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CONTINUOUS-TRACE k-GRAPH C^*-ALGEBRAS

DANNY CRYTSER

ABSTRACT. A characterization is given for directed graphs that yield graph C^*-algebras with continuous trace. This is established for row-finite graphs with no sources first using a groupoid approach, and extended to the general case via the Drinen-Tomforde desingularization. A characterization of continuous-trace AF C^*-algebras is obtained. Partial results are given to characterize higher-rank graphs that yield C^*-algebras with continuous trace.

1. Introduction. From a directed graph E, one can construct a graph C^*-algebra $C^*(E)$, generated by a universal family of projections and partial isometries that satisfy certain Cuntz-Krieger relations. Many important properties of this C^*-algebra, e.g., simplicity and K-theory, are governed by graph-theoretic properties of E. From E, we can also construct an étale *path groupoid* G_E that models the shift dynamics of infinite paths in E; the groupoid C^*-algebra of G_E is canonically isomorphic to $C^*(E)$. This allows for the use of tools from the theory of groupoid C^*-algebras when studying graph C^*-algebras. In this paper, we give an example of this approach, applying the main result of **[14]** to the path groupoid in order to characterize all graph C^*-algebras with continuous trace.

The path groupoid is easiest to use if the graph is *non-singular*, in the sense that each vertex is the range of a finite non-empty set of edges. Therefore, we first work in the non-singular case; we then use the *Drinen-Tomforde desingularization* to extend our results to graphs with singular vertices. Desingularization works by taking a graph E and returning a non-singular graph $\widetilde{E}$ such that $C^*(E)$ sits inside $C^*(\widetilde{E})$ as a full corner; this implies that $C^*(E)$ has continuous trace if and only if $C^*(\widetilde{E})$ does. As an application, we use a result from **[19]** to char-

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acterize continuous trace AF algebras in terms of their Bratteli diagrams.

In the last section, we consider higher-rank graph C^*-algebras with continuous trace. Higher-rank graphs are categories which generalize the category of finite directed paths within a directed graph. They have C^*-algebras defined along the same lines as graph C^*-algebras. We include necessary background on the theory of higher-rank graph algebras. Again, the use of groupoids is crucial. The higher-rank case is more complicated, and we are only able to give partial results. In particular, giving a combinatorial description of when the isotropy groups continuously vary for a k-graph path groupoid seems out of reach, so we focus instead on the principal case. We note a simple necessary condition on a higher-rank graph for its associated C^*-algebra to have continuous trace, a corollary of a result from [**5**].

While this paper was in preparation, we were made aware of a related paper of Hazlewood [**9**] which contains similar results to ours. In particular, [**9,** Theorems 6.2.13, 6.4.11, 6.5.22] are in some sense cycle-free/principal versions of Theorems 3.8 and 5.15. (In particular, our finite sets of ancestry pairs play a role similar to that of $F \times F$ in the statement of [**9,** Theorem 6.2.13].) The results in [**9**] also show that some desingularization as in Section 4 in the present paper is possible for the k-graph case (although it seems that resolving infinite receivers is somewhat more difficult). The desingularization results of [**6, 20**] are also of much interest in the k-graph case, although we have not applied them here and instead have focused on non-singular k-graphs. Some related results regarding type theory for the groupoid C^*-algebra were obtained in [**2**], and the interested reader should refer to that paper for a more general, groupoid-theoretic approach.

2. Continuous-trace C^*-algebras, graph algebras, groupoids. In this section, we review prerequisite material on continuous-trace C^*-algebras, graph C^*-algebras and groupoids.

Let A be a C^*-algebra, and let $\widehat{A}$ denote its spectrum of unitary equivalence classes of irreducible representations. There is a canonical map $\widehat{A} \to \mathrm{Prim} A$, given by $[\pi] \mapsto \ker \pi$, and this map is used to define the topology on $\widehat{A}$ as in [**17**]. If $\widehat{A}$ is Hausdorff in this topology, then the map $[\pi] \to \ker \pi$ is a bijection. In this case, for $a \in A$ and $s = [\pi] \in \widehat{A}$, we denote by $a(s)$ the element $a + \ker \pi \in A/\ker \pi$; in this way, A

is fibered over $\widehat{A}$. For an element a in a C^*-algebra A and a unitary equivalence class $s = [\pi] \in \widehat{A}$, the element we define the *rank* of $a(s)$ to be the rank of $\pi(a)$, and we say that $a(s)$ is a *projection* if and only if $\pi(a)$ is a projection. (Both of these notions are well defined up to unitary equivalence.)

Definition 2.1 ([**17,** Definition 5.13]). Let A be a C^*-algebra with Hausdorff spectrum $\widehat{A}$. Then, A is said to have *continuous trace* (or be continuous trace) if, for each $t \in \widehat{A}$, there exist an open set $U \subset \widehat{A}$ containing t and an element $a \in A$ such that $a(s)$ is a rank one projection for every $s \in U$.

For an introduction to graph C^*- algebras, see [**16**]. The reader who is already familiar with graph C^*-algebras may disregard the following, standard definitions.

Definition 2.2. A (*directed*) *graph* E is an ordered quadruple $E = (E^0, E^1, r, s)$, where the E^0 and E^1 are countable sets, called the *vertices* and *edges*, and $r, s : E^1 \to E^0$ are maps, called the *range* and *source* maps.

A vertex v is called an *infinite receiver* if there are infinitely many edges in E^1 with range v; a vertex is called a *source* if it receives no sources. A vertex is *regular* if it is neither an infinite receiver or a source; otherwise, it is called *singular*. A graph is *row-finite* if it has no infinite receivers and has *no sources* if every vertex receives an edge.

The finite path space E^* consists of all finite sequences $e_1 \cdots e_n$ in E^1 such that $s(e_i) = r(e_{i+1})$ for $i = 1, \ldots, n-1$. The range of the path $e_1 \cdots e_n$ is defined to be $r(e_1)$, and its source is $s(e_n)$. If $\mu = e_1 \cdots e_n$ is a finite path, then we define the length to be n and write $|\mu| = n$. The vertices are included in the finite path space as the paths of length zero. If $\lambda = e_1 \cdots e_n$ and $\mu = f_1 \cdots f_m$ are finite paths with $s(\lambda) = r(\mu)$, we can concatenate them to form $\lambda\mu = e_1 \cdots e_n f_1 \cdots f_m \in E^*$. A path λ *contains* the path ν if there are paths μ, π (possibly of length zero) with $\lambda = \mu\nu\pi$.

A *cycle* in a directed graph is a path $\lambda \in E^* \setminus E^0$ with $r(\lambda) = s(\lambda)$. An *entrance* to the cycle $\lambda = e_1 \cdots e_n$ is an edge e such that $r(e) = e_k$ and $e \neq e_k$ for some $k \in \{1, \ldots, n\}$. A *simple cycle* is a cycle λ such that there do not exist cycles μ, ν with $\lambda = \mu\nu$ (in [**12**], called *loops*).

A graph E is said to satisfy Condition (L) if every cycle in E has an entrance, and E satisfies Condition (K) if there is no vertex in E which is the range of exactly one simple cycle. A path is *cycle-free* if it contains no cycles.

The *infinite path space* is $E^\infty = \{e_1 e_2 \cdots | s(e_i) = r(e_{i+1}) \text{ for all } i \geq 1\}$. If $\lambda = e_1 \cdots e_n \in E^*$ and $x = f_1 \cdots \in E^\infty$, then $\lambda x = e_1 \cdots e_n f_1 \cdots \in E^\infty$. The range of $x = e_1 e_2 \cdots \in E^\infty$ is defined as $r(x) := r(e_1)$. The shift map $\sigma : E^\infty \to E^\infty$ removes the first edge from an infinite path: $\sigma(e_1 e_2 \cdots) = e_2 e_3 \cdots$. Composing σ with itself yields additional maps $\sigma^2, \sigma^3, \ldots,$ from E^∞ to E^∞.

