defined as

$$\begin{aligned}&\mathcal{I}(\omega) = \mathcal{I}(u, v)\\ &\quad = \int_{\mathbb{R}^N} F(x, u)\, dx + \int_{\mathbb{R}^N} G(x, v)\, dx + \frac{1}{q}\int_{\mathbb{R}^N} \Gamma(x)|u|^q|v|^q dx.\end{aligned}$$

Note that $\mathcal{I}$ is of $\mathcal{C}^1$-class.

Since (V_3), then $\|\cdot\|_{V_1}$ and $\|\cdot\|_{V_2}$ are equivalent to the standard H^s-norm, i.e., there exists a constant $C > 0$ such that

$$C^{-1}\|u\|_{H^s(\mathbb{R}^N)} \le \|u\|_{V_1} \le C\|u\|_{H^s(\mathbb{R}^N)},$$

and

$$C^{-1}\|v\|_{H^s(\mathbb{R}^N)} \le \|v\|_{V_2} \le C\|v\|_{H^s(\mathbb{R}^N)}.$$

In this paper, we use the method of Nehari manifold minimization to obtain a bounded minimizing sequence and adopt the approach of Jeanjean-Tanaka **[8]** to obtain a decomposition of the bounded Palais-Smale sequence. The Nehari manifold method goes back to Nehari's work **[16, 17]** when he considered a boundary value problem for a certain nonlinear second order ordinary differential equation in an interval (a, b) and showed that it has a nontrivial solution which may be found by constrained minimization of the Euler-Lagrange functional corresponding to the problem. The Nehari manifold method has the advantage of not requiring an Ambrosetti-Rabinowitz type condition, which is customary when dealing with the Palais-Smale condition. For a unified approach of the Nehari manifold method and application to the existence of solutions to the nonlinear boundary value problem, the interested reader may refer to Szukin and Weth **[23]**. For its application to the existence of ground states for functionals with nonhomogeneous principal part and complete continuity of $\mathcal{I}'$, the reader may refer to Figueiredo and Quoirin **[7]**. Note that, in our problem, $\mathcal{I}$ no longer satisfies the conditions in **[23]**.

Our main result in this paper is the following.

Theorem 1.1. *Suppose that* (V_1)–(V_3), (Γ), (F_1)–(F_4) *and* (G_1)–(G_4) *are satisfied. Then,* (1.1) *has a ground state* $\omega \in E \setminus \{0\}$, *i.e.,* ω *is a critical point of* $\mathcal{J}$ *such that* $\mathcal{J}(\omega) = \inf_{\mathcal{N}} \mathcal{J}$.

2. Preliminaries.

Remark 2.1.

(I_1) Set

$$\varphi(t) = \frac{t^2-1}{2}\mathcal{I}'(\omega)(\omega) + \mathcal{I}(\omega) - \mathcal{I}(t\omega), \qquad \omega \in \mathcal{N},\ t \in (0,\infty) \setminus \{1\}. \tag{2.1}$$

Observe that $\mathcal{I}'(\omega)(\omega) = \|\omega\|^2 > 0$ and $\varphi'(t) = t\mathcal{I}'(\omega)(\omega) - \mathcal{I}'(t\omega)(\omega) < 0$ for $t > 1$, and $\varphi'(t) = t\mathcal{I}'(\omega)(\omega) - \mathcal{I}'(t\omega)(\omega) > 0$ for $t < 1$, $\varphi(1) = 0$. Hence, $\varphi(t) < \varphi(1) = 0$. Therefore, we may assume that

$$(1-t)(t\mathcal{I}'(\omega)(\omega) - \mathcal{I}'(t\omega)(\omega)) > 0$$

for ω such that $\mathcal{I}'(\omega)(\omega) > 0$ and $t \in (0,\infty) \setminus \{1\}$.

(I_2) Note that the following condition (J_3) in Theorem 2.2 is equivalent to the condition: $\omega \in \mathcal{N}$ is the unique maximum point of $\mathcal{J}(t\omega)$ for $t \in (0,+\infty)$. Indeed, for $\omega \in \mathcal{N}$, it holds that

$$\begin{aligned}\mathcal{J}(t\omega) &= \left(\mathcal{J}(t\omega) - \mathcal{J}(\omega) - \frac{t^2-1}{2}\mathcal{J}'(\omega)(\omega)\right) + \mathcal{J}(\omega) \\ &= \varphi(t) + \mathcal{J}(\omega) < \mathcal{J}(\omega)\end{aligned} \tag{2.2}$$

if and only if $\varphi(t) < 0$.

In order to obtain a bounded Palais-Smale sequence, we need the following theorem [**4**]. However, for the reader's convenience, we give a sketch of the proof and remind the reader that our model is a nonlinear fractional Schrödinger system.

Theorem 2.2. [**4**] *Suppose that*:

(J_1) *there is an* $r > 0$ *such that* $a := \inf_{\|\omega\|=r} \mathcal{J}(\omega) > \mathcal{J}(0) = 0$;

(J_2) *there is a* $q \geq 2$ *such that* $\mathcal{I}(t_n\omega_n)/t_n^{2q} \to \infty$ *for* $\omega_n \to \omega$ *on* $E \setminus \{0\}$ *and any* $t_n \to \infty$ *as* $n \to \infty$;

(J_3) *the following holds*:

$$\frac{t^2-1}{2}\mathcal{I}'(\omega)(\omega) - \mathcal{I}(t\omega) + \mathcal{I}(\omega) < 0,$$

for $t \in (0,\infty) \setminus \{1\}$ *and* $\omega \in \mathcal{N}$.

(J_4) $\mathcal{J}$ *is coercive on* $\mathcal{N}$, *i.e., for* $(\omega_n) \subset \mathcal{N}$, *there is a* $\mathcal{J}(\omega_n) \to \infty$ *as* $\|\omega_n\| \to \infty$.

Then, $\inf_{\mathcal{N}} \mathcal{J} > 0$, *and there exists a bounded minimizing sequence for* $\mathcal{J}$ *on* $\mathcal{N}$, *i.e., there is a sequence* $(\omega_n) \subset \mathcal{N}$ *such that* $\mathcal{J}(\omega_n) \to \inf_{\mathcal{N}} \mathcal{J}$ *and* $\mathcal{J}'(\omega_n) \to 0$.

Remark 2.3. Observing (J_1) and (J_2), there are $\omega \neq 0$ and $t > 0$ such that $\mathcal{J}(t\omega) < 0$, and $\mathcal{J}$ has the classical Mountain Pass geometry **[1, 24]**. Then, we can find a Palais-Smale sequence, although we do not know whether it is a bounded sequence or whether it is included in $\mathcal{N}$. Therefore, we assume coercivity in ($\mathcal{J}_4$) to obtain the boundedness.

Proof of Theorem 2.2. Observe (2.2) and the map

$$\varphi(t) : [0,+\infty) \longrightarrow \mathbb{R} :$$

$$\varphi(t) = \mathcal{J}(t\omega) - \mathcal{J}(\omega) \quad \text{for } t \in [0,+\infty) \text{ and } \omega \neq 0.$$

In view of (J_1)–(J_2), we have that

$$\varphi(0) = \mathcal{J}(0) - \mathcal{J}(\omega) < \mathcal{J}\left(\frac{r}{\|\omega\|}\omega\right) - \mathcal{J}(\omega) = \varphi\left(\frac{r}{\|\omega\|}\right),$$

and $\varphi(t) \to -\infty$ as $t \to \infty$. There is a maximum point $t(\omega) > 0$ of φ which is a critical point, i.e., $\mathcal{J}'(t(\omega)\omega)(t(\omega)\omega) = 0$ for $t(\omega)\,\omega \in \mathcal{N}$. Therefore, we consider the functional $\widehat{m} : E \setminus \{0\} \to \mathcal{N}$,

$$\widehat{m}(\omega) = t(\omega)\omega \quad \text{for } \omega \neq 0.$$

Take $t_n = t(\omega_n)$ for $n \geq 0$ and $\omega_n \to \omega_0 \neq 0$. Observe that $\widehat{m}(\omega_n) = t_n\omega_n$. If $t_n \to \infty$, we have

$$\begin{aligned} o(1) &= \mathcal{J}(\omega_n)/t_n^{2q} \leq \mathcal{J}(\widehat{m}(\omega_n))/t_n^{2q} \\ &= \frac{1}{2}t_n^{2-2q}\|\omega_n\|^2 - \mathcal{I}(t_n\omega_n)/t_n^{2q} \longrightarrow -\infty \end{aligned} \tag{2.3}$$

by (2.2) and (J_2). In view of (2.3), we get a contradiction. Therefore, we may assume that $t_n \to t_0 \geq 0$. Thus,

$$\begin{aligned}\mathcal{J}(t(\omega_0)\omega_0) \geq \mathcal{J}(t_0\omega_0) &= \lim_{n\to\infty} \mathcal{J}(t_n\omega_n)\\ &\geq \lim_{n\to\infty} \mathcal{J}(t(\omega_0)\omega_n) = \mathcal{J}(t(\omega_0)\omega_0).\end{aligned}$$

Then, we obtain $t_0 = t(\omega_0)$, and $\widehat{m}$ is continuous.

Define $m = \widehat{m}|_{S^1}$, where S^1 is the unit sphere in E. The inverse function is given by $m^{-1}(\theta) = \theta/\|\theta\|$ for $\theta \in \mathcal{N}$. Then, m is a homeomorphism and

$$c := \inf_{\omega\in S^1}(\mathcal{J}\circ m)(\omega) = \inf_{\omega\in\mathcal{N}}\mathcal{J}(\omega) \geq \inf_{\omega\in\mathcal{N}}\mathcal{J}\left(\frac{r}{\|\omega\|}\omega\right) \geq a > 0$$

by (2.2). Take $z \in E$ such that $\omega_n := \omega_0 + nz \in E$. In view of

$$\widehat{m}(\omega) = \frac{\|\widehat{m}(\omega)\|}{\|\omega\|}\omega = t(\omega)\omega,$$

the mean value theorem implies that

$$\begin{aligned}&(\mathcal{J}\circ\widehat{m})(\omega_n) - (\mathcal{J}\circ\widehat{m})(\omega_0)\\ &\quad= \mathcal{J}(t_n\omega_n) - \mathcal{J}(t_0\omega_0)\\ &\quad\leq \mathcal{J}(t_n\omega_n) - \mathcal{J}(t_n\omega_0)\\ &\quad= \mathcal{J}'(t_n[\omega_0 + \tau_n(\omega_n - \omega_0)])t_n(\omega_n - \omega_0)\\ &\quad= t_0\mathcal{J}'(\widehat{m}(\omega_0))nz + o(n) \quad \text{as } n \to 0\end{aligned} \tag{2.4}$$

with some $\tau_n \in (0,1)$. In a similar manner, we obtain

$$\begin{aligned}&(\mathcal{J}\circ\widehat{m})(\omega_n) - (\mathcal{J}\circ\widehat{m})(\omega_0)\\ &\quad= \mathcal{J}(t_n\omega_n) - \mathcal{J}(t_0\omega_0)\\ &\quad\geq \mathcal{J}(t_0\omega_n) - \mathcal{J}(t_0\omega_0)\\ &\quad= \mathcal{J}'(t_0[\omega_0 + \eta_n(\omega_n - \omega_0)])t_0(\omega_n - \omega_0)\\ &\quad= t_0\mathcal{J}'(\widehat{m}(\omega_0))nz + o(n) \quad \text{as } n \to 0\end{aligned} \tag{2.5}$$

with some $\eta_n \in (0,1)$. In view of (2.4) and (2.5), it follows that the directional derivative $(\mathcal{J}\circ\widehat{m})'(\omega_0)z$ exists and is given by

$$\begin{aligned}(\mathcal{J}\circ\widehat{m})'(\omega_0)z &= \lim_{n\to 0}\frac{(\mathcal{J}\circ\widehat{m})(\omega_n)-(\mathcal{J}\circ\widehat{m})(\omega_0)}{n}\\ &= t_0\mathcal{J}'(\widehat{m}(\omega_0))z=\frac{\|\widehat{m}(\omega_0)\|}{\|\omega_0\|}\mathcal{J}'(\widehat{m}(\omega_0))z.\end{aligned}$$

We get that $\mathcal{J}\circ m : S^1\to\mathbb{R}$ is of $\mathcal{C}^1$-class.

