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CONSTRUCTING MONOMIAL IDEALS WITH A GIVEN MINIMAL RESOLUTION

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ABSTRACT. This paper gives a description of various recent results, which construct monomial ideals with a given minimal free resolution. We show that these results are all instances of coordinatizing a finite atomic lattice, as found in [**11**]. Subsequently, we explain how, in some of these cases [**5, 6**] where questions still remain, this point of view can be applied. We also prove an equivalence for trees between the notion of *maximal* defined in [**6**] and the notion of being maximal in a Betti stratum.

1. Introduction. In recent years, there have been a number of papers, see, for example, [**5, 6, 9, 10, 13, 14**], where the authors focused on constructing monomial ideals with a specified minimal resolution, typically described as being supported on a specific CW-complex via the construction in [**2**]. Many of these constructions may be interpreted as "coordinatizing" a finite atomic lattice via the construction found in [**11**]. The main purpose of this paper is to bring attention to this fact through three particular cases, as found in [**5, 6, 14**].

In particular, a number of unanswered questions still remain which are motivated by the latter two papers. We believe consideration of the structure of the corresponding lcm lattices will help answer some of these questions. Specifically, this is due to the fact that the lcm lattice of a monomial ideal encodes important data which is obscured only by considering the cell complex that supports the resolution. More generally, we believe that, for many questions concerning monomial ideals, it is important to consider this additional data. For example, in the recent work [**7**], the authors give a characterization of which finite atomic lattices can be the lcm lattices of monomial ideals with

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pure resolutions. For their work, it is significant that they work in the context of the lcm lattice rather than a cell complex that supports the resolution.

This paper is structured as follows. First, in Section 2 we present the necessary background on finite atomic lattices and coordinatizations. Proposition 3.2 in Section 3 completes the characterization of coordinatizations found in [**11,** Theorem 3.2]. It should be noted that, at present, equivalent results have been independently proved [**9**]. However, we include our proof here for completeness since the language is consistent with that of [**11**].

Section 4 gives a description of the "nearly Scarf" construction of [**14**] and the "minimal squarefree" construction of [**15**] as coordinatizations. This section then shows that the construction found in [**5**] is also an example of a coordinatization. Further, we offer context for how the ideals in [**5**] fit in with the nearly Scarf ideals and the minimal squarefree ideals, and how this can be useful for considering the questions posed in [**5**].

Finally, in Section 5, we give a reformulation of the ideas in [**6**] in terms of the underlying lcm lattice of these ideals. We also show that, for trees, there is an explicit description of the finite atomic lattice and coordinatization that yields the ideals constructed in [**6**]. Using this description, we are able to show that, for trees, the *maximal* ideals constructed in [**6**] are also maximal in their Betti stratum. Further, we discuss how these ideals could be useful for understanding minimal resolutions of monomial ideals in general.

2. Preliminaries.

2.1. Lattices. A *lattice* is a set $(P, <)$ with an order relation $<$, which is transitive and antisymmetric, satisfying the following properties:

(1) P has a maximum element denoted by $\widehat{1}$;

(2) P has a minimum element denoted by $\widehat{0}$;

(3) Every pair of elements a and b in P has a join $a \vee b$, which is the least upper bound of the two elements;

(4) Every pair of elements a and b in P has a meet $a \wedge b$, which is the greatest lower bound of the two elements.

A related object is a *finite meet-semilattice*, a finite poset for which every pair of elements has a meet. Here, we will most often work with a meet-semilattice P that is equal to a set $L - \{\widehat{0}\}$, where L is a lattice.

We define an *atom* of a lattice P to be an element $x \in P$ such that x covers $\widehat{0}$, i.e., $x > \widehat{0}$, and there is no element a such that $x > a > \widehat{0}$. We will denote the set of atoms as atoms(P).

Definition 2.1. If P is a lattice and every element in $P - \{\widehat{0}\}$ is the join of atoms, then P is an *atomic lattice*. Further, if P is finite, then it is a *finite atomic lattice*.