Remark 2.3. If λ is a cycle and $\lambda = \alpha\beta$, where neither α nor β contains a cycle, then λ is simple.

Definition 2.4. Let E be a directed graph. Then, the *graph C^*-algebra* of E, denoted $C^*(E)$, is the universal C^*-algebra generated by projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ satisfying the following Cuntz-Krieger relations:

(i) $s_e^* s_e = p_{s(e)}$ for any $e \in E^1$;
(ii) $s_e s_e^* \leq p_{r(e)}$ for any $e \in E^1$;
(iii) $s_e^* s_f = 0$ for distinct $e, f \in E^1$;
(iv) if v is regular, then $p_v = \sum_{r(e)=v} s_e s_e^*$.

We also include the basic definitions for groupoids. A concise definition of a groupoid is a small category with inverses; we include a more detailed definition.

A *groupoid* is a set G along with a subset $G^{(2)} \subset G \times G$ of *composable pairs* and two functions: composition $\circ : G^{(2)} \to G$ (written $(\alpha, \beta) \to \alpha\beta$) and an involution ${}^{-1} : G \to G$ (written $\gamma \to \gamma^{-1}$) such that the following hold.

(A) (Associativity). If $(\gamma, \eta), (\eta, \zeta) \in G^{(2)}$, then $(\gamma\eta, \zeta), (\gamma, \eta\zeta) \in G^{(2)}$, also, and $\gamma(\eta\zeta) = (\gamma\eta)\zeta$.
(B) (Inverses). $(\gamma, \gamma^{-1}) \in G^{(2)}$ for all $\gamma \in G$, and $\gamma^{-1}(\gamma\eta) = \eta$ and $(\gamma\eta)\eta^{-1} = \gamma$ for $(\gamma, \eta) \in G^{(2)}$.

Elements satisfying $g = g^2 \in G$ are called *units* of G, and the set of all such units is denoted $G^{(0)} \subset G$ and called the *unit space* of G. There are maps $r, s : G \to G^{(0)}$, defined by

$$r(\gamma) = \gamma\gamma^{-1}, \qquad s(\gamma) = \gamma^{-1}\gamma$$

that are called, respectively, the *range* and *source* maps. These maps orient G as a category, with units serving as objects: $(\alpha, \beta) \in G^{(2)}$ if and only if $s(\alpha) = r(\beta)$. For a given unit $u \in G^{(0)}$ there is an associated group $G(u) = \{\gamma \in G : r(\gamma) = s(\gamma) = u\}$; this is called the *isotropy* or *stabilizer group* of u. The union of all isotropy groups in G forms a subgroupoid of G, called $\mathrm{Iso}(G)$, the *isotropy bundle* of G. A groupoid is called *principal* (or an *equivalence relation*) if $\mathrm{Iso}(G) = G^{(0)}$, that is, if no unit has non-trivial stabilizer group.

A topological groupoid is a groupoid G endowed with a topology so that the composition and inversion operations are continuous (the domain of $\circ$ is equipped with the relative product topology). A topological groupoid is étale if the topology is locally compact and the range and source maps are local homeomorphisms. Most of the groupoids encountered in this paper will be Hausdorff and second countable (the possible exceptions being the orbit groupoids R_G of Definition 2.11). Note that, if G is étale, then each range fiber $r^{-1}(u)$ is discrete in the relative topology (likewise for source fibers). Hence, the intersection of any compact subset of G with a given range fiber or source fiber is finite.

In order to define a C^*-algebra from an étale groupoid G, it is necessary to specify a $*$-algebra structure on $C_c(G)$. This is given by

$$(f * g)(\gamma) = \sum_{(\alpha,\beta)\in G^{(2)}:\alpha\beta=\gamma} f(\alpha)g(\beta);$$

compactness of supports ensures that this sum gives a well defined element of $C_c(G)$. (The most important notion here is that the counting measures on range fibers form a *Haar system*, which is necessary for any topological groupoid to define a C^*-algebra, see **[18]**.) We do not include all of the details for how to place a norm on $C_c(G)$; these may be found in **[18]**. In brief, there are two distinguished C^*-norms $||\cdot||$, $||\cdot||_r$ on $C_c(G)$, and completion in these yields the full groupoid C^*-algebra $C^*(G)$ and the reduced groupoid $C_r^*(G)$, respectively. As in the group case, when G is amenable, these two C^*-norms coincide and $C^*(G) = C_r^*(G)$.

Definition 2.5 (**[11]**)**.** Let E be a graph, and let E^∞ denote its infinite path space. For paths $x, y \in E^\infty$ and $k \in \mathbb{Z}$, we write $x \sim_n y$ if there exist $p, q \in \mathbb{N}$ such that $\sigma^p x = \sigma^q y$ and $p - q = n$. Then, the *path groupoid* G_E is the set $\{(x,n,y) \in E^\infty \times \mathbb{Z} \times E^\infty : x \sim_n y\}$ equipped with operations $(x,n,y)(y,m,z) = (x, m+n, z)$ and $(x,n,y)^{-1} = (y,-n,x)$. The unit space is identified with E^∞ via the mapping $x \mapsto (x,0,x)$ so that the range and source maps are given by $r(x,n,y) = x$ and $s(x,n,y) = y$.

Remark 2.6. If $G = G_E$, then the isotropy group of an infinite path x is either trivial ($\sigma^p x = \sigma^q x$ implies $p = q$) or infinite cyclic (in which case $x = \alpha(\lambda^\infty)$ for some finite path α and cycle λ).

The topology on G_E is generated by basic open sets of the form

$$Z(\alpha,\beta) = \{(\alpha z, |\alpha| - |\beta|, \beta z) \in G_E : r(z) = s(\alpha)\}$$

where $\alpha, \beta \in E^*$ with $s(\alpha) = s(\beta)$. The topology defined above restricts to the relative product topology on $G_E^{(0)} = E^\infty \subset \prod_{\mathbb{N}} E^1$, if we treat E^1 as a discrete space. We will refer to this topology on E^∞ using basic compact-open sets of the form $Z(\alpha) = \{\alpha x \mid x \in E^\infty, r(x) = s(\alpha)\}$.

It is noted in **[12]** that $Z(\alpha,\beta) \cap Z(\gamma,\delta) = \emptyset$ unless $(\alpha,\beta) = (\gamma\epsilon, \delta\epsilon)$, or vice versa. This topology makes G_E into an étale groupoid (**[12,** Proposition 2.6]) since the restriction of the range map to the basic sets is a homeomorphism, and furthermore, each basic set is compact. Thus, G_E has a canonical Haar system $\{\lambda_x\}_{x \in E^\infty}$ consisting of counting measures on the source fibers. As G_E is an étale groupoid, it has a groupoid C^*-algebra $C^*(G_E)$, as described above.

The next theorem has been modified from its original statement to fit our orientation convention.

Theorem 2.7 (**[12,** Theorem 4.2])**.** *For any row-finite graph with no sources E, we have $C^*(E) \cong C^*(G_E)$ via an isomorphism carrying s_e to $\mathbf{1}_{Z(e,s(e))} \in C_c(G)$ and p_v to $\mathbf{1}_{Z(v,v)}$.*

It is a fact that G_E is always an *amenable* groupoid such that we have $C^*(G_E) = C^*_r(G_E)$. In order to describe graph C^*-algebras with continuous trace, we need to know when the isotropy groups $G(u)$ continuously vary with respect to the unit $u \in G^{(0)}$. First, the topology on the set of isotropy groups must be defined.