By the Ekeland variational principle **[24,** Theorem 2.4], we find a minimizing sequence $(\theta_n)\subset S^1$ for $\mathcal{J}\circ m$ such that $(\mathcal{J}\circ m)'(\theta_n)\to 0$. Let $\omega_n=m(\theta_n)\in\mathcal{N}$, and then we obtain that $\mathcal{J}'(\omega_n)\to 0$.

For $\omega_n\in\mathcal{N}$, in view of

$$\begin{aligned}c &\le \mathcal{J}(\omega_n)\\ &= \frac{1}{2}\|\omega_n\|^2-\int_{\mathbb{R}^N}F(x,u_n)-\int_{\mathbb{R}^N}G(x,v_n)-\frac{1}{q}\int_{\mathbb{R}^N}\Gamma(x)|u_n|^q|v_n|^q\\ &\le \frac{1}{2}\|\omega_n\|^2,\end{aligned}\tag{2.6}$$

then $\|\omega_n\|\ge\sqrt{2c}$ for some $c>0$. In view of (J_2), we have

$$\mathcal{J}(t_n\omega_n)=\frac{1}{2}t_n^2\|\omega_n\|^2-\mathcal{I}(t_n\omega_n)\longrightarrow-\infty$$

for any $t_n\to\infty$ and $\omega_n\to\omega_0\ne 0$, and hence, $\mathcal{J}(\omega_n)<\infty$. We get $\sup_n\|\omega_n\|<\infty$ by the coercivity of $\mathcal{J}$, and hence, (ω_n) is a bounded minimizing sequence for $\mathcal{J}$ on $\mathcal{N}$ such that $\mathcal{J}'(\omega_n)\to 0$. □

3. Decomposition of bounded Palais-Smale sequences.

Lemma 3.1. *Suppose that* (V_1)–(V_3), (Γ), (F_1)–(F_4) *and* (G_1)–(G_4) *are satisfied. Then,* (J_1)–(J_4) *hold.*

Proof.

(J_1) Observe (F_1) and (F_2), and fixing $\varepsilon>0$ for some C_ε, we have $F(x,u)\le\varepsilon|u|^2+C_\varepsilon|u|^p$. For some constant $C>0$, applying the fractional Sobolev embedding theorem **[18,** Theorem 6.7], we have

$$\int_{\mathbb{R}^N}F(x,u)\,dx\le C(\varepsilon\|u\|^2+C_\varepsilon\|u\|^p).$$

In a similar manner, we obtain

$$\int_{\mathbb{R}^N} G(x,v)\,dx \le C(\varepsilon\|v\|^2 + C_\varepsilon\|v\|^p).$$

The Hölder inequality and the Sobolev embedding theorem imply that

$$\frac{1}{q}\int_{\mathbb{R}^N} \Gamma(x)|u|^q|v|^q dx \le \frac{1}{q}|\Gamma|_\infty |u|_{2q}^q |v|_{2q}^q \le C\max\{\|u\|^{2q}, \|v\|^{2q}\}.$$

Then, for $\|\omega\| \le r$, there is an $r > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} F(x,u)\,dx + \int_{\mathbb{R}^N} G(x,v)\,dx + \frac{1}{q}\int_{\mathbb{R}^N} \Gamma(x)|u|^q|v|^q dx \\
&\quad \le C(\varepsilon\|\omega\|^2 + C_\varepsilon(\|u\|^p + \|v\|^p)) + C\|\omega\|^{2q} \\
&\quad \le C(\varepsilon\|\omega\|^2 + C_\varepsilon\|\omega\|^p) + C\|\omega\|^{2q} \\
&\quad \le \frac{1}{4}\|\omega\|^2 \le \frac{1}{4}r^2.
\end{aligned}$$

For $\|\omega\| = r$, we have

$$\mathcal{J}(\omega) \ge \frac{1}{2}\|\omega\|^2 - \frac{1}{4}\|\omega\|^2 = \frac{1}{4}r^2 > 0.$$

(J_2) Note that

$$\begin{aligned}
\mathcal{I}(t_n\omega_n)/t_n^{2q} &= \int_{\mathbb{R}^N} \frac{F(x,t_nu_n)}{t_n^{2q}|u_n|^{2q}}|u_n|^{2q}dx + \int_{\mathbb{R}^N} \frac{G(x,t_nv_n)}{t_n^{2q}|v_n|^{2q}}|v_n|^{2q}dx \\
&\quad + \frac{1}{q}\int_{\mathbb{R}^N} \Gamma(x)|u_n|^q|v_n|^q \longrightarrow \infty.
\end{aligned}$$

Indeed, we obtain

$$\frac{F(x,t_nu_n)}{t_n^{2q}|u_n|^{2q}} \longrightarrow \infty, \qquad \frac{G(x,t_nv_n)}{t_n^{2q}|v_n|^{2q}} \longrightarrow \infty,$$

by (F_3) and (G_3). On the other hand, we have $\liminf_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2q}dx \ge \int_{\mathbb{R}^N}|u|^{2q}dx > 0$ or $\liminf_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{2q}dx \ge \int_{\mathbb{R}^N}|v|^{2q}dx > 0$ by Fatou's lemma. Hölder's inequality and Sobolev's embedding theorem imply that

$$\begin{aligned}
\frac{1}{q}\int_{\mathbb{R}^N} \Gamma(x)|u|^q|v|^q dx &\le \frac{1}{q}|\Gamma|_\infty |u|_{2q}^q |v|_{2q}^q \\
&\le C\max\{\|u\|^{2q}, \|v\|^{2q}\} < \infty.
\end{aligned}$$

(J_3) Fix ω such that $\mathcal{I}'(\omega)(\omega) > 0$; hence,

$$\mathcal{I}'(\omega)(\omega) = \int_{\mathbb{R}^N} f(x,u)u\,dx + \int_{\mathbb{R}^N} g(x,v)v\,dx + 2\int_{\mathbb{R}^N} \Gamma(x)|u|^q|v|^q dx > 0,$$

in view of

$$\begin{aligned} &t\mathcal{I}'(\omega)(\omega) - \mathcal{I}'(t\omega)(\omega) \\ &\quad = t\left(\int_{\mathbb{R}^N} f(x,u)u\,dx + \int_{\mathbb{R}^N} g(x,v)v\,dx + 2\int_{\mathbb{R}^N} \Gamma(x)|u|^q|v|^q dx\right) \\ &\qquad - \left(\int_{\mathbb{R}^N} f(x,tu)u\,dx + \int_{\mathbb{R}^N} g(x,tv)v\,dx + 2t^{2q-1}\int_{\mathbb{R}^N} \Gamma(x)|u|^q|v|^q dx\right). \end{aligned}$$

For $t < 1$, observing that (F_4) and (G_4) hold, we have

$$\begin{aligned} &t\mathcal{I}'(\omega)(\omega) - \mathcal{I}'(t\omega)(\omega) \\ &\quad > t^{2q-1}\left(\int_{\mathbb{R}^N} f(x,u)u\,dx + \int_{\mathbb{R}^N} g(x,v)v\,dx + 2\int_{\mathbb{R}^N} \Gamma(x)|u|^q|v|^q dx\right) \\ &\qquad - \left(\int_{\mathbb{R}^N} f(x,tu)u\,dx + \int_{\mathbb{R}^N} g(x,tv)v\,dx + 2t^{2q-1}\int_{\mathbb{R}^N} \Gamma(x)|u|^q|v|^q dx\right) \\ &\quad = t^{2q-1}\left(\int_{\mathbb{R}^N} f(x,u)u\,dx + \int_{\mathbb{R}^N} g(x,v)v\,dx \right. \\ &\qquad\qquad \left. - \int_{\mathbb{R}^N} \frac{f(x,tu)u}{t^{2q-1}}\,dx - \int_{\mathbb{R}^N} \frac{g(x,tv)v}{t^{2q-1}}\right) dx > 0, \end{aligned}$$

and, for $t > 1$, we have

$$\begin{aligned} &t\mathcal{I}'(\omega)(\omega) - \mathcal{I}'(t\omega)(\omega) \\ &\quad < t^{2q-1}\left(\int_{\mathbb{R}^N} f(x,u)u\,dx + \int_{\mathbb{R}^N} g(x,v)v\,dx + 2\int_{\mathbb{R}^N} \Gamma(x)|u|^q|v|^q dx\right) \\ &\qquad - \left(\int_{\mathbb{R}^N} f(x,tu)u\,dx + \int_{\mathbb{R}^N} g(x,tv)v\,dx + 2t^{2q-1}\int_{\mathbb{R}^N} \Gamma(x)|u|^q|v|^q dx\right) \\ &\quad = t^{2q-1}\left(\int_{\mathbb{R}^N} f(x,u)u\,dx + \int_{\mathbb{R}^N} g(x,v)v\,dx \right. \\ &\qquad\qquad \left. - \int_{\mathbb{R}^N} \frac{f(x,tu)u}{t^{2q-1}}\,dx - \int_{\mathbb{R}^N} \frac{g(x,tv)v}{t^{2q-1}}\right) dx < 0. \end{aligned}$$

From Remark 2.1, (J_3) holds.