Definition 2.8 (**[17]**)**.** Let X be a topological space. Consider the collection $F(X)$ of all closed subsets of X; the *Fell topology* on $F(X)$ is defined by the requirement that a net $(Y_i)_{i\in I} \subset F(X)$ converges to $Y \in F(X)$ exactly when:

(i) if elements y_i are chosen in Y_i such that $y_i \to z$, then z belongs to Y; and
(ii) for any element $y \in Y$, and for any subnet (Y_{i_j}) of (Y_i), there exists a subnet $(Y_{i_{j_k}})$ of (Y_{i_j}) and elements $y_{i_{j_k}} \in Y_{i_{j_k}}$ such that $y_{i_{j_k}} \to y$.

We say that the topological groupoid G has *continuous isotropy* if the isotropy map $G^{(0)} \to F(G)$ defined by $x \mapsto G(x)$ is continuous. By $\Sigma^{(0)}$, we denote the space of closed *subgroups* of G (that is, subsets which form groups under the inherited operations), equipped with the Fell topology.

We will not need to handle the Fell topology directly in this paper due to the following result which describes continuous isotropy for graph algebras.

Theorem 2.9 (**[8]**)**.** *Let E be a row-finite graph with no sources. Then, G_E has continuous isotropy if and only if no cycle in E has an entrance.*

Definition 2.10. A topological groupoid G is *proper* if the orbit map $\Phi_G : G \to G^{(0)} \times G^{(0)}$ given by $g \to (r(g), s(g))$ is proper (where the codomain is equipped with the relative product topology).

Definition 2.11. Let G be a groupoid. Let $\pi_R : G \to G^{(0)} \times G^{(0)}$ be given by $\pi_R(g) = (r(g), s(g))$. Then, the *orbit groupoid* of G, denoted by $R_G = R$, is the image of π_R, where the groupoid operations are:

$$(u, v)(v, w) = (u, w)$$
$$(u, v)^{-1} = (v, u).$$

The unit space of R is identified with the unit space of G. The range and source maps are naturally identified with the projections onto the first and second factors.

Definition 2.12. The topology on $R = R_G$ is the *quotient topology* induced by the above map $\pi_R : G \to R$.

Remark 2.13. If the groupoid G is principal, in the sense that $G(x) = \{x\}$ for every unit $x \in G^{(0)}$, then the map π_R is a groupoid isomorphism.

The following commutative diagram serves to keep the relevant groupoids and spaces distinct. Note that, as a set map, Φ_R is merely an inclusion. However, R carries a different topology from the product topology on $G^{(0)} \times G^{(0)}$; thus, we distinguish between the two.

$$\begin{tikzcd} G \arrow[dd, "\pi"'] \arrow[dr, "\Phi_G"] & \\ & G^{(0)} \times G^{(0)}. \\ R \arrow[ur, "\Phi_R"'] & \end{tikzcd}$$

Remark 2.14. In some sources, such as [**14**], the orbit groupoid is denoted by $R = \mathcal{G}/\mathrm{A}$ to indicate that it is the quotient of a groupoid by the (in this case, abelian) isotropy bundle.

The next theorem from [**14**] allows us to determine whether a groupoid C^*-algebra $C^*(G)$ is continuous-trace from properties of its groupoid G.

Theorem 2.15 ([**14,** Theorem 1.1])**.** *Let G be a second-countable locally compact Hausdorff groupoid with unit space $G^{(0)}$, abelian isotropy and Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. Then, $C^*(G,\lambda)$ has continuous trace if and only if:*

(i) *the stabilizer map $u \mapsto G(u)$ is continuous from $G^{(0)}$ to $\Sigma^{(0)}$;*
(ii) *the action of R on $G^{(0)}$ is proper.*

3. Continuous-trace graph algebras. The path groupoid of a directed graph E is made of infinite paths, and the open sets are described by finite path prefixes. It is not surprising, then, that the characterization of proper path groupoids is stated in terms of a certain finiteness condition on paths. *For this section, the standing assumption is that E is a row-finite graph with no sources.*

Definition 3.1. Let $v, w \in E^0$ be vertices in a directed graph E (where we allow $v = w$). An *ancestry pair* for v and w is a pair of paths (λ, μ) such that $r(\lambda) = v$, $r(\mu) = w$, and $s(\lambda) = s(\mu)$. A *minimal ancestry pair* is an ancestry pair (λ, μ) such that, whenever $(\lambda, \mu) = (\lambda'\nu, \mu'\nu)$ for some ancestry pair (λ', μ') and path ν, then we must necessarily have $\nu = s(\lambda) = s(\mu)$. An ancestry pair (λ, μ) is *cycle-free* if neither λ nor μ contains a cycle. The graph E has *finite ancestry* if every pair of vertices v, w has at most finitely many cycle-free minimal ancestry pairs. (We include the possibility that $v = w$.)

Remark 3.2. Note that it is not necessary that any two vertices of E have an ancestry pair in order for E to have finite ancestry.

Much of our analysis below hinges on an understanding of the topology on R_G when $G = G_E$ is the path groupoid of E. The compact-open basic sets $Z(\alpha, \beta) \subset G_E$ project onto compact-open basic sets $\pi_R(Z(\alpha, \beta))$ when $\pi_R : G \to R_G$ is the quotient map. The following lemma describes how the latter sets overlap in the case when (α, β) is further required to be an ancestry pair. (Recall that a cycle λ is simple if it cannot be written as the product of two other cycles.)

Lemma 3.3. *Let E be a directed graph, and let (α, β) and (γ, δ) be two distinct cycle-free minimal ancestry pairs in E. If $\pi_R(Z(\alpha, \beta)) \cap \pi_R(Z(\gamma, \delta)) \neq \emptyset$, then one of the two must hold*:

(i) *there are path factorizations $\alpha = \gamma\alpha'$ and $\delta = \beta\delta'$ such that $\lambda = \alpha'\delta'$ is a simple cycle*; *or*

(ii) *there are path factorizations $\gamma = \alpha\gamma'$ and $\beta = \delta\beta'$ such that $\lambda = \gamma'\beta'$ is a simple cycle.*

Proof. By the definition of π_R, we have $\pi_R(Z(\alpha, \beta)) = \{(\alpha w, \beta w) : w \in E^\infty, r(w) = s(\alpha)\}$; thus, if $\pi_R(Z(\alpha, \beta)) \cap \pi_R(Z(\gamma, \delta)) \neq \emptyset$, then there must exist infinite paths $w, z \in E^\infty$ such that $r(w) = s(\alpha) = s(\beta)$, $r(z) = s(\gamma) = s(\delta)$ and $(\alpha w, \beta w) = (\gamma z, \delta z)$ (simply as ordered pairs of infinite paths).

Claim 3.4. *We must have* $|\alpha| \geq |\gamma|$ *and* $|\beta| \leq |\delta|$*, or* $|\alpha| \leq |\gamma|$ *and* $|\beta| \geq |\delta|$*.*

Proof. Suppose that $|\alpha| \geq |\gamma|$ and $|\beta| \geq |\delta|$. Then, the equations $\alpha w = \gamma z$ and $\beta w = \delta z$ imply that we can factor α and β, respectively, as $\alpha = \gamma\alpha'$ and $\beta = \delta\beta'$. Then, we see that $\gamma\alpha' w = \gamma z$ and $\delta\beta' w = \delta z$, from which we see that $\alpha' w = \beta' w$. If $\alpha' = \beta'$, then, by minimality of (α, β), we conclude that $(\alpha, \beta) = (\gamma, \delta)$, establishing the claim. If $\alpha' \neq \beta'$, then the equation $\alpha' w = \beta' w$ forces one of α' or β' to be a cycle, contradicting the assumption that α and β are cycle-free. The case where $|\alpha| \leq |\gamma|$ and $|\beta| \leq |\delta|$ follows by symmetry, establishing the claim. □

Now assume, without loss of generality, that $|\alpha| \geq |\gamma|$ and $|\beta| \leq |\delta|$, in addition to the initial assumption that $(\alpha w, \beta w) = (\gamma z, \delta z)$. We can write path factorizations $\alpha = \gamma\alpha'$ and $\delta = \beta\delta'$. Let $\lambda = \alpha'\delta'$, which is indeed a finite path as $s(\alpha') = s(\alpha) = s(\beta) = r(\delta')$, and moreover, a cycle as $r(\alpha') = s(\gamma) = s(\delta) = s(\delta')$. Since neither α nor δ contains a cycle, it must be the case that λ is a simple cycle. □

Lemma 3.5. *If no cycle of E has an entrance, then, for a fixed cycle-free minimal ancestry pair* (α, β)*, there are at most finitely many cycle-free minimal ancestry pairs* (γ, δ) *such that*

$$\pi_R(Z(\alpha, \beta)) \cap \pi_R Z((\gamma, \delta)) \neq \emptyset.$$

Proof. The assumption that no cycle of E has an entrance implies that any vertex in E is the range of at most one simple cycle. Note that, in Lemma 3.3, the cycle λ has range and source vertex equal to $s(\alpha)$. If λ is the unique simple cycle with $r(\lambda) = s(\lambda) = s(\alpha) = s(\beta)$, and, if (γ, δ) is an ancestry pair such that $\pi_R(Z(\alpha, \beta)) \cap \pi_R Z((\gamma, \delta)) \neq \emptyset$, then it is possible to recover (γ, δ) from the factorization of λ as in Lemma 3.3. Specifically, if (i) holds in Lemma 3.3, then, knowing the factorization $\lambda = \alpha'\delta'$ (that is, the ordered pair of paths (α', δ') such that $\alpha'\delta' = \lambda$) immediately yields γ (truncate the first $|\alpha'|$ edges from α) and δ $(= \beta\delta')$; the same is true if (ii) holds. Thus, specifying a factorization of λ is enough to specify the ancestry pair (γ, δ). However, the cycle λ has only finitely many possible factorizations; thus, there are only finitely many cycle-free minimal ancestry pairs (γ, δ) such that $\pi_R(Z(\alpha, \beta)) \cap \pi_R Z((\gamma, \delta)) \neq \emptyset$. $\square$

Lemma 3.6. *Let (λ, μ) be an ancestry pair. Then, there is a unique minimal ancestry pair (α, β) such that $Z(\lambda, \mu) \subset Z(\alpha, \beta)$ in G_E. If (λ, μ) is cycle-free, then so is (α, β).*

Proof. First, we prove uniqueness. If $Z(\lambda, \mu) \subset Z(\alpha, \beta) \cap Z(\alpha', \beta')$ for minimal ancestry pairs (α, β) and (α', β'), then, by [**12,** Lemma 2.5], we must have that $(\alpha, \beta) = (\alpha'\epsilon, \beta'\epsilon)$, or vice versa, for some choice of path ϵ. By minimality, we see that ϵ must have length zero so that $(\alpha, \beta) = (\alpha', \beta')$, proving uniqueness.

Next, we prove existence. If (λ, μ) is not minimal, then choose ϵ of maximum length such that we can write $(\lambda, \mu) = (\alpha\epsilon, \beta\epsilon)$ for some α, β. Then, we see that (α, β) is a minimal ancestry pair and $Z(\lambda, \mu) \subset Z(\alpha, \beta)$. For the final claim, it is clear that, if (λ, μ) is cycle-free, then (α, β) constructed as above will be as well. $\square$

Lemma 3.7. *Let E be a directed graph in which no cycle has an entrance, and let (λ, μ) be an ancestry pair. Then, there is a cycle-free minimal ancestry pair (α, β) such that $\pi_R(Z(\lambda, \mu)) \subset \pi_R(Z(\alpha, \beta))$.*

Proof. First, take a minimal ancestry pair (α, β) such that $Z(\lambda, \mu) \subset Z(\alpha, \beta)$ as in Lemma 3.6. Then, $\pi_R(Z(\alpha, \beta) \subset \pi_R(Z(\lambda, \mu))$; thus, we focus on the minimal ancestry pair (α, β). If neither α nor β contains a cycle, then we are finished. Now, assume that τ is the longest cycle that α contains; then, there exist $\tau', \tau'' \in E^*$ such that α can be written as $\alpha = \alpha'\tau\tau'$ and $\tau = \tau'\tau''$. It is evident that $\alpha'\tau'$ contain *no* cycle and that any infinite path x, which can be written as $x = \alpha z$, can also be written as $x = \alpha'\tau' z$ (by the assumption that no cycle has an entrance). Replacing α with the (cycle-free) path $\alpha'\tau'$, we see that $(\alpha'\tau', \beta)$ is a minimal ancestry pair whose first coordinate contains no cycle, and $\pi_R(Z(\alpha'\tau', \beta)) = \pi_R(Z(\alpha, \beta))$.

Repeating this process, we can find β', σ' such that $\pi_R(Z(\alpha, \beta)) = \pi_R(Z(\alpha'\tau', \beta'\sigma'))$ and such that $(\alpha'\tau', \beta'\sigma')$ is a cycle-free minimal ancestry pair. Since $\pi_R(Z(\alpha, \beta))$ contains $\pi_R(Z(\lambda, \mu))$, this establishes the lemma. $\square$

Theorem 3.8. *Let E be a row-finite directed graph with no sources. Then, $C^*(E)$ has continuous trace if and only if both*

(i) *no cycle of E has an entrance, and*
(ii) *E has finite ancestry.*

Proof. Suppose no cycle of E has an entrance and E has finite ancestry. We show that Φ_R in Definitions 2.10 and 2.11 is proper. The collection of sets $Z(v) \times Z(w) \subset E^\infty \times E^\infty$ form a compact-open cover for $E^\infty \times E^\infty$. Thus, it suffices to prove that $\Phi_R^{-1}(Z(v) \times Z(w))$ is compact for any vertices v and w. It is not difficult to see that

$$\Phi_G^{-1}(Z(v) \times Z(w)) = \bigcup_{(\alpha,\beta)\in\mathcal{M}} Z(\alpha,\beta),$$

where $\mathcal{M}$ is the set of all minimal ancestry pairs for v and w. We can partition these pairs into two families: $\mathcal{C}$, the set of minimal ancestry pairs for v and w containing cycles and $\mathcal{D}$, the set of cycle-free minimal ancestry pairs.

By definition, we have that $\Phi_R^{-1}(Z(v) \times Z(w)) = \pi_R(\Phi^{-1}(Z(v) \times Z(w)))$. Thus, we can write

$$\pi_R(\Phi^{-1}(Z(v) \times Z(w))) = \pi_R\bigg(\bigcup_{(\alpha,\beta)\in\mathcal{C}} Z(\alpha,\beta)\bigg) \cup \pi_R\bigg(\bigcup_{(\lambda,\mu)\in\mathcal{D}} Z(\lambda,\mu)\bigg).$$

However, for each $(\alpha,\beta) \in \mathcal{C}$, we have that $\pi_R(Z(\alpha,\beta)) \subset \pi_R(Z(\lambda,\mu))$ for some $(\lambda,\mu) \in \mathcal{D}$. Thus, $\Phi_R^{-1}(Z(v) \times Z(w)) = \cup_{(\lambda,\mu)\in\mathcal{D}} \pi_R(Z(\lambda,\mu))$. Each $\pi_R(Z(\lambda,\mu))$ is compact by the continuity of π_R and compactness of $Z(\lambda,\mu)$, and $\mathcal{D}$ is finite by the assumption that E has finite ancestry. Hence, Φ_R is a proper map, and the C^*-algebra has continuous trace by **[14]**.