(J_4) Suppose that $(\omega_n) \subset \mathcal{N}$ is a sequence such that $\|\omega_n\| \to \infty$ for $n \to \infty$. In view of $\omega = (u, v)$, for $u \geq 0$, we have

$$f(x,u)u = 2q\int_0^u \frac{f(x,u)}{u^{2q-1}} s^{2q-1} ds \geq 2q \int_0^u \frac{f(x,s)}{s^{2q-1}} s^{2q-1} ds = 2qF(x,u)$$

by (F_4). For $u < 0$, observe that

$$\begin{aligned} f(x,u)u &= -2q\int_u^0 \frac{f(x,u)}{u^{2q-1}} s^{2q-1} ds \geq -2q \int_u^0 \frac{f(x,s)}{s^{2q-1}} s^{2q-1} ds \\ &= 2qF(x,u). \end{aligned}$$

In a similar way, we obtain

$$g(x,v)v \geq 2qG(x,v).$$

Then,

$$\begin{aligned} \mathcal{J}(\omega_n) &= \frac{1}{2}\|\omega_n\|^2 - \int_{\mathbb{R}^N} F(x,u_n)\,dx \\ &\quad - \int_{\mathbb{R}^N} G(x,v_n)\,dx - \frac{1}{q}\int_{\mathbb{R}^N} \Gamma(x)|u_n|^q|v_n|^q dx \\ &= \left(\frac{1}{2} - \frac{1}{2q}\right)\|\omega_n\|^2 - \int_{\mathbb{R}^N} F(x,u_n)\,dx \\ &\quad + \frac{1}{2q}\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx - \int_{\mathbb{R}^N} G(x,v_n)\,dx \\ &\quad + \frac{1}{2q}\int_{\mathbb{R}^N} g(x,v_n)v_n\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2q}\right)\|\omega_n\|^2 \longrightarrow \infty \quad \text{as } n \to \infty, \end{aligned}$$

which completes the proof of (J_4). $\square$

We shall deduce a decomposition of bounded Palais-Smale sequences which is a key step in the proof of Theorem 1.1. Motivated by **[4, 5]**, we mainly adopt the method of Jeanjean and Tanaka in **[8,** Theorem 5.1]. Denote the functional $\mathcal{J}_{\text{per}} : E \to \mathbb{R}$ as

$$
\begin{aligned}
\mathcal{J}_{\mathrm{per}}(\omega) = {} & \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} V^1_{\mathrm{per}}(x)|u|^2 \\
& + \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} V^2_{\mathrm{per}}(x)|v|^2\,dx \\
& - \int_{\mathbb{R}^N} F(x,u)\,dx - \int_{\mathbb{R}^N} G(x,v)\,dx.
\end{aligned} \tag{3.1}
$$

Theorem 3.2. *Suppose that* (V_1)–(V_3), (Γ), (F_1)–(F_4) *and* (G_1)–(G_4) *are satisfied. Let* (ω_n) *be a bounded Palais-Smale sequence for* $\mathcal{J}$. *Then, passing to a subsequence of* (ω_n), *there exist an integer* $l > 0$ *and sequences* $(y^k_n) \subset \mathbb{Z}^N$, $\omega^k \in E$, $k = 1, \ldots, l$, *such that*:

(a) $\omega_n \rightharpoonup \omega_0$ *and* $\mathcal{J}'(\omega_0) = 0$;

(b) $|y^k_n| \to \infty$ *and* $|y^k_n - y^{k'}_n| \to \infty$ *for* $k \neq k'$;

(c) $\omega^k \neq 0$ *and* $\mathcal{J}'_{\mathrm{per}}(\omega^k) = 0$ *for each* $1 \leq k \leq l$;

(d) $\omega_n - \omega_0 - \sum_{k=1}^{l} \omega^k(x - y^k_n) \to 0$ *in* E *as* $n \to \infty$;

(e) $\mathcal{J}(\omega_n) \to \mathcal{J}(\omega_0) + \sum_{k=1}^{l} \mathcal{J}_{\mathrm{per}}(\omega^k)$.

In this section, our proof is based on Lions' lemma **[10]**. As in (2.6), there exists a $\rho > 0$ such that each nontrivial critical point ω of $\mathcal{J}$ satisfies $\|\omega\| \geq \rho$.

Proof of Theorem 3.2. The proof consists of six steps as follows.

Step 1. We may find a subsequence of (ω_n), and $\omega_0 \in E$ is a critical point of $\mathcal{J}$ such that $\omega_n \rightharpoonup \omega_0$. Since (ω_n) is bounded, there is an $\omega_0 \in E$ such that $\omega_n \rightharpoonup \omega_0$ up to a subsequence. Moreover, up to a subsequence, we may assume that $\omega_n(x) \to \omega(x)$ for almost every $x \in \mathbb{R}^N$. Let $\phi = (\varphi, \psi)$, φ, $\psi \in C_0^\infty(\mathbb{R}^N)$. Take $Q := \mathbb{R}^{2N} \setminus [(\mathbb{R}^N \setminus \operatorname{Supp}\varphi) \times (\mathbb{R}^N \setminus \operatorname{Supp}\varphi)]$ and $M := \mathbb{R}^{2N} \setminus [(\mathbb{R}^N \setminus \operatorname{Supp}\psi) \times (\mathbb{R}^N \setminus \operatorname{Supp}\psi)]$. Observe that:

$$
\begin{aligned}
& \mathcal{J}'(\omega_n)\phi - \mathcal{J}'(\omega_0)\phi \\
& \quad = \int_Q \frac{[(u_n(x)-u_0(x))-(u_n(y)-u_0(y))](\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \\
& \quad\quad + \int_M \frac{[(v_n(x)-v_0(x))-(v_n(y)-v_0(y))](\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
&+\int\limits_{\operatorname{Supp}\varphi} V_1(x)(u_n-u_0)\varphi\,dx+\int\limits_{\operatorname{Supp}\psi} V_2(x)(v_n-v_0)\psi\,dx\\
&-\int\limits_{\operatorname{Supp}\varphi}(f(x,u_n)-f(x,u_0))\varphi\,dx-\int\limits_{\operatorname{Supp}\psi}(g(x,v_n)-g(x,v_0))\psi\,dx\\
&-\Bigg(\int\limits_{\operatorname{Supp}\varphi}\Gamma(x)|u_n|^{q-2}u_n|v_n|^q\varphi\,dx-\int\limits_{\operatorname{Supp}\varphi}\Gamma(x)|u_0|^{q-2}u_0|v_0|^q\varphi\,dx\Bigg)\\
&-\Bigg(\int\limits_{\operatorname{Supp}\psi}\Gamma(x)|v_n|^{q-2}v_n|u_n|^q\psi\,dx-\int\limits_{\operatorname{Supp}\psi}\Gamma(x)|v_0|^{q-2}v_0|u_0|^q\psi\,dx\Bigg).
\end{aligned}
$$

The weak convergence $\omega_n \rightharpoonup \omega_0$ implies that

$$
\begin{aligned}
&\int_Q \frac{[(u_n(x)-u_0(x))-(u_n(y)-u_0(y))](\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad+\int_M \frac{[(v_n(x)-v_0(x))-(v_n(y)-v_0(y))](\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad+\int\limits_{\operatorname{Supp}\varphi} V_1(x)(u_n-u_0)\varphi\,dx+\int\limits_{\operatorname{Supp}\psi} V_2(x)(v_n-v_0)\psi\,dx\\
&\quad\longrightarrow 0.
\end{aligned}
$$

For measurable set $E\subset \operatorname{Supp}\varphi$, we have

$$
\int_E |f(x,u_n)\varphi|\,dx \le C(|\varphi\chi_E|_2|u_n\chi_E|_2+|\varphi\chi_E|_p|u_n\chi_E|_p^{p-1})
$$

by the Hölder inequality and (F_1). Therefore, $f(x,u_n)\varphi$ is uniformly integrable and, by the Vitali convergence theorem, we have

$$
\int\limits_{\operatorname{Supp}\varphi}(f(x,u_n)-f(x,u_0))\varphi\,dx \longrightarrow 0.
$$

Similarly, we have

$$
\int\limits_{\operatorname{Supp}\psi}(g(x,v_n)-g(x,v_0))\psi\,dx \longrightarrow 0.
$$

From the Hölder inequality, we have

$$\int_E \Gamma(x)|u_n|^{q-2}u_n|v_n|^q\varphi\,dx \le |\Gamma|_\infty \max\{|u_n\chi_E|_{2q}^{2q-1}, |v_n\chi_E|_{2q}^{2q-1}\}|\varphi\chi_E|_{2q},$$

and then, using the Vitali convergence theorem, we obtain

$$\int_{\operatorname{Supp}\varphi} \Gamma(x)|u_n|^{q-2}u_n|v_n|^q\varphi\,dx - \int_{\operatorname{Supp}\varphi} \Gamma(x)|u_0|^{q-2}u_0|v_0|^q\varphi\,dx \longrightarrow 0.$$

In a similar way, we have

$$\int_{\operatorname{Supp}\psi} \Gamma(x)|v_n|^{q-2}v_n|u_n|^q\psi\,dx - \int_{\operatorname{Supp}\psi} \Gamma(x)|v_0|^{q-2}v_0|u_0|^q\psi\,dx \longrightarrow 0.$$

Hence, $\mathcal{J}'(\omega_n)\phi - \mathcal{J}'(\omega_0)\phi \to 0$. Then, we obtain that $\mathcal{J}'(\omega_0)\phi = 0$ by (ω_n) is a Palais-Smale sequence.

Step 2. Take $\omega_n^1 = \omega_n - \omega_0$, $u_n^1 = u_n - u_0$, $v_n^1 = v_n - v_0$. Suppose that

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |\omega_n^1|^2 dx \longrightarrow 0. \tag{3.3}$$

Then, $\omega_n \to \omega_0$ and (a)–(e) hold for $l = 0$. In view of

$$\sup_{z\in\mathbb{R}^N}\int_{B(z,1)}|\omega_n^1|^2dx = \sup_{z\in\mathbb{R}^N}\bigg(\int_{B(z,1)}|u_n^1|^2dx + \int_{B(z,1)}|v_n^1|^2dx\bigg) \longrightarrow 0,$$

we have that

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n^1|^2 dx \longrightarrow 0 \tag{3.4}$$

and

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |v_n^1|^2 dx \longrightarrow 0. \tag{3.5}$$

Observing that

$$\begin{aligned}\mathcal{J}'(\omega_n)\omega_n^1 &= \int_{\mathbb{R}^{2N}} \frac{|u_n^1(x) - u_n^1(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &\quad + \int_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(u_n^1(x) - u_n^1(y))}{|x-y|^{N+2s}}\,dx\,dy\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N} V_1(x)|u_n^1|^2\,dx+\int_{\mathbb{R}^N} V_1(x)u_0u_n^1\,dx\\
&+\int_{\mathbb{R}^{2N}} \frac{|v_n^1(x)-v_n^1(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&+\int_{\mathbb{R}^{2N}} \frac{(v_0(x)-v_0(y))(v_n^1(x)-v_n^1(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&+\int_{\mathbb{R}^N} V_2(x)|v_n^1|^2\,dx+\int_{\mathbb{R}^N} V_2(x)v_0v_n^1\,dx\\
&-\int_{\mathbb{R}^N} f(x,u_n)u_n^1\,dx-\int_{\mathbb{R}^N} g(x,v_n)v_n^1\,dx\\
&-\int_{\mathbb{R}^N}\Gamma(x)|u_n|^{q-2}u_n|v_n|^q u_n^1\,dx-\int_{\mathbb{R}^N}\Gamma(x)|v_n|^{q-2}v_n|u_n|^q v_n^1\,dx,
\end{aligned}
$$

we obtain

(3.6)

$$
\begin{aligned}
\|\omega_n^1\|^2 = {}& \mathcal{J}'(\omega_n)\omega_n^1-\int_{\mathbb{R}^{2N}} \frac{(u_0(x)-u_0(y))(u_n^1(x)-u_n^1(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&-\int_{\mathbb{R}^N} V_1(x)u_0u_n^1\,dx-\int_{\mathbb{R}^{2N}}\frac{(v_0(x)-v_0(y))(v_n^1(x)-v_n^1(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&-\int_{\mathbb{R}^N} V_2(x)v_0v_n^1dx+\int_{\mathbb{R}^N} f(x,u_n)u_n^1\,dx\\
&+\int_{\mathbb{R}^N} g(x,v_n)v_n^1\,dx+\int_{\mathbb{R}^N}\Gamma(x)|u_n|^{q-2}u_n|v_n|^q u_n^1\,dx\\
&+\int_{\mathbb{R}^N}\Gamma(x)|v_n|^{q-2}v_n|u_n|^q v_n^1\,dx.
\end{aligned}
$$