Now, suppose that $C^*(E)$ has continuous trace. Then, the isotropy groups of G must vary continuously so that no cycle of E has an entrance. Thus, we only need show that E has finite ancestry. Suppose that v and w are two vertices, and let $\{(\alpha_k,\beta_k) : k \in \mathrm{A}\}$ be an enumeration of the cycle-free minimal ancestry pairs for v and w. As in the proof of the sufficiency, we can write $\Phi_R^{-1}(Z(v) \times Z(w)) = \cup_k \pi_R(Z(\alpha_k,\beta_k))$. By properness of Φ_R, we must be able to extract a finite subcover. However, for any finite subset $B \subset A$, the set

$$B' = \left\{ k \in A : \pi_R(Z(\alpha_k, \beta_k)) \cap \left(\bigcup_{j \in B} \pi_R(Z(\alpha_j, \beta_j)) \right) \neq \emptyset \right\}$$

is finite by Lemma 3.5. This implies that A is finite. Thus, E has finite ancestry. $\square$

4. Arbitrary graphs. The previous theorem is given only in the context of row-finite graphs with no sources. In this section, we will remove the requirement that all graphs be row-finite and have no sources, through the use of the Drinen-Tomforde desingularization.

Definition 4.1. A *tail* at the vertex v is an infinite path $e_1 e_2 \cdots$ with range $v = r(e_1)$.

Briefly, the Drinen-Tomforde desingularization adds a tail to each singular vertex. If the singular vertex v is an infinite receiver, it takes all the edges with range v and redirects each to a different vertex on the infinite tail (the choice of which vertices receive which edges requires us to use a desingularization). This produces a new graph $\widetilde{E}$, which has no singular vertices. For details, see [**4, 16**]. Note that we have reversed the edge orientation of [**4**] to fit with the higher-rank graphs considered in the next section.

Theorem 4.2 ([**4,** Theorem 2.11]). *Let E be an (arbitrary) directed graph. Let $\widetilde{E}$ be a desingularization for E. Then, $C^*(E)$ embeds in $C^*(\widetilde{E})$ as a full corner so that $C^*(E)$ is Morita equivalent to $C^*(\widetilde{E})$.*

The basic technical lemma necessary for the analysis in this section is a bijection between finite paths in a singular graph and certain finite paths in its desingularization. (We have omitted the part about infinite paths.)

Lemma 4.3 ([**4,** Lemma 2.6]). *Let E be a directed graph, and let $\widetilde{E}$ be a desingularization. Then, there is a bijection*

$$\phi : E^* \longrightarrow \{\beta \in \widetilde{E}^* : s(\beta), r(\beta) \in E^0\}.$$

The map ϕ preserves source and range.

Remark 4.4. The construction of the bijection ϕ in **[4]** has the following homomorphism property: for paths $\alpha, \beta, \lambda \in E^*$, we have $\lambda = \alpha\beta$ if and only if $\phi(\lambda) = \phi(\alpha)\phi(\beta)$. Furthermore, λ is cycle-free in the sense of Definition 2.2 if and only if $\phi(\lambda)$ is cycle-free.

Lemma 4.5. *Let E be a directed graph, and let $\widetilde{E}$ be a desingularization for E. Then, no cycle of E has an entrance if and only if no cycle of $\widetilde{E}$ has an entrance.*

Proof. Suppose that no cycle of $\widetilde{E}$ has an entrance. Let $\lambda = e_1 \cdots e_n$ be a cycle in E, and let $\widetilde{\lambda} = \phi(\lambda) = f_1 \cdots f_m$ be the corresponding path in $\widetilde{E}$, with $r(\widetilde{\lambda}) = r(\lambda)$ and $s(\widetilde{\lambda}) = s(\lambda)$. Suppose that e is an edge in E with $r(e) = r(e_k)$, and yet, $e \neq e_k$. Then, $\widetilde{e} = \phi(e)$ is a path in $\widetilde{E}$ with $r(\widetilde{e}) = r(\phi(e_k))$ and $\widetilde{e} \neq \widetilde{e}_k$ (here, we are using the fact that ϕ is a bijection). Thus, $\widetilde{e}$ is an entrance to the cycle $\widetilde{\lambda}$.

Suppose that no cycle of E has an entrance, and let $\mu = f_1 f_2 \cdots f_n$ be a cycle in $\widetilde{E}$. If $s(\mu)$ belongs to E^0, then $\phi^{-1}(\mu)$ is a cycle in E. Furthermore, we know that no vertex of E^0 on $\phi^{-1}(\mu)$ can be singular since then the cycle $\phi^{-1}(\mu)$ would have an entrance. Thus, μ consists solely of edges in E and does not meet any singular vertices or tails. The only edges in $\widetilde{E}$ that meet μ are images under ϕ of edges from E, and we know that μ has no entrances in E; thus, μ has no entrances.

Now, we show that, under the assumption that no cycle of E has an entrance, there is no cycle of $\widetilde{E}$ whose source vertex lies on an infinite tail added in the desingularization. Suppose that μ is a cycle in $\widetilde{E}$ with source on such an infinite tail. Since no infinite tail contains a cycle, we can write $\mu = f_1 \cdots f_k d_1 \cdots d_j$, where $d_1 \cdots d_j$ is the largest path in the infinite tail containing $s(\mu)$ such that $d_1 \cdots d_j$ is contained in μ. Then, $r(d_1)$ must be the vertex to which the infinite tail is attached, i.e., $r(d_1) \in E^0$. Consider the cycle $\mu' = d_1 \cdots d_j f_1 \cdots f_k$. This begins and ends in E^0; thus, it equals $\phi(\lambda)$ for some cycle λ in E. This cycle cannot meet any singular vertices in E (or else it would have an entrance); hence, it must be the case that $\lambda = \phi(\lambda)$. However, $s(\mu)$ belongs to λ, contradicting our assumption that $s(\mu)$ belongs to an infinite tail. Combining this with the previous part shows that, if no cycle of E has an entrance, then no cycle of $\widetilde{E}$ has an entrance. $\square$

In the proof of the next technical lemma, we will sometimes refer to a pair (α, β) as an ancestry pair in a graph, which means that it is an ancestry pair for the vertices $r(\alpha)$, $r(\beta)$.

Lemma 4.6. *Let E be a directed graph, and let $\widetilde{E}$ be a desingularization of E. Then $\widetilde{E}$ has finite ancestry if and only if E has finite ancestry.*

Proof.

(If). Suppose that E has finite ancestry, and let $v, w \in \widetilde{E}^0$ be two vertices in the desingularization and $A_{v,w} = \{\alpha_i, \beta_i\}$ the set of all distinct cycle-free minimal ancestry pairs (in $\widetilde{E}$) for v and w.

Let σ be the shortest path in $\widetilde{E}$ from v to a vertex in E^0, and likewise, let τ be the shortest path in $\widetilde{E}$ from w to a vertex in E^0. Let (α, β) be a cycle-free minimal ancestry pair for v and w; consider the ancestry pair $(\sigma\alpha, \tau\beta)$: this is minimal unless either $\alpha = v$ and $\beta = \beta'\sigma$ or $\beta = w$ and $\alpha = \alpha'\tau$, in which case the pair $(r(\sigma), \tau\beta')$ or $(\sigma\alpha', r(\tau))$ will be minimal, respectively. Define $\psi(\alpha, \beta) = (\sigma\alpha, \tau\beta)$, if the latter ancestry pair is minimal, and define $\psi(\alpha, \beta)$ to be either $(r(\sigma), \tau\beta')$ or $(\sigma\alpha', r(\tau))$, depending on the case above. The map ψ is an injection from the cycle-free minimal ancestry pairs for v and w into the set of cycle-free minimal ancestry pairs for $r(\sigma)$ and $r(\tau)$. It is easy to see that any *minimal* ancestry pair in $\widetilde{E}$ for vertices that are both in E^0 must have common source vertex also in E^0. Thus, we can consider the composition

$$(\alpha, \beta) \longmapsto \psi(\alpha, \beta) = (\lambda, \mu) \longmapsto (\phi(\lambda), \phi(\mu)).$$

This carries the set $A_{v,w}$ injectively into the set of cycle-free minimal ancestry pairs in E for $r(\sigma)$, $r(\tau)$, according to Remark 4.4. Since the latter set is finite by our assumptions on E, it must be the case that $A_{v,w}$ is finite so that E has finite ancestry.