Since $\mathcal{J}'(\omega_0)\omega_n^1=0$, then

$$
\begin{aligned}
\|\omega_n^1\|^2 = {}& \mathcal{J}'(\omega_n)\omega_n^1+\int_{\mathbb{R}^N} f(x,u_n)u_n^1\,dx-\int_{\mathbb{R}^N} f(x,u_0)u_n^1\,dx\\
&+\int_{\mathbb{R}^N} g(x,v_n)v_n^1\,dx-\int_{\mathbb{R}^N} g(x,v_0)v_n^1\,dx\\
&+\int_{\mathbb{R}^N}\Gamma(x)|u_n|^{q-2}u_n|v_n|^q u_n^1\,dx-\int_{\mathbb{R}^N}\Gamma(x)|u_0|^{q-2}u_0|v_0|^q u_n^1\,dx\\
&+\int_{\mathbb{R}^N}\Gamma(x)|v_n|^{q-2}v_n|u_n|^q v_n^1\,dx-\int_{\mathbb{R}^N}\Gamma(x)|v_0|^{q-2}v_0|u_0|^q v_n^1\,dx.
\end{aligned}
$$

Then, since (ω_n^1) is bounded,

$$\|\mathcal{J}'(\omega_n)(\omega_n^1)\| \le \|\mathcal{J}'(\omega_n)\|\|\omega_n^1\| \longrightarrow 0.$$

The Hölder inequality and (F_1) imply that

$$\left|\int_{\mathbb{R}^N} f(x,u_n)u_n^1\,dx\right| \le \varepsilon|u_n|_2|u_n^1|_2 + C_\varepsilon|u_n|_p^{p-1}|u_n^1|_p.$$

For each $4 \le p < 2^*$, in view of (3.4), it follows that $u_n^1 \to 0$ in $L^p(\mathbb{R}^N)$ by Lions' lemma [**10,** Lemma I.1]. Thus, $\int_{\mathbb{R}^N} f(x,u_n)u_n^1\,dx \to 0$. In a similar manner, we have

$$\int_{\mathbb{R}^N} f(x,u_0)u_n^1\,dx \longrightarrow 0, \qquad \int_{\mathbb{R}^N} g(x,v_n)v_n^1\,dx \longrightarrow 0$$

and

$$\int_{\mathbb{R}^N} g(x,v_0)v_n^1\,dx \longrightarrow 0.$$

From the Hölder inequality,

$$\int_{\mathbb{R}^N} \Gamma(x)|u_n|^{q-2}u_n|v_n|^q u_n^1\,dx \le |\Gamma|_\infty \max\{|u_n|_{2q}^{2q-1}, |v_n|_{2q}^{2q-1}\}|u_n^1|_{2q}.$$

In view of (3.4) and, since $4 \le 2q < 2^*$, $u_n^1 \to 0$ in $L^{2q}(\mathbb{R}^N)$ by Lions' lemma [**10,** Lemma I.1]. Therefore,

$$\int_{\mathbb{R}^N} \Gamma(x)|u_n|^{q-2}u_n|v_n|^q u_n^1\,dx \longrightarrow 0.$$

Similarly, we have

$$\int_{\mathbb{R}^N} \Gamma(x)|u_0|^{q-2}u_0|v_0|^q u_n^1\,dx \longrightarrow 0,$$
$$\int_{\mathbb{R}^N} \Gamma(x)|v_n|^{q-2}v_n|u_n|^q v_n^1\,dx \longrightarrow 0$$

and

$$\int_{\mathbb{R}^N} \Gamma(x)|v_0|^{q-2}v_0|u_0|^q v_n^1\,dx \longrightarrow 0.$$

Finally, we have $\|\omega_n^1\| \to 0$, which completes the proof of Step 2.

Step 3. Suppose that there is a sequence $(z_n) \subset \mathbb{Z}^N$ such that

$$\liminf_{n\to\infty} \int_{B(z_n,1+\sqrt{N})} |\omega_n^1|^2 dx > 0. \tag{3.7}$$

Then, there is an $\omega^0 \in E$ such that:

(i) $|z_n| \to \infty$,

(ii) $\omega_n(x+z_n) \rightharpoonup \omega^0 \neq 0$, $\omega^0 = (u^0, v^0)$, $u^0, v^0 \in H^s(\mathbb{R}^N)$,

(iii) $\mathcal{J}'_{\text{per}}(\omega^0) = 0$. Indeed,

(a) due to $\omega_n \rightharpoonup \omega_0$ and (3.7), we deduce that $|z_n| \to \infty$.

(b) Observe that $\omega_n(x+z_n)$ in a bounded space; therefore, there is an $\omega^0 \in E$ such that $\omega_n(x+z_n) \rightharpoonup \omega^0 \neq 0$.

(c) Take $\gamma_n = \omega_n(x+z_n)$, $\gamma_n = (\alpha_n, \beta_n)$, $\alpha_n = u_n(x+z_n)$ and $\beta_n = v_n(x+z_n)$ for each $\phi = (\varphi, \psi)$, $\varphi, \psi \in C_0^\infty(\mathbb{R}^N)$. In view of:

$$
\begin{aligned}
\mathcal{J}'(\omega_n)(\phi(x-z_n)) &= \mathcal{J}'(u_n, v_n)(\varphi(x-z_n), \psi(x-z_n)) \\
&= \int_{\mathbb{R}^N} \nabla\alpha_n \nabla\varphi\, dx + \int_{\mathbb{R}^N} V_1(x+z_n)\alpha_n\varphi\, dx \\
&\quad + \int_{\mathbb{R}^N} \nabla\beta_n \nabla\psi\, dx + \int_{\mathbb{R}^N} V_2(x+z_n)\beta_n\psi\, dx \\
&\quad - \int_{\mathbb{R}^N} f(x, \alpha_n)\varphi\, dx - \int_{\mathbb{R}^N} g(x, \beta_n)\psi\, dx \\
&\quad - \int_{\mathbb{R}^N} \Gamma(x+z_n)|\alpha_n|^{q-2}\alpha_n|\beta_n|^q\varphi\, dx \\
&\quad - \int_{\mathbb{R}^N} \Gamma(x+z_n)|\beta_n|^{q-2}\beta_n|\alpha_n|^q\psi\, dx,
\end{aligned} \tag{3.8}
$$

there is a $B > 0$ such that

$$
\begin{aligned}
\|\mathcal{J}'(\omega_n)(\phi(x-z_n))\| &\le \|\mathcal{J}'(\omega_n)\|\|\phi(x-z_n)\| \\
&\le B\|\mathcal{J}'(\omega_n)\|\|\phi\|_{H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N)} \longrightarrow 0.
\end{aligned}
$$

In view of (Γ), for almost every $x \in \mathbb{R}^N$, $\Gamma(x+z_n) \to 0$ as $|z_n| \to \infty$. Hence,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \Gamma(x+z_n)|\alpha_n|^{q-2}\alpha_n|\beta_n|^q\varphi\, dx \\
&\qquad + \int_{\mathbb{R}^N} \Gamma(x+z_n)|\beta_n|^{q-2}\beta_n|\alpha_n|^q\psi\, dx \longrightarrow 0.
\end{aligned} \tag{3.9}
$$

For almost every $x \in \mathbb{R}^N$, $V_{\text{loc}}^1(x+z_n) \to 0$ as $|z_n| \to \infty$. Then,

$$\int_{\mathbb{R}^N} V_{\text{loc}}^1(x+z_n)\alpha_n\varphi\,dx = \int_{\mathbb{R}^N} (V_1(x+z_n) - V_{\text{per}}^1(x+z_n))\alpha_n\varphi\,dx \longrightarrow 0. \tag{3.10}$$

In a similar way, we get

$$\int_{\mathbb{R}^N} V_{\text{loc}}^2(x+z_n)\beta_n\psi\,dx = \int_{\mathbb{R}^N} (V_2(x+z_n) - V_{\text{per}}^2(x+z_n))\beta_n\psi\,dx \longrightarrow 0.$$

In view of (3.8), (3.9) and (3.10), we obtain

$$\begin{aligned}\mathcal{J}_{\text{per}}'(\omega_n)(\phi) = \mathcal{J}'(\omega_n)(\phi(x-z_n))\\ &+ \int_{\mathbb{R}^N} \Gamma(x+z_n)|\alpha_n|^{q-2}\alpha_n|\beta_n|^q\varphi\,dx\\ &+ \int_{\mathbb{R}^N} \Gamma(x+z_n)|\beta_n|^{q-2}\beta_n|\alpha_n|^q\psi\,dx\\ &- \int_{\mathbb{R}^N} V_{\text{loc}}^1(x+z_n)\alpha_n\varphi\,dx\\ &- \int_{\mathbb{R}^N} V_{\text{loc}}^2(x+z_n)\beta_n\psi\,dx \longrightarrow 0.\end{aligned}$$

On the other hand, in a similar manner as in Step 1, we obtain

$$\mathcal{J}_{\text{per}}'(\omega_n)\phi - \mathcal{J}_{\text{per}}'(\omega^0)\phi \longrightarrow 0.$$

Therefore,

$$\mathcal{J}_{\text{per}}'(\omega^0) = 0.$$

Step 4. Suppose that there exist $m \geq 1$, $\omega^k = (u^k, v^k) \in E$, $(y_n^k) \subset \mathbb{Z}^N$ for $1 \leq k \leq m$ such that $|y_n^k| \to \infty$, $|y_n^k - y_n^{k'}| \to \infty$ for $k \neq k'$; $\omega_n(x+y_n^k) \to \omega^k \neq 0$ for each $1 \leq k \leq m$ and $\mathcal{J}_{\text{per}}'(\omega^k) = 0$ for each $1 \leq k \leq m$. Then:

(1) if

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} \left|\omega_n - \omega_0 - \sum_{k=1}^m \omega^k(x-y_n^k)\right|^2 dx \longrightarrow 0$$

for $n \to \infty$, then $\|\omega_n - \omega_0 - \sum_{k=1}^m \omega^k(x-y_n^k)\| \to 0$;

(2) if there exist $(z_n) \subset \mathbb{Z}^N$ such that

$$\liminf_{n\to\infty} \int_{B(z_n,1+\sqrt{N})} \left|\omega_n - \omega_0 - \sum_{k=1}^{m} \omega^k(x-y_n^k)\right|^2 dx > 0, \tag{3.11}$$

then there is an $\omega^{m+1} \in E$ such that (up to subsequences):

(i) $|z_n| \to \infty$, $|z_n - y_n^k| \to \infty$, for $1 \le k \le m$,
(ii) $\omega_n^{m+1}(x+z_n) \rightharpoonup \omega^{m+1} \neq 0$,
(iii) $\mathcal{J}'_{\mathrm{per}}(\omega^{m+1}) = 0$.