(Only if). Suppose that $\widetilde{E}$ has finite ancestry, let v and w be two vertices of E and let $A_{v,w} = \{(\alpha_i, \beta_i)\}$ be the set of all cycle-free minimal ancestry pairs for v and w. Then, $\{(\phi(\alpha_i), \phi(\beta_i))\}$ is, again following Remark 4.4, a set of cycle-free minimal ancestry pairs for $\phi(v)$ and $\phi(w)$, and thus, it must be finite. Then, the fact that ϕ is a bijection shows that $A_{v,w}$ is finite. $\square$

Theorem 4.7. *Let E be an arbitrary graph. Then, $C^*(E)$ has continuous trace if and only if both*

(i) *no cycle of E has an entrance*; *and*
(ii) *E has finite ancestry.*

Proof. We begin by fixing a desingularization $\widetilde{E}$ of E. If no cycle of E has an entrance and E has finite ancestry, then Lemmas 4.5 and 4.6 tell us that the same is true of $\widetilde{E}$. Then, Theorem 3.8 states that $C^*(\widetilde{E})$ has continuous trace. Theorem 4.2 and the fact that the class of continuous-trace C^*-algebras is closed under Morita equivalence then give that $C^*(E)$ has continuous trace.

Now, suppose that $C^*(E)$ has continuous trace. Then, $C^*(\widetilde{E})$ has continuous trace as in the previous part of the proof. By Theorem 3.8, we see that no cycle of $\widetilde{E}$ has an entrance and $\widetilde{E}$ has finite ancestry. Lemmas 4.5 and 4.6 again yield that E satisfies the same conditions. $\square$

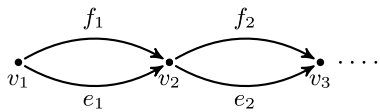
Corollary 4.8. *If E is a graph with no cycles, then $C^*(E)$ has continuous trace if and only if E has finite ancestry.*

Corollary 4.8 can be applied to AF algebras. Drinen showed that every AF algebra arises as the C^*-algebra of a locally finite pointed directed graph [**3**]. Tyler gave a useful complementary result, showing that, if E is a Bratteli diagram for an AF algebra A, then there is another Bratteli diagram KE for A such that (treating the diagrams as directed graphs) $C^*(KE)$ contains A and $C^*(E)$ as complementary full corners [**19**]. Thus, in particular, A and $C^*(E)$ are Morita equivalent.

Cautionary note. The paper [**19**] uses a different edge orientation convention; therefore, we *reverse* all edges of Bratteli diagrams when computing their graph C^*-algebras.

Corollary 4.9. *Let A be an* AF *algebra, and let E_{rev} be a Bratteli diagram for A with all edges reversed. Then, A has continuous trace if and only if E_{rev} has finite ancestry.*

Example 4.10. Let $A = \bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$ be the UHF algebra of type 2^{∞}. The familiar Bratteli diagram for A (with labels) is



If we treat this diagram as a directed graph E and reverse all of the edges, then $C^*(E_{\text{rev}})$ is Morita equivalent to A. The graph E_{rev} fails to have finite ancestry: for each k, we have the cycle-free minimal ancestry pair $(f_1 e_2 f_3 \cdots e_{2k}, e_1 f_2 e_3 \cdots f_{2k})$ for v_1, v_1. Thus, A does not have continuous trace. (As is well known, we can actually reach a stronger conclusion, namely, that A does not have Hausdorff spectrum, see **[8]**.)

5. Higher-rank graphs. In this section, we partially extend the results of Section 4 to the realm of higher-rank graphs. We have not completely described which higher-rank graph C^*-algebras have continuous trace. However, we do characterize the higher-rank graphs with *principal* path groupoids which yield continuous-trace C^*-algebras. The jump in combinatorial complexity from the graph to the k-graph case is noteworthy. In addition, we provide some negative results regarding the *generalized cycles* of **[5]**. In particular, a generalized cycle with entry leads to an infinite projection, which cannot occur if the algebra has Hausdorff spectrum.

Remark 5.1. In this section, the semigroup $\mathbb{N}^k$ is treated as a category with a single object, 0.

Definition 5.2 (**[10]**)**.** A *higher-rank graph*, or *k-graph*, consists of a countable category Λ equipped with a *degree functor* $d : \Lambda \to \mathbb{N}^k$, which satisfies the following factorization property: if $d(\lambda) = m + n$ for some $m, n \in \mathbb{N}^k$, then $\lambda = \mu\nu$ for some unique μ, ν such that $d(\mu) = m$ and $d(\nu) = n$. The *vertices* Λ^0 of Λ are identified with the objects. The elements of Λ are referred to as *paths*. For fixed degree $n \in \mathbb{N}^k$, the paths of degree n are denoted by Λ^n. We refer to paths of degree 0 as *vertices* in the k-graph. The range and source maps $r, s : \Lambda \to \Lambda^0$ are defined so that $r(\lambda)\lambda = \lambda s(\lambda) = \lambda$ for all λ.

We can construct a C^*-algebra from a higher-rank graph much like the procedure for graph C^*-algebras; however, some additional hypotheses must be added in order to ensure the result is not trivial.

The hypotheses used here are not the weakest which define a meaningful C^*-algebra, but they let us easily use the groupoid machinery.

Definition 5.3 ([**10**])**.** For $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, let $v\Lambda^n = r^{-1}(v) \cap \Lambda^n$. We say that Λ is *row-finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$, $n \in \mathbb{N}^k$. We say that Λ *has no sources* if $v\Lambda^n$ is non-empty for all $v \in \Lambda^0$, $n \in \mathbb{N}^k$.

Definition 5.4 ([**10**])**.** Let Λ be a row-finite k-graph with no sources. Then, the *higher-rank graph* C^*-algebra of Λ, denoted $C^*(\Lambda)$, is the universal C^*-algebra generated by a family of partial isometries $\{s_\lambda\}_{\lambda \in \Lambda}$ satisfying:

(i) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;
(ii) if $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$, then $s_\lambda s_\mu = s_{\lambda\mu}$;
(iii) $s_\lambda^* s_\lambda = s_{s(\lambda)}$;
(iv) for any $v \in \Lambda^0$ and any degree $n \in \mathbb{N}^k$, we have $s_v = \sum_{\lambda \in \Lambda^n : r(\lambda) = v} s_\lambda s_\lambda^*$.

Similarly to the graph case, we study continuous-trace higher-rank graph C^*-algebras by constructing a path groupoid and applying groupoid results. The following k-graph is used to define infinite paths in k-graphs.

Definition 5.5. Let Ω_k be the category of all pairs $\{(m, n) : m \leq n\} \subset \mathbb{N}^k \times \mathbb{N}^k$, where $m \leq n$ if $m_i \leq n_i$ for all $i = 1, \ldots, k$. The composition is given by $(m, n)(n, p) = (m, p)$. The degree functor is given by $d(m, n) = n - m$. The objects are all pairs of the form (m, m). If Λ is a k-graph, then an *infinite path* in Λ is a degree preserving functor $x : \Omega_k \to \Lambda$. The collection of infinite paths in Λ is denoted Λ^∞.

Let Λ be a k-graph, and let x be an infinite path in Λ. For any $p \in \mathbb{N}^k$, we define $\sigma^p x$ to be the infinite path given by $\sigma^p x(m, n) = x(m + p, n + p)$. The *range* of an infinite path $x \in \Lambda^\infty$ is defined to be the vertex $x(0, 0)$ and is denoted by $r(x)$. If $\lambda \in \Lambda$ and $x \in \Lambda^\infty$ with $s(\lambda) = r(x)$, then there is a unique path $y = \lambda x \in \Lambda^\infty$ such that $\sigma^{d(\lambda)} y = x$ and $y(0, d(\lambda)) = \lambda$.

Now, we can define the higher-rank version of the path groupoid. As noted in [**10**], the no sources assumption implies that every vertex $v \in \Lambda^0$ is the range of at least one infinite path $x \in \Lambda^\infty$.

Definition 5.6 ([**10**])**.** Let Λ be a k-graph. For paths $x, y \in \Lambda^\infty$ and $n \in \mathbb{Z}^k$, write $x \sim_n y$ if there exist $p, q \in \mathbb{N}^k$ such that $\sigma^p x = \sigma^q y$ and $p - q = n$. The *path groupoid* of Λ is $G_\Lambda = \{(x, n, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty : x \sim_n y\}$, with operations given by $(x, n, y)(y, m, z) = (x, m+n, z)$ and $(x, n, y)^{-1} = (y, -n, x)$.

The topology on G_Λ is defined in the same way as the topology on G_E, for E a graph. The basic open sets take the form

$$Z(\alpha, \beta) = \{(x, d(\alpha) - d(\beta), y) \in G_\Lambda : \sigma^{d(\alpha)}(x) = \sigma^{d(\beta)}(y)\}.$$

The topology on G_Λ generated by these sets transforms it into a locally compact Hausdroff étale groupoid with unit space Λ^∞, see [**10,** Proposition 2.8]. The relative topology on the unit space can be described by open sets of the form $Z(\alpha) = \{x \in \Lambda^\infty : x(0, d(\alpha)) = \alpha\}$.

Theorem 5.7 ([**10,** Corollary 3.5])**.** *Let Λ be a row-finite k-graph with no sources, and let G_Λ be its path groupoid. Then, $C^*(\Lambda) \cong C^*(G_\Lambda)$.*

Recall that, if $u \in G^{(0)}$, then $G(x) = \{\gamma \in G : r(\gamma) = s(\gamma) = u\}$ is the stabilizer subgroup at u.

Definition 5.8. A row-finite k-graph Λ with no sources is called *principal* if the path groupoid G_Λ is principal, that is, if, for all $x \in \Lambda^\infty$ and $p, q \in \mathbb{N}^k$, the equation $\sigma^p x = \sigma^q x$ implies that $p = q$.

Remark 5.9. As noted in [**7**], for principal groupoids G, the map onto the orbit groupoid R is an isomorphism.

We modify our definition of ancestry pair to the k-graph situation.

Definition 5.10. Let Λ be a row-finite k-graph with no sources, and let $v, w \in \Lambda^0$ be two vertices. Then, an *ancestry pair* for v, w is a pair $(\lambda, \mu) \in \Lambda \times \Lambda$ such that $r(\lambda) = v$, $r(\mu) = w$, and $s(\lambda) = s(\mu) = w$. An ancestry pair (λ, μ) is *minimal* if $(\lambda, \mu) = (\lambda'\nu, \mu'\nu)$ implies $\nu = s(\lambda)$. We say that Λ has *strong finite ancestry* if each pair of vertices has at most finitely many minimal ancestry pairs.

Remark 5.11. Strong finite ancestry implies finite ancestry in the 1-graph case. In fact, a 1-graph E having strong finite ancestry is equivalent to E having finite ancestry and no directed cycles.

Consider the map $p : G_\Lambda \to \Lambda^\infty \times \Lambda^\infty$, given by $(x, n, y) \mapsto (x, y)$. The image of this map forms a groupoid under $(x, y)(y, z) = (x, z)$ and $(x, y)^{-1} = (y, x)$, and we denote this groupoid by R_Λ. We give R_Λ the quotient topology induced by the map p. For the following lemma, recall that a groupoid H is proper if the map $h \to (r(h), s(h)) \in H^{(0)} \times H^{(0)}$ is proper.

Lemma 5.12. *Suppose that Λ is a row-finite k-graph with no sources. If Λ has strong finite ancestry, then R_Λ is proper.*

Proof. We adopt the notation of Theorem 3.8. As in the proof of Theorem 3.8, we see that

$$\Phi_R^{-1}(Z(v) \times Z(w)) = \bigcup_{(\alpha,\beta)\in\mathcal{M}} \pi_R(Z(\alpha, \beta)),$$

where $\mathcal{M}$ is the set of minimal finite ancestry pairs for arbitrary vertices v and w. Strong finite ancestry then implies that Φ_R is proper so that R_Λ is proper. $\square$

The next lemma is used to show that strong finite ancestry is necessary for a *principal* k-graph to yield a C^*-algebra with continuous trace.

Remark 5.13. In the proof of the following lemma, if $1 \leq j \leq k$, then e_j refers to the jth standard basis vector in $\mathbb{N}^k$. Moreover, if α is a path and $0 \leq m \leq n \leq d(\alpha)$, we use $\alpha(m, n)$ to refer to the unique path of degree $n - m$ such that $\alpha = \alpha'\alpha(m, n)\alpha''$ for paths $\alpha' \in \Lambda^m$, $\alpha'' \in \Lambda^{d(\alpha)-n}$.

Lemma 5.14. *Let Λ be a principal row-finite k-graph with no sources. Let (α, β) and (γ, δ) be two distinct minimal ancestry pairs in Λ. Then, $Z(\alpha, \beta) \cap Z(\gamma, \delta) = \emptyset$.*

Proof. We claim that it suffices to show that, if $Z(\alpha,\beta)\cap Z(\gamma,\delta) \neq \emptyset$, then either $d(\alpha) \geq d(\gamma)$ and $d(\beta) \geq d(\delta)$, or $d(\gamma) \geq d(\alpha)$ and $d(\delta) \geq d(\beta)$. For, suppose that $(\alpha z, n, \beta z) = (\gamma w, n, \delta w)$, $d(\alpha) \geq d(\gamma)$ and $d(\beta) \geq d(\delta)$. Then, $\alpha z = \gamma\alpha' z = \gamma w$ so that $\alpha' z = w$ (where $\alpha' = \alpha(d(\gamma), d(\alpha)))$. We also have $\beta z = \delta\beta' z = \delta w$ so that $\beta' z = w$ (where $\beta' = \beta(d(\delta), d(\beta)))$. If $d(\alpha') = d(\beta')$, then this shows that $\alpha' = \beta'$ so that $(\alpha, \beta) = (\gamma\alpha', \beta\alpha')$, contradicting minimality of the pair (α, β). If $d(\alpha') \neq d(\beta')$, then the equation $\sigma^{d(\alpha')}w = z = \sigma^{d(\beta')}w$ contradicts the assumption that Λ is principal. The other case follows by symmetry, establishing the claim.

If the intersection $Z(\alpha, \beta) \cap Z(\gamma, \delta)$ is nonempty, then we must have $(\alpha z, n, \beta z) = (\gamma w, n, \delta w)$ for some $z, w \in \Lambda^\infty$, where $n = d(\alpha) - d(\beta) = d(\gamma) - d(\delta)$; note that this implies $d(\alpha) - d(\gamma) = d(\beta) - d(\delta)$. Thus, if $d(\alpha) \geq d(\gamma)$, we also have $d(\beta) \geq d(\delta)$, reducing to the claim. Hence, we assume by way of contradiction that there are indices $j, \ell \leq k$ such that

$$d(\alpha)_j - d(\gamma)_j > 0 \quad \text{and} \quad d(\gamma)_\ell - d(\alpha)_\ell > 0. \tag{5.1}$$

(Note that $d(\beta)_j - d(\delta)_j$ and $d(\delta)_\ell - d(\beta)_\ell$ are both positive as well.)