(a) Denote $\xi_n = \omega_n - \omega_0 - \sum_{k=1}^m \omega^k(x-y_n^k)$ and $\xi_n = (\theta_n, \rho_n)$. Then, we have $\theta_n = u_n - u_0 - \sum_{k=1}^m u^k(x-y_n^k)$ and $\rho_n = v_n - v_0 - \sum_{k=1}^m v^k(x-y_n^k)$. If

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} \left|\omega_n - \omega_0 - \sum_{k=1}^{m} \omega^k(x-y_n^k)\right|^2 dx \longrightarrow 0 \quad \text{for } n \to \infty,$$

we obtain

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |\theta_n|^2 \longrightarrow 0 \quad \text{for } n \to \infty \tag{3.12}$$

and

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |\rho_n|^2 \longrightarrow 0 \quad \text{for } n \to \infty. \tag{3.13}$$

Direct computations yield

(3.14)

$$\begin{aligned}
\mathcal{J}'(\omega_n)\xi_n &= \int_{\mathbb{R}^{2N}} \frac{|\theta_n(x)-\theta_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&+ \int_{\mathbb{R}^{2N}} \frac{(u_0(x)-u_0(y))(\theta_n(x)-\theta_n(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&+ \int_{\mathbb{R}^{2N}} \frac{(\sum_{k=1}^m u^k(x-y_n^k) - \sum_{k=1}^m u^k(y-y_n^k))(\theta_n(x)-\theta_n(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&+ \int_{\mathbb{R}^N} V_1(x)|\theta_n|^2\,dx + \int_{\mathbb{R}^N} V_1(x)u_0\theta_n\,dx \\
&+ \int_{\mathbb{R}^N} V_1(x)\sum_{k=1}^m u^k(x-y_n^k)\theta_n\,dx + \int_{\mathbb{R}^{2N}} \frac{|\rho_n(x)-\rho_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^{2N}} \frac{(v_0(x)-v_0(y))(\rho_n(x)-\rho_n(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&+\int_{\mathbb{R}^{2N}} \frac{(\sum_{k=1}^m v^k(x-y_n^k)-\sum_{k=1}^m v^k(y-y_n^k))(\rho_n(x)-\rho_n(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&+\int_{\mathbb{R}^N} V_2(x)|\rho_n|^2\,dx + \int_{\mathbb{R}^N} V_2(x)v_0\rho_n\,dx \\
&+\int_{\mathbb{R}^N} V_2(x)\sum_{k=1}^m v^k(x-y_n^k)\rho_n dx - \int_{\mathbb{R}^N} f(x,u_n)\theta_n dx - \int_{\mathbb{R}^N} g(x,v_n)\rho_n dx \\
&-\int_{\mathbb{R}^N} \Gamma(x)|u_n|^{q-2}u_n|v_n|^q\theta_n dx - \int_{\mathbb{R}^N} \Gamma(x)|v_n|^{q-2}v_n|u_n|^q\rho_n dx
\end{aligned}
$$

and

(3.15)

$$
\begin{aligned}
\mathcal{J}'(\omega_0)\xi_n = &\int_{\mathbb{R}^{2N}} \frac{(u_0(x)-u_0(y))(\theta_n(x)-\theta_n(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&+\int_{\mathbb{R}^N} V_1(x)u_0\theta_n\,dx \\
&+\int_{\mathbb{R}^{2N}} \frac{(v_0(x)-v_0(y))(\rho_n(x)-\rho_n(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&+\int_{\mathbb{R}^N} V_2(x)v_0\rho_n\,dx - \int_{\mathbb{R}^N} f(x,u_0)\theta_n dx - \int_{\mathbb{R}^N} g(x,v_0)\rho_n dx \\
&-\int_{\mathbb{R}^N} \Gamma(x)|u_0|^{q-2}u_0|v_0|^q\theta_n dx - \int_{\mathbb{R}^N} \Gamma(x)|v_0|^{q-2}v_0|u_0|^q\rho_n dx.
\end{aligned}
$$

Combining (3.14) and (3.15) yields

$$
\begin{aligned}
\|\xi_n\|^2 = &\mathcal{J}'(\omega_n)(\xi_n) \\
&-\sum_{k=1}^m \int_{\mathbb{R}^{2N}} \frac{(u^k(x-y_n^k)-u^k(y-y_n^k))(\theta_n(x)-\theta_n(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&-\sum_{k=1}^m \int_{\mathbb{R}^N} V_1(x)u^k(x-y_n^k)\theta_n\,dx \\
&-\sum_{k=1}^m \int_{\mathbb{R}^{2N}} \frac{(v^k(x-y_n^k)-v^k(y-y_n^k))(\rho_n(x)-\rho_n(y))}{|x-y|^{N+2s}}\,dx\,dy
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^N} V_2(x)\Big(\sum_{k=1}^m v^k(x-y_n^k)\Big)\rho_n\,dx \\
&+ \int_{\mathbb{R}^N} f(x,u_n)\theta_n\,dx - \int_{\mathbb{R}^N} f(x,u_0)\theta_n\,dx \\
&+ \int_{\mathbb{R}^N} g(x,v_n)\rho_n\,dx - \int_{\mathbb{R}^N} g(x,v_0)\rho_n\,dx \\
&+ \int_{\mathbb{R}^N} \Gamma(x)|u_n|^{q-2}u_n|v_n|^q\theta_n\,dx - \int_{\mathbb{R}^N} \Gamma(x)|u_0|^{q-2}u_0|v_0|^q\theta_n\,dx \\
&+ \int_{\mathbb{R}^N} \Gamma(x)|v_n|^{q-2}v_n|u_n|^q\rho_n\,dx - \int_{\mathbb{R}^N} \Gamma(x)|v_0|^{q-2}v_0|u_0|^q\rho_n\,dx.
\end{aligned}
$$

Since $\mathcal{J}'_{\mathrm{per}}(\omega^k)\xi_n = 0$, iterating (3.15) with ω^k instead of ω_0 and then combining it with (3.16), $k = 1, \ldots, m$, it follows that

(3.17)

$$
\begin{aligned}
\|\xi_n\|^2 = \mathcal{J}'(\omega_n)(\xi_n) &- \sum_{k=1}^m \int_{\mathbb{R}^N} V^1_{\mathrm{loc}}(x)u^k(x-y_n^k)\theta_n\,dx \\
&- \sum_{k=1}^m \int_{\mathbb{R}^N} V^2_{\mathrm{loc}}(x)v^k(x-y_n^k)\rho_n\,dx \\
&+ \int_{\mathbb{R}^N} f(x,u_n)\theta_n\,dx - \int_{\mathbb{R}^N} f(x,u_0)\theta_n\,dx \\
&- \sum_{k=1}^m \int_{\mathbb{R}^N} f(x,u^k)\theta_n(x+y_n^k)\,dx \\
&+ \int_{\mathbb{R}^N} g(x,v_n)\rho_n\,dx - \int_{\mathbb{R}^N} g(x,v_0)\rho_n\,dx \\
&- \sum_{k=1}^m \int_{\mathbb{R}^N} g(x,v^k)\rho_n(x+y_n^k)\,dx \\
&+ \int_{\mathbb{R}^N} \Gamma(x)|u_n|^{q-2}u_n|v_n|^q\theta_n\,dx - \int_{\mathbb{R}^N} \Gamma(x)|u_0|^{q-2}u_0|v_0|^q\theta_n\,dx \\
&- \sum_{k=1}^m \int_{\mathbb{R}^N} \Gamma(x)|u^k(x-y_n^k)|^{q-2}u^k(x-y_n^k)|v^k(x-y_n^k)|^q\theta_n\,dx \\
&+ \int_{\mathbb{R}^N} \Gamma(x)|v_n|^{q-2}v_n|u_n|^q\rho_n\,dx - \int_{\mathbb{R}^N} \Gamma(x)|v_0|^{q-2}v_0|u_0|^q\rho_n\,dx \\
&- \sum_{k=1}^m \int_{\mathbb{R}^N} \Gamma(x)|v^k(x-y_n^k)|^{q-2}v^k(x-y_n^k)|u^k(x-y_n^k)|^q\rho_n\,dx.
\end{aligned}
$$

Note that
$$\|\mathcal{J}'(\omega_n)(\xi_n)\| \le \|\mathcal{J}'(\omega_n)\|\|\xi_n\| \longrightarrow 0$$
and
$$\int_{\mathbb{R}^N} V^1_{\mathrm{loc}}(x)u^k(x-y^k_n)\theta_n\,dx = \int_{\mathbb{R}^N} V^1_{\mathrm{loc}}(x+y^k_n)u^k(x)\theta_n(x+y^k_n)\,dx \longrightarrow 0$$
by (V_2). In a similar way,
$$\int_{\mathbb{R}^N} V^2_{\mathrm{loc}}(x)v^k(x-y^k_n)\rho_n\,dx = \int_{\mathbb{R}^N} V^2_{\mathrm{loc}}(x+y^k_n)v^k(x)\rho_n(x+y^k_n)\,dx \longrightarrow 0.$$
Since $\Gamma(x+y^k_n) \to 0$ as $y^k_n \to \infty$, then
$$\begin{aligned}\int_{\mathbb{R}^N} &\Gamma(x)|u^k(x-y^k_n)|^{q-2}u^k(x-y^k_n)|v^k(x-y^k_n)|^q\theta_n\,dx\\ &= \int_{\mathbb{R}^N} \Gamma(x+y^k_n)|u^k(x)|^{q-2}u^k(x)|v^k(x)|^q\theta_n(x+y^k_n)\,dx \longrightarrow 0,\end{aligned}$$
and
$$\begin{aligned}\int_{\mathbb{R}^N} &\Gamma(x)|v^k(x-y^k_n)|^{q-2}v^k(x-y^k_n)|u^k(x-y^k_n)|^q\rho_n\,dx\\ &= \int_{\mathbb{R}^N} \Gamma(x+y^k_n)|v^k(x)|^{q-2}v^k(x)|u^k(x)|^q\rho_n(x+y^k_n)\,dx \longrightarrow 0.\end{aligned}$$
Observe that, from Lions' lemma [**10,** Lemma I.1], (3.12) and (3.13), we have $\theta_n \to 0$ and $\rho_n \to 0$ in $L^p(\mathbb{R}^N)$ for each $4 \le p < 2^*$. Then, Hölder's inequality implies that
$$\left|\int_{\mathbb{R}^N} f(x,u_0)\theta_n\,dx\right| \le \varepsilon|u_0|_2|\theta_n|_2 + C_\varepsilon|u_0|_p^{p-1}|\theta_n|_p \longrightarrow 0. \tag{3.18}$$
As above, in view of Lions' lemma, we have $\theta_n \to 0$ and $\rho_n \to 0$ in $L^{2q}(\mathbb{R}^N)$, for each $4 \le 2q < 2^*$. Then, applying Hölder's inequality, we obtain
$$\begin{aligned}\int_{\mathbb{R}^N} &\Gamma(x)|u_n|^{q-2}u_n|v_n|^q\theta_n\,dx\\ &\le |\Gamma|_\infty \max\{|u_n|_{2q}^{2q-1}, |v_n|_{2q}^{2q-1}\}|\theta_n|_{2q} \longrightarrow 0.\end{aligned} \tag{3.19}$$
Similarly to (3.18) and (3.19), we prove that all of the integrals in (3.17) tend to 0. Hence, $\|\xi_n\| \to 0$, which completes the proof of (a).

(b) For some $(z_n) \subset \mathbb{Z}^N$, suppose that

$$\liminf_{n\to\infty} \int_{B(z_n,1+\sqrt{N})} \left|\omega_n - \omega_0 - \sum_{k=1}^{m} \omega^k(x-y_n^k)\right|^2 dx > 0.$$

(i) Since $\omega_n - \omega_0 - \sum_{k=1}^{m} \omega^k(x-y_n^k) \rightharpoonup 0$ and (3.11), we easily obtain $|z_n| \to \infty$ and $|z_n - y_n^k| \to \infty$ for $1 \le k \le m$.

(ii) It is standard that $\omega_n(x+z_n)$ in a bounded space, and hence, there is an $\omega^{m+1} \in E$ such that $\omega_n(x+z_n) \rightharpoonup \omega^{m+1} \neq 0$.

(iii) Similarly to Step 3, let $\gamma_n = \omega_n(x+z_n)$ and $\gamma_n = (\alpha_n, \beta_n)$ for $\phi = (\varphi, \psi)$. Then, similarly to Step 1, we have

$$\mathcal{J}'_{\rm per}(\gamma_n)(\phi) - \mathcal{J}'_{\rm per}(\omega^{m+1})(\phi) \longrightarrow 0.$$

Thus, the proof of (iii) is similar to Step 3. We have $\mathcal{J}'_{\rm per}(\gamma_n)(\phi) \to 0$. Therefore, $\mathcal{J}'_{\rm per}(\omega^{m+1}) = 0$, which completes the proof of (b).

Step 5. Summary of Steps 1–4. From Step 1, we obtain $\omega_n \rightharpoonup \omega_0$ and $\mathcal{J}'(\omega_0) = 0$, which completes the proof of (a). Suppose that condition (3.3) is satisfied. Then, $\omega_n \to \omega_0$, and the theorem is true for $l = 0$. Otherwise, for some $(y_n) \subset \mathbb{R}^N$, we have

$$\liminf_{n\to\infty} \int_{B(y_n,1)} |\omega_n^1|^2 dx > 0.$$

Then, we find $z_n \in \mathbb{Z}^N$ for each $y_n \in \mathbb{R}^N$ such that

$$B(y_n, 1) \subset B(z_n, 1+\sqrt{N}).$$

Therefore,

$$\liminf_{n\to\infty} \int_{B(z_n,1+\sqrt{N})} |\omega_n^1|^2 dx \ge \liminf_{n\to\infty} \int_{B(y_n,1)} |\omega_n^1|^2 dx > 0.$$

Thus, in view of Step 3, we find ω^0 such that (i)–(iii) are true. Take $\omega^1 = \omega^0$ and $y_n^1 = z_n$. Suppose that (1) of Step 4 holds with $m = 1$. Then, (b)–(d) hold. Otherwise, (2) of Step 4 holds. Then, take $\omega^2 = \omega^0$ and $y_n^2 = z_n$. Next, iterate Step 4. It is sufficient to prove that the procedure is complete after a finite number of steps. Indeed, for each $m \ge 1$, observe that

$$\lim_{n\to\infty} \|\omega_n\|^2 - \|\omega_0\|^2 - \sum_{k=1}^{m} \|\omega^k\|^2 = \lim_{n\to\infty} \left\|\omega_n - \omega_0 - \sum_{k=1}^{m} \omega^k(x-y_n^k)\right\|^2 \ge 0.$$

There is an $\eta > 0$, and ω^k is critical point of $\mathcal{J}_{\text{per}}$ such that $\|\omega^k\| \geq \eta$. Therefore, condition (1) in Step 4 will hold by a finite number of steps.

Step 6. We will show that (e) holds, i.e.,

$$\mathcal{J}(\omega_n) \longrightarrow \mathcal{J}(\omega_0) + \sum_{k=1}^{l} \mathcal{J}_{\text{per}}(\omega^k).$$

Take $\omega_n(x) = \omega_n(x) - \omega_0(x) + \omega_0(x)$. Then,

$$\begin{aligned}
\mathcal{J}(\omega_n) = {} & \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|(u_n(x) - u_0(x)) - (u_n(y) - u_0(y))|^2}{|x-y|^{N+2s}}\, dx\, dy \\
& + \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \\
& + \int_{\mathbb{R}^{2N}} \frac{[(u_n(x) - u_0(x)) - (u_n(y) - u_0(y))](u_0(x) - u_0(y))}{|x-y|^{N+2s}}\, dx\, dy \\
& + \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|(v_n(x) - v_0(x)) - (v_n(y) - v_0(y))|^2}{|x-y|^{N+2s}}\, dx\, dy \\
& + \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|v_0(x) - v_0(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \\
& + \int_{\mathbb{R}^{2N}} \frac{[(v_n(x) - v_0(x)) - (v_n(y) - v_0(y))](v_0(x) - v_0(y))}{|x-y|^{N+2s}}\, dx\, dy \\
& + \frac{1}{2} \int_{\mathbb{R}^N} V_1(x)|u_n - u_0|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V_1(x)|u_0|^2 dx \\
& + \int_{\mathbb{R}^N} V_1(x)(u_n - u_0)u_0\, dx + \frac{1}{2} \int_{\mathbb{R}^N} V_2(x)|v_n - v_0|^2 dx \\
& + \frac{1}{2} \int_{\mathbb{R}^N} V_2(x)|v_0|^2 dx + \int_{\mathbb{R}^N} V_2(x)(v_n - v_0)v_0\, dx \\
& - \int_{\mathbb{R}^N} F(x, u_n)\, dx - \int_{\mathbb{R}^N} G(x, v_n)\, dx - \frac{1}{q} \int_{\mathbb{R}^N} \Gamma(x)|u_n|^q |v_n|^q dx.
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathcal{J}(\omega_n) = {} & \mathcal{J}(\omega_0) + \mathcal{J}_{\text{per}}(\omega_n - \omega_0) \\
& + \int_{\mathbb{R}^{2N}} \frac{[(u_n(x) - u_0(x)) - (u_n(y) - u_0(y))](u_0(x) - u_0(y))}{|x-y|^{N+2s}}\, dx\, dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N} V_1(x)(u_n-u_0)u_0\,dx\\
&+\int_{\mathbb{R}^{2N}} \frac{[(v_n(x)-v_0(x))-(v_n(y)-v_0(y))](v_0(x)-v_0(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&+\int_{\mathbb{R}^N} V_2(x)(v_n-v_0)v_0\,dx+\frac{1}{2}\int_{\mathbb{R}^N} V_{\mathrm{loc}}^1(x)(u_n-u_0)^2dx\\
&+\frac{1}{2}\int_{\mathbb{R}^N} V_{\mathrm{loc}}^2(x)(v_n-v_0)^2dx+\int_{\mathbb{R}^N} F(x,u_n-u_0)\,dx\\
&+\int_{\mathbb{R}^N} F(x,u_0)\,dx-\int_{\mathbb{R}^N} F(x,u_n)\,dx+\int_{\mathbb{R}^N} G(x,v_0)\,dx\\
&+\int_{\mathbb{R}^N} G(x,v_n-v_0)\,dx-\int_{\mathbb{R}^N} G(x,v_n)\,dx\\
&+\frac{1}{q}\int_{\mathbb{R}^N}\Gamma(x)|u_0|^q|v_0|^qdx-\frac{1}{q}\int_{\mathbb{R}^N}\Gamma(x)|u_n|^q|v_n|^qdx.
\end{aligned}
\tag{3.20}
$$

The weak convergence theorem implies that

$$
\begin{aligned}
\int_{\mathbb{R}^{2N}} &\frac{[(u_n(x)-u_0(x))-(u_n(y)-u_0(y))](u_0(x)-u_0(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&+\int_{\mathbb{R}^N} V_1(x)(u_n-u_0)u_0\,dx\\
&+\int_{\mathbb{R}^{2N}} \frac{[(v_n(x)-v_0(x))-(v_n(y)-v_0(y))](v_0(x)-v_0(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&+\int_{\mathbb{R}^N} V_2(x)(v_n-v_0)v_0\,dx\longrightarrow 0.
\end{aligned}
$$

Let $E\subset\mathbb{R}^N$ be a measurable set. Since

$$\frac{1}{2}\int_E V_{\mathrm{loc}}^1(x)(u_n-u_0)^2\,dx\le\frac{1}{2}|V_{\mathrm{loc}}^1|_\infty|(u_n-u_0)\chi_E|_2^2$$

and

$$\frac{1}{2}\int_E V_{\mathrm{loc}}^2(x)(v_n-v_0)^2\,dx\le\frac{1}{2}|V_{\mathrm{loc}}^2|_\infty|(v_n-v_0)\chi_E|_2^2,$$

by the Vitali convergence theorem, we have

$$\frac{1}{2}\int_{\mathbb{R}^N} V_{\mathrm{loc}}^1(x)(u_n-u_0)^2\,dx\longrightarrow\frac{1}{2}\int_{\mathbb{R}^N} V_{\mathrm{loc}}^1(x)(u_0-u_0)^2dx=0$$

and

$$\frac{1}{2}\int_{\mathbb{R}^N} V_{\text{loc}}^2(x)(v_n - v_0)^2\,dx \longrightarrow \frac{1}{2}\int_{\mathbb{R}^N} V_{\text{loc}}^2(x)(v_0 - v_0)^2 dx = 0.$$

We denote the function $H : \mathbb{R}^N \times [0,1] \to \mathbb{R}$, given by $H(x,t) = F(x, u_n - tu_0)$. Observe that

$$F(x, u_n - u_0) - F(x, u_n) = H(x,1) - H(x,0) = \int_0^1 \frac{\partial H}{\partial s}(x,s)\,ds.$$

Then,

$$\begin{aligned}(3.21)\qquad &\int_{\mathbb{R}^N}[F(x,u_n-u_0) - F(x,u_n) + F(x,u_0)]\,dx \\ &\quad = \int_{\mathbb{R}^N}\left[\int_0^1 \frac{\partial H}{\partial s}(x,s)\,ds + F(x,u_0)\right]dx \\ &\quad = \int_0^1\int_{\mathbb{R}^N} -f(x,u_n - su_0)u_0\,dx\,ds + \int_{\mathbb{R}^N} F(x,u_0)\,dx.\end{aligned}$$

Let $E \subset \mathbb{R}^N$ be a measurable set. From the Hölder inequality, there is a $C > 0$ such that

$$\begin{aligned}&\int_E |f(x,u_n - su_0)u_0|\,dx \\ &\quad \le C\left(\int_E |u_n - su_0||u_0|\,dx + \int_E |u_n - su_0|^{r-1}|u_0|\,dx\right) \\ &\quad \le C(|(u_n - su_0)\chi_E|_2|u_0\chi_E|_2 + |(u_n - su_0)\chi_E|_r^{r-1}|u_0\chi_E|_r).\end{aligned}$$