First, consider the jth coordinate: as $d(\gamma)_j \leq d(\alpha)_j$, we can factor α as $\alpha(0, d(\gamma)_j e_j)\alpha(d(\gamma)_j e_j, d(\alpha))$ and γ as $\gamma(0, d(\gamma)_j e_j)\gamma(d(\gamma)_j e_j, d(\gamma))$. Set $\alpha' = \alpha(d(\gamma)_j e_j, d(\alpha))$ and $\gamma' = \gamma(d(\gamma)_j e_j, d(\gamma))$. Now, observe that $\alpha' z = \gamma' w$; we denote this infinite path by x, and remark that $\sigma^{d(\alpha')}x = z$ and $\sigma^{d(\gamma')}x = w$. Subtract $d(\gamma)_j e_j + d(\delta)_j e_j$ on both sides of $d(\alpha) + d(\delta) = d(\beta) + d(\gamma)$ to obtain

$$d(\alpha) - d(\gamma)_j e_j + d(\delta) - d(\delta)_j e_j = d(\beta) - d(\delta)_j e_j + d(\gamma) - d(\gamma)_j e_j. \tag{5.2}$$

Now, observe that

$$\begin{aligned} \sigma^{d(\delta)-d(\delta)_j e_j} z &= \sigma^{d(\delta)-d(\delta)_j e_j}\sigma^{d(\alpha)-d(\gamma)_j e_j} x \\ &= \sigma^{d(\delta)-d(\delta)_j e_j + d(\alpha)-d(\gamma)_j e_j} x \\ &= \sigma^{d(\beta)-d(\delta)_j e_j + d(\gamma)-d(\gamma)_j e_j} x \\ &= \sigma^{d(\beta)-d(\delta)_j e_j} w, \end{aligned}$$

where we have used (5.2) in the third equation. A similar calculation shows that $\sigma^{d(\alpha)-d(\alpha)_\ell e_\ell} w = \sigma^{d(\gamma)-d(\alpha)_\ell e_\ell} z$. Applying $\sigma^{d(\delta)-d(\delta)_j e_j}$ to both sides of $\sigma^{d(\gamma)-d(\alpha)_\ell e_\ell} z = \sigma^{d(\alpha)-d(\alpha)_\ell e_\ell} w$, we obtain

$$\begin{aligned}\sigma^{d(\delta)-d(\delta)_j e_j + d(\gamma)-d(\gamma)_\ell e_\ell} z &= \sigma^{d(\delta)-d(\delta)_j e_j + d(\alpha)-d(\alpha)_\ell e_\ell} w \\ &= \sigma^{d(\gamma)+d(\beta)-d(\delta)_j e_j - d(\alpha)_\ell e_\ell} w \\ &= \sigma^{d(\gamma)-d(\alpha)_\ell e_\ell} \sigma^{d(\beta)-d(\delta)_j e_j} w \\ &= \sigma^{d(\gamma)-d(\alpha)_\ell e_\ell} \sigma^{d(\delta)-d(\delta)_j e_j} z,\end{aligned}$$

where we have used $d(\alpha)+d(\delta)=d(\beta)+d(\gamma)$ in the second equation. We have assumed that Λ is principal, so we must have that

$$d(\delta)-d(\delta)_j e_j + d(\gamma) - d(\gamma)_\ell e_\ell = d(\gamma) - d(\alpha)_\ell e_\ell + d(\delta) - d(\delta)_j e_j.$$

Cancelation now yields $d(\gamma)_\ell e_\ell = d(\alpha)_\ell e_\ell$, contradicting our assumption that $d(\gamma)_\ell > d(\alpha)_\ell$ in (5.1). $\square$

Theorem 5.15. *Let Λ be a principal row-finite k-graph with no sources. Then, $C^*(\Lambda)$ has continuous trace if and only if Λ has strong finite ancestry.*

Proof.

(i) Since Λ is principal, the path groupoid G_Λ trivially has continuous isotropy. Lemma 5.12 implies that $G_\Lambda = R_\Lambda$ is proper. Thus, [**14**, Theorem 1.1] implies that $C^*(\Lambda)$ has continuous trace.

(ii) Since Λ is strictly aperiodic, we can identify the groupoids G_Λ and R_Λ. Let v and w be two vertices of Λ. Then,

$$\Phi^{-1}(Z(v)\times Z(w)) = \bigcup_{(\alpha,\beta)\in\mathcal{M}} Z(\alpha,\beta)$$

as in the proof of Theorem 3.8. Lemma 5.14 implies that the sets $Z(\alpha,\beta)$ are pairwise disjoint and open. Thus, $\mathcal{M}$ must be finite by compactness of $\Phi^{-1}(Z(v)\times Z(w))$, which implies that Λ has strong finite ancestry. $\square$

Remark 5.16. Theorem 5.15 is not as complete as Theorem 3.8; a complete description of those k-graphs which define continuous-trace C^*-algebras as in Theorem 3.8 seems out of reach since it is difficult to give a condition on k-graphs which is equivalent to continuity of stabilizers in the path groupoid. Any such condition used must be at least as strong as the condition [**1,** Theorem 4.4] which describes the k-graphs that have closed interior isotropy.

Desingularization is, in general, much more complicated for higher-rank graphs, cf., [**6**], so it seems perhaps unlikely that this could be easily extended to higher-rank graphs with sources. However, we can give some necessary conditions for a k-graph to satisfy in order that its C^*-algebra have continuous trace. The following definition is somewhat modified from [**5**].

Definition 5.17 ([**5**])**.** Let Λ be a row-finite graph with no sources. Then, a pair $(\lambda, \mu) \in \Lambda \times \Lambda$ is called a *generalized cycle* if $\lambda \neq \mu$, $r(\lambda) = r(\mu)$, $s(\lambda) = s(\mu)$ and $Z(\lambda) \subset Z(\mu)$. We say that a generalized cycle (λ, μ) has an *entrance* if (μ, λ) is *not* a generalized cycle, that is, if $Z(\lambda) \subsetneq Z(\mu)$.

Recall that a projection p in a C^*-algebra A is infinite if there exists a $v \in A$ with $v^*v = p$ and $vv^* < p$, that is, if it is Murray-von Neumann equivalent to a proper subprojection of itself.

Lemma 5.18 ([**5,** Corollary 3.8])**.** *If Λ contains a generalized cycle with entrance, then $C^*(\Lambda)$ contains an infinite projection.*

The following, simple observation is probably not new but is proven here for ease of reference.

Lemma 5.19. *If A is a C^*-algebra containing an infinite projection, then A does not have continuous trace.*

Proof. Let p be a projection in A with a proper subprojection q such that $p \sim q$. Take an irreducible representation $\pi : A \to B(H)$ such that $\pi(p - q) \neq 0$. Then, $\pi(q) < \pi(q)$ are equivalent projections in $B(H)$. All compact projections are of finite rank so it cannot be the case that the range of π lies within the compacts. Since every irreducible representation of a C^*-algebra with continuous trace has range within the compact operators [**15,** Theorem 6.1.11], we see that A does not have continuous trace. $\square$

Corollary 5.20. *If Λ is a row-finite k-graph with no sources that contain a generalized cycle with entrance, then $C^*(\Lambda)$ does not have continuous trace.*

It is somewhat unsatisfactory that the question of when a higher-rank graph yields a continuous-trace C^*-algebra should have such a partial answer in comparison with the graph case. The main question to answer is the following.

Question 5.21. *For which (row-finite, source-free) k-graphs is it the case that the path groupoid G_Λ has continuous stabilizer subgroupoid?*

It is known (see **[13]**) that, for an étale groupoid G, the stabilizer map is continuous at a unit $u \in G$ if and only if $\mathrm{Iso}(G)_u = \mathrm{Iso}(G)^\circ_u$, where $\mathrm{Iso}(G)^\circ$ is the interior of the isotropy subgroupoid (itself an étale groupoid). Therefore, Question 5.21 can be reformulated as follows.

Question 5.22. *For which (row-finite, source-free) k-graphs Λ is the isotropy subgroupoid open in G_Λ?*

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St. Lawrence University, Mathematics Department, 118 Valentine Hall, 23 Romoda Drive, Canton, NY 13617
Email address: **dannycrytser@gmail.com**