Hence, $f(x, u_n - su_0)u_0$ is uniformly integrable, and

$$\int_0^1\int_{\mathbb{R}^N} -f(x,u_n - su_0)u_0\,dx\,ds \longrightarrow \int_0^1\int_{\mathbb{R}^N} -f(x,u_0 - su_0)u_0\,dx\,ds$$

by the Vitali convergence theorem. On the other hand,

$$\begin{aligned}(3.22)\qquad &\int_0^1\int_{\mathbb{R}^N} -f(x,u_0 - su_0)u_0\,dx\,ds \\ &\quad = \int_{\mathbb{R}^N}\int_0^1 -f(x,u_0 - su_0)u_0\,ds\,dx = \int_{\mathbb{R}^N}\int_0^1 \frac{\partial}{\partial s}[F(x,u_0 - su_0)]\,ds\,dx \\ &\quad = \int_{\mathbb{R}^N}(F(x,0) - F(x,u_0))\,dx = \int_{\mathbb{R}^N} -F(x,u_0)\,dx.\end{aligned}$$

In view of (3.21) and (3.22), we obtain

$$\int_{\mathbb{R}^N} [F(x, u_n - u_0) - F(x, u_n) + F(x, u_0)]\, dx \longrightarrow \int_{\mathbb{R}^N} [F(x, u_0) - F(x, u_0)]\, dx = 0. \tag{3.23}$$

In a similar manner, we have

$$\int_{\mathbb{R}^N} [G(x, v_n - v_0) - G(x, v_n) + G(x, v_0)]\, dx \longrightarrow 0.$$

Let $E \subset \mathbb{R}^N$ be a measurable set. Since

$$\frac{1}{q} \int_E \Gamma(x) |u_n|^q |v_n|^q\, dx \le \frac{1}{q} |\Gamma|_\infty |u_n \chi_E|_{2q}^q |v_n \chi_E|_{2q}^q,$$

by the Vitali convergence theorem,

$$\frac{1}{q} \int_{\mathbb{R}^N} \Gamma(x) |u_0|^q |v_0|^q\, dx - \frac{1}{q} \int_{\mathbb{R}^N} \Gamma(x) |u_n|^q |v_n|^q\, dx \longrightarrow 0.$$

From (3.20), in order to complete the proof of Step 6, it is sufficient to prove that

$$\mathcal{J}_{\text{per}}(\omega_n - \omega_0) \longrightarrow \sum_{k=1}^{l} \mathcal{J}_{\text{per}}(\omega^k).$$

Note that

$$\begin{aligned}
\mathcal{J}_{\text{per}}(\omega_n - \omega_0) = {} & \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|(u_n(x) - u_0(x)) - (u_n(y) - u_0(y))|^2}{|x - y|^{N+2s}}\, dx\, dy \\
& + \frac{1}{2} \int_{\mathbb{R}^N} V^1_{\text{per}}(x) (u_n - u_0)^2 dx \\
& + \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|(v_n(x) - v_0(x)) - (v_n(y) - v_0(y))|^2}{|x - y|^{N+2s}}\, dx\, dy \\
& + \frac{1}{2} \int_{\mathbb{R}^N} V^2_{\text{per}}(x) (v_n - v_0)^2 dx \\
& - \int_{\mathbb{R}^N} F(x, u_n - u_0)\, dx - \int_{\mathbb{R}^N} G(x, v_n - v_0)\, dx.
\end{aligned}$$

Take $c_m^n = \omega_n - \omega_0 - \sum_{k=1}^m \omega^k(x - y_n^k)$ and $c_m^n = (a_m^n, b_m^n)$. Recalling (3.23), we have proved that

$$\int_{\mathbb{R}^N} [F(x, a_0^n) - F(x, u_n) + F(x, u_0)]\, dx \longrightarrow 0.$$

Then, putting u^1 instead of u_0 and $a_0^n(x + y_n^1)$ instead of u_n, we have

$$\int_{\mathbb{R}^N} [F(x, a_1^n) - F(x, a_0^n) + F(x, u^1)]\, dx \longrightarrow 0. \tag{3.24}$$

In a similar manner, putting u^2 instead of u_0 and $a_1^n(x + y_n^2)$ instead of u_n, we obtain

$$\int_{\mathbb{R}^N} [F(x, a_2^n) - F(x, a_1^n) + F(x, u^2)]\, dx \longrightarrow 0. \tag{3.25}$$

Combining (3.24) and (3.25), we have

$$\int_{\mathbb{R}^N} [F(x, a_2^n) - F(x, a_0^n) + F(x, u^1) + F(x, u^2)]\, dx \longrightarrow 0.$$

Then, we iterate the procedure. Letting u^l instead of u_0 and $a_{l-1}^n(x + y_n^l)$ instead of u_n, we obtain

$$\int_{\mathbb{R}^N} [F(x, a_l^n) - F(x, a_{l-1}^n) + F(x, u^l)]\, dx \longrightarrow 0.$$

Using mathematical induction, we can easily verify that

$$\begin{gathered}
\int_{\mathbb{R}^N} [F(x, a_l^n) - F(x, a_{l-2}^n) + F(x, u^{l-1}) + F(x, u^l)]\, dx \longrightarrow 0, \\
\vdots \\
\int_{\mathbb{R}^N} F(x, a_l^n)\, dx - \int_{\mathbb{R}^N} F(x, a_0^n)\, dx + \sum_{k=1}^{l} \int_{\mathbb{R}^N} F(x, u^k)\, dx \longrightarrow 0.
\end{gathered}$$

Observing that $a_l^n \to 0$, we have

$$\int_{\mathbb{R}^N} F(x, a_l^n)\, dx \longrightarrow 0.$$

Since $a_0^n = u_n - u_0$, then

$$\int_{\mathbb{R}^N} F(x, u_n - u_0)\, dx \longrightarrow \sum_{k=1}^{l} \int_{\mathbb{R}^N} F(x, u^k)\, dx. \tag{3.26}$$

In a similar manner, we have

$$\int_{\mathbb{R}^N} G(x, v_n - v_0)\, dx \longrightarrow \sum_{k=1}^{l} \int_{\mathbb{R}^N} G(x, v^k)\, dx. \tag{3.27}$$

Observe that

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|(u_n(x) - u_0(x) - \sum_{k=1}^{l} u^k(x - y_n^k))}{|x-y|^{N+2s}}\, dx\, dy \\
&\qquad \times \frac{-(u_n(y) - u_0(y) - \sum_{k=1}^{l} u^k(y - y_n^k))|^2}{|x-y|^{N+2s}}\, dx\, dy \\
&\qquad + \int_{\mathbb{R}^{2N}} \frac{|(v_n(x) - v_0(x) - \sum_{k=1}^{l} v^k(x - y_n^k))}{|x-y|^{N+2s}}\, dx\, dy \\
&\qquad \times \frac{-(v_n(y) - v_0(y) - \sum_{k=1}^{l} v^k(y - y_n^k))|^2}{|x-y|^{N+2s}}\, dx\, dy \\
&\qquad + \int_{\mathbb{R}^N} V_1(x) \Big| u_n(x) - u_0(x) - \sum_{k=1}^{l} u^k(x - y_n^k) \Big|^2 dx \\
&\qquad + \int_{\mathbb{R}^N} V_2(x) \Big| v_n(x) - v_0(x) - \sum_{k=1}^{l} v^k(x - y_n^k) \Big|^2 dx \longrightarrow 0,
\end{aligned} \tag{3.28}$$

which is equivalent to

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|(u_n(x) - u_0(x)) - (u_n(y) - u_0(y))|^2}{|x-y|^{N+2s}}\, dx\, dy \\
&\quad + \sum_{k=1}^{l} \int_{\mathbb{R}^{2N}} \frac{|u^k(x - y_n^k) - u^k(y - y_n^k)|^2}{|x-y|^{N+2s}}\, dx\, dy \\
&\quad - 2 \int_{\mathbb{R}^{2N}} \sum_{k=1}^{l} \frac{[(u_n(x) - u_0(x)) - (u_n(y) - u_0(y))]}{|x-y|^{N+2s}}\, dx\, dy \\
&\quad \times \frac{(u^k(x - y_n^k) - u^k(y - y_n^k))}{|x-y|^{N+2s}}\, dx\, dy \\
&\quad + \sum_{k \neq k'} \int_{\mathbb{R}^{2N}} \frac{(u^k(x - y_n^k) - u^k(y - y_n^k))(u^{k'}(x - y_n^{k'}) - u^{k'}(y - y_n^{k'}))}{|x-y|^{N+2s}}\, dx\, dy
\end{aligned}$$

$$
\begin{aligned}
&+\int\limits_{\mathbb{R}^{2N}} \frac{|(v_n(x)-v_0(x))-(v_n(y)-v_0(y))|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&+\sum_{k=1}^{l}\int\limits_{\mathbb{R}^{2N}} \frac{|v^k(x-y_n^k)-v^k(y-y_n^k)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&-2\int\limits_{\mathbb{R}^{2N}}\sum_{k=1}^{l} \frac{[(v_n(x)-v_0(x))-(v_n(y)-v_0(y))]}{|x-y|^{N+2s}}\,dx\,dy\\
\times&\,\frac{(v^k(x-y_n^k)-v^k(y-y_n^k))}{|x-y|^{N+2s}}\,dx\,dy\\
&+\sum_{k\neq k'}\int\limits_{\mathbb{R}^{2N}} \frac{(v^k(x-y_n^k)-v^k(y-y_n^k))(v^{k'}(x-y_n^{k'})-v^{k'}(y-y_n^{k'}))}{|x-y|^{N+2s}}\,dx\,dy\\
&+\int\limits_{\mathbb{R}^N} V_{\mathrm{per}}^1(x)\Big(u_n-u_0-\sum_{k=1}^{l}u^k(x-y_n^k)\Big)^2dx\\
&+\int\limits_{\mathbb{R}^N} V_{\mathrm{per}}^2(x)\Big(v_n-v_0-\sum_{k=1}^{l}v^k(x-y_n^k)\Big)^2dx\\
&+\int\limits_{\mathbb{R}^N} V_{\mathrm{loc}}^1(x)\Big(u_n-u_0-\sum_{k=1}^{l}u^k(x-y_n^k)\Big)^2dx\\
&+\int\limits_{\mathbb{R}^N} V_{\mathrm{loc}}^2(x)\Big(v_n-v_0-\sum_{k=1}^{l}v^k(x-y_n^k)\Big)^2dx \longrightarrow 0.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} V_{\mathrm{loc}}^1(x)\Big(u_n-u_0-\sum_{k=1}^{l}u^k(x-y_n^k)\Big)^2dx\right|\\
&\qquad\qquad \le |V_{\mathrm{loc}}^1|_\infty\left|u_n-u_0-\sum_{k=1}^{l}u^k(x-y_n^k)\right|_2^2 \longrightarrow 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} V_{\mathrm{loc}}^2(x)\Big(v_n-v_0-\sum_{k=1}^{l}v^k(x-y_n^k)\Big)^2dx\right|\\
&\qquad\qquad \le |V_{\mathrm{loc}}^2|_\infty\left|v_n-v_0-\sum_{k=1}^{l}v^k(x-y_n^k)\right|_2^2 \longrightarrow 0.
\end{aligned}
$$

Due to $\omega_n^l(x+y_n^l) \rightharpoonup \omega^l(x)$, we deduce that $u_n^l(x+y_n^l) \rightharpoonup u^l(x)$ and $v_n(x+y_n^l) \rightharpoonup v^l(x)$. Then,

$$\begin{aligned}
&-2\int_{\mathbb{R}^{2N}} \sum_{k=1}^{l} \frac{[(u_n(x)-u_0(x))-(u_n(y)-u_0(y))]}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad\times \frac{(u^k(x-y_n^k)-u^k(y-y_n^k))}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad= -2\sum_{k=1}^{l}\int_{\mathbb{R}^{2N}} \frac{|u^k(x-y_n^k)-u^k(y-y_n^k)|^2}{|x-y|^{N+2s}}\,dx\,dy + o(1),
\end{aligned}$$

and

$$\begin{aligned}
&-2\int_{\mathbb{R}^{2N}} \sum_{k=1}^{l} \frac{[(v_n(x)-v_0(x))-(v_n(y)-v_0(y))]}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad\times \frac{(v^k(x-y_n^k)-v^k(y-y_n^k))}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad= -2\sum_{k=1}^{l}\int_{\mathbb{R}^{2N}} \frac{|v^k(x-y_n^k)-v^k(y-y_n^k)|^2}{|x-y|^{N+2s}}\,dx\,dy + o(1).
\end{aligned}$$

Due to $|y_n^k - y_n^{k'}| \to \infty$ for $k \neq k'$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{(u^k(x-y_n^k)-u^k(y-y_n^k))(u^{k'}(x-y_n^{k'})-u^{k'}(y-y_n^{k'}))}{|x-y|^{N+2s}}\,dx\,dy \\
&= \int_{\mathbb{R}^{2N}} \frac{(u^k(x)-u^k(y))(u^{k'}(x+y_n^k-y_n^{k'})-u^{k'}(y+y_n^k-y_n^{k'}))}{|x-y|^{N+2s}}\,dx\,dy \longrightarrow 0,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{(v^k(x-y_n^k)-v^k(y-y_n^k))(v^{k'}(x-y_n^{k'})-v^{k'}(y-y_n^{k'}))}{|x-y|^{N+2s}}\,dx\,dy \\
&= \int_{\mathbb{R}^{2N}} \frac{(v^k(x)-v^k(y))(v^{k'}(x+y_n^k-y_n^{k'})-v^{k'}(y+y_n^k-y_n^{k'}))}{|x-y|^{N+2s}}\,dx\,dy \longrightarrow 0.
\end{aligned}$$

Therefore, (3.28) is equivalent to

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|(u_n(x)-u_0(x))-(u_n(y)-u_0(y))|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad -\sum_{k=1}^{l}\int_{\mathbb{R}^{2N}} \frac{|u^k(x-y_n^k)-u^k(y-y_n^k)|^2}{|x-y|^{N+2s}}\,dx\,dy
\end{aligned}$$

$$\begin{aligned}
&+\int_{\mathbb{R}^{2N}} \frac{|(v_n(x)-v_0(x))-(v_n(y)-v_0(y))|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&-\sum_{k=1}^{l}\int_{\mathbb{R}^{2N}} \frac{|v^k(x-y_n^k)-v^k(y-y_n^k)|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&+\int_{\mathbb{R}^N} V_{\mathrm{per}}^1(x)|u_n-u_0|^2\,dx+\sum_{k=1}^{l}\int_{\mathbb{R}^N} V_{\mathrm{per}}^1(x)(u^k(x-y_n^k))^2\,dx \\
&+\sum_{k\neq k'} V_{\mathrm{per}}^1(x)u^k(x-y_n^k)u^{k'}(x-y_n^{k'})\,dx \\
&-2\sum_{k=1}^{l}\int_{\mathbb{R}^N} V_{\mathrm{per}}^1(x)(u_n-u_0)u^k(x-y_n^k)\,dx \\
&+\int_{\mathbb{R}^N} V_{\mathrm{per}}^2(x)|v_n-v_0|^2\,dx+\sum_{k=1}^{l}\int_{\mathbb{R}^N} V_{\mathrm{per}}^2(x)(v^k(x-y_n^k))^2\,dx \\
&+\sum_{k\neq k'} V_{\mathrm{per}}^2(x)v^k(x-y_n^k)v^{k'}(x-y_n^{k'})\,dx \\
&\quad-2\sum_{k=1}^{l}\int_{\mathbb{R}^N} V_{\mathrm{per}}^2(x)(v_n-v_0)v^k(x-y_n^k)\,dx \longrightarrow 0.
\end{aligned}$$

Since $\omega_n(x+y_n^k)\rightharpoonup\omega^k(x)$, we obtain

$$\begin{aligned}
&-2\sum_{k=1}^{l}\int_{\mathbb{R}^N} V_{\mathrm{per}}^1(x)(u_n-u_0)u^k(x-y_n^k)\,dx \\
&\qquad\qquad=-2\sum_{k=1}^{l}\int_{\mathbb{R}^N} V_{\mathrm{per}}^1(x)(u^k(x-y_n^k))^2\,dx+o(1),
\end{aligned}$$

and

$$\begin{aligned}
&-2\sum_{k=1}^{l}\int_{\mathbb{R}^N} V_{\mathrm{per}}^2(x)(v_n-u_0)v^k(x-y_n^k)\,dx \\
&\qquad\qquad=-2\sum_{k=1}^{l}\int_{\mathbb{R}^N} V_{\mathrm{per}}^2(x)(v^k(x-y_n^k))^2\,dx+o(1).
\end{aligned}$$

Since $|y_n^k-y_n^{k'}|\to\infty$ for $k\neq k'$, then

$$\int_{\mathbb{R}^N} V_{\mathrm{per}}^1(x)u^k(x-y_n^k)u^{k'}(x-y_n^{k'})\,dx \\ = \int_{\mathbb{R}^N} V_{\mathrm{per}}^1(x)u^k(x)u^{k'}(x+y_n^k-y_n^{k'})\,dx \longrightarrow 0,$$

and

$$\int_{\mathbb{R}^N} V_{\mathrm{per}}^2(x)v^k(x-y_n^k)v^{k'}(x-y_n^{k'})\,dx \\ = \int_{\mathbb{R}^N} V_{\mathrm{per}}^2(x)v^k(x)v^{k'}(x+y_n^k-y_n^{k'})\,dx \longrightarrow 0.$$

Hence, (3.28) is equivalent to

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{|(u_n(x)-u_0(x))-(u_n(y)-u_0(y))|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &- \sum_{k=1}^{l}\int_{\mathbb{R}^{2N}} \frac{|u^k(x-y_n^k)-u^k(y-y_n^k)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &+ \int_{\mathbb{R}^{2N}} \frac{|(v_n(x)-v_0(x))-(v_n(y)-v_0(y))|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &- \sum_{k=1}^{l}\int_{\mathbb{R}^{2N}} \frac{|v^k(x-y_n^k)-v^k(y-y_n^k)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &+ \int_{\mathbb{R}^N} V_{\mathrm{per}}^1(x)|u_n-u_0|^2dx + \int_{\mathbb{R}^N} V_{\mathrm{per}}^2(x)|v_n-v_0|^2dx \\ &- \sum_{k=1}^{l}\int_{\mathbb{R}^N} V_{\mathrm{per}}^1(x)(u^k(x-y_n^k))^2dx \\ &- \sum_{k=1}^{l}\int_{\mathbb{R}^N} V_{\mathrm{per}}^2(x)(v^k(x-y_n^k))^2dx \longrightarrow 0. \end{aligned} \tag{3.29}$$

Combining (3.26), (3.27) and (3.29), we have

$$\mathcal{J}_{\mathrm{per}}(\omega_n-\omega_0) - \sum_{k=1}^{l}\mathcal{J}_{\mathrm{per}}(\omega^k(x)) \\ = \mathcal{J}_{\mathrm{per}}(\omega_n-\omega_0) - \sum_{k=1}^{l}\mathcal{J}_{\mathrm{per}}(\omega^k(x-y_n^k)) \longrightarrow 0.$$

Moreover,

$$\mathcal{J}_{\mathrm{per}}(\omega_n - \omega_0) \longrightarrow \sum_{k=1}^{l} \mathcal{J}_{\mathrm{per}}(\omega^k).$$

This completes the proof of Step 6 and, therefore, Theorem 3.2. $\square$

4. Proof of main theorem.

Proof of Theorem 1.1. For $\mathcal{J}$ on $\mathcal{N}$, we may find a bounded minimizing sequence $(\omega_n) \subset \mathcal{N}$ by Theorem 2.2. We consider

$$c := \inf_{\mathcal{N}} \mathcal{J}$$

and

$$c_{\mathrm{per}} := \inf\{\mathcal{J}_{\mathrm{per}}(\omega) : \omega \in E \setminus \{0\}, \mathcal{J}'_{\mathrm{per}}(\omega) = 0\}.$$

By Theorem 3.2, there is an $\omega_{\mathrm{per}} \neq 0$ which is a critical point of J_{per} such that $\mathcal{J}_{\mathrm{per}}(\omega_{\mathrm{per}}) = c_{\mathrm{per}}$. Take $t > 0$ such that $t\omega_{\mathrm{per}} \in \mathcal{N}$. It follows that

$$c_{\mathrm{per}} = \mathcal{J}_{\mathrm{per}}(\omega_{\mathrm{per}}) \geq \mathcal{J}_{\mathrm{per}}(t\omega_{\mathrm{per}}) > \mathcal{J}(t\omega_{\mathrm{per}}) \geq c > 0.$$

There is an ω_0 such that $\mathcal{J}'(\omega_0) = 0$ by Theorem 3.2. Moreover,

$$\mathcal{J}(\omega_n) \longrightarrow \mathcal{J}(\omega_0) + \sum_{k=1}^{l} \mathcal{J}_{\mathrm{per}}(\omega^k).$$

Observe that

$$c = \mathcal{J}(\omega_0) + \sum_{k=1}^{l} \mathcal{J}_{\mathrm{per}}(\omega^k) \geq \mathcal{J}(\omega_0) + lc_{\mathrm{per}}.$$

We obtain $l = 0$, and ω_0 is a ground state. This completes the proof of Theorem 1.1. $\square$

Remark 4.1. For the problem,

$$\begin{cases} (-\Delta)^s u + V_{\mathrm{per}}(x)u = f(x,u) + \Gamma(x)|u|^{q-2}u|v|^q & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v + V_{\mathrm{per}}(x)v = f(x,v) + \Gamma(x)|v|^{q-2}v|u|^q & \text{in } \mathbb{R}^N, \\ u, v \in H^s(\mathbb{R}^N), \end{cases} \tag{4.1}$$

an interesting problem remains: how to find a solution $\omega = (u, v)$ such that $u \not\equiv 0$ and $v \not\equiv 0$.

Assume that u_0 is a ground state solution of

$$(-\Delta)^s u + V_{\mathrm{per}}(x)u = f(x,u). \tag{4.2}$$

Let $y_n \in \mathbb{Z}^N$ and $y_n \to \infty$, and set $u_n = u_0(x - y_n)$. Then, u_n is also a ground state solution of 4.2. Since $\Gamma(x) \to 0$ as $|x| \to \infty$, $\Gamma(x+y_n) \to 0$ as $n \to \infty$ for all x in a bounded domain. The techniques in [**13,** Section 6] may be helpful in solving the above problem, and we will study it in the future.

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