Given a lattice P, elements $x \in P$ are *meet-irreducible* if $x \neq a \wedge b$ for any $a > x, b > x$. The set of meet-irreducible elements in P is denoted mi(P). Given an element $x \in P$, the *order ideal* of x is the set $\lfloor x \rfloor = \{a \in P \mid a \leqslant x\}$. Similarly, the *filter* of x is $\lceil x \rceil = \{a \in P \mid x \leqslant a\}$. We can also define *intervals* (open and closed, respectively) between two elements a and b of P as follows:

$$(a, b) = \{c \in P \mid a < c < b\}$$

and

$$[a, b] = \{c \in P \mid a \leq c \leq b\}.$$

There are two different simplicial complexes that can be associated with a finite atomic meet-semilattice P (or any poset for that matter). One is the *order complex* $\Delta(P)$, which is the complex whose vertices correspond to elements of P and facets correspond to maximal chains of P. For finite atomic lattices, such as those discussed here, we also have (a specific instance of) the *cross cut complex*, denoted $\Gamma(P)$. In $\Gamma(P)$, vertices correspond to atoms of P, and simplices correspond to subsets of atoms which have a join or meet in P. It is known that $\Delta(P)$ is homotopy equivalent to $\Gamma(P)$ [**3**].

2.2. Coordinatizations. One of the main results of [**15,** Theorem 5.1] is to show that every finite atomic lattice is in fact the lcm lattice of a monomial ideal. Phan's result is constructive in that, given a finite atomic lattice, he constructs a monomial ideal with that lattice. This result was generalized by a modified construction [**11**], which also

showed that, with the modified construction, all monomial ideals (not merely one instance of a monomial ideal with a given lattice) can be obtained. For convenience, we include here a brief description of this work.

Define a *labeling* of a finite atomic lattice P as any assignment of non-trivial monomials $\mathcal{M} = \{m_{p_1}, \ldots, m_{p_t}\}$ to some set of elements $p_i \in P$. It will be convenient to think of unlabeled elements as having the label 1. Define the monomial ideal $M_{\mathcal{M}}$ as the ideal generated by monomials

$$x(a) = \prod_{p \in \lceil a \rceil^c} m_p \tag{2.1}$$

for each $a \in \text{atoms}(P)$, where $\lceil a \rceil^c$ means taking the complement of $\lceil a \rceil$ in P. We say that the labeling $\mathcal{M}$ is a *coordinatization* if the lcm lattice of $M_{\mathcal{M}}$ is isomorphic to P.

The next theorem, [**11,** Theorem 3.2], gives a criteria for when a labeling is a coordinatization.

Theorem 2.2. *Any labeling $\mathcal{M}$ of elements in a finite atomic lattice P by monomials satisfying the following two conditions will yield a coordinatization of P.*

(C1) *If $p \in \text{mi}(P)$, then $m_p \neq 1$, i.e., all meet-irreducibles are labeled.*

(C2) *If $\gcd(m_p, m_q) \neq 1$ for some $p, q \in P$, then p and q must be comparable, i.e., each variable only appears in monomials along one chain in P.*

Note that Theorem 2.2 is not an "if and only if" statement; Section 3 addresses this issue. A main ingredient in Section 3 is the following discussion of deficit labelings from [**11**].

In order to complete our introduction to coordinatizations, we show that every monomial ideal can be obtained via a specific coordinatization of its lcm lattice. Let M be a monomial ideal with n generators, and let L_M be its lcm lattice. For notational purposes, let L_M be the set consisting of elements denoted l_p, which represent the monomials occurring in L_M. Now, define the abstract finite atomic lattice P where the elements in P are formal symbols p satisfying the relations $p < p'$

if and only if $l_p < l_{p'}$ in L_M, in other words, P is the abstract finite atomic lattice isomorphic to L_M obtained simply by omitting the data of the monomials in L_M. Define a labeling of P by letting $\mathcal{D}$ be the set consisting of monomials m_p for each $p \in P$, where

$$m_p = \frac{\gcd\{l_t \mid t > p\}}{l_p}. \tag{2.2}$$

By convention, $\gcd\{l_t \mid t > p\}$ for $p = \widehat{1}$ is defined to be $l_{\widehat{1}}$. Note that m_p is a monomial since, clearly, l_p divides l_t for all $t > p$.

This labeling may be used to prove that every monomial ideal can be realized as a coordinatization of its lcm lattice, as shown in the next proposition [**11,** Proposition 3.6].

Proposition 2.3. *Given M, a monomial ideal with* lcm *lattice P_M, if P is the abstract finite atomic lattice such that P and P_M are isomorphic as lattices, then the labeling $\mathcal{D}$ of P as defined by* (2.2) *is a coordinatization, and the resulting monomial ideal $M_{P,\mathcal{D}} = M$.*

Example 2.4. In Figure 1, we show a coordinatization of a finite atomic lattice together with its labeling. The labeling shown is in fact a coordinatization. It produces the monomial ideal $M = (bc^2d, acd, a^2d, a^2bc^2)$, where the monomials are ordered according to the atoms in Figure 1. In this case, computing $\mathcal{D}$ as defined by (2.2), returns the original labeling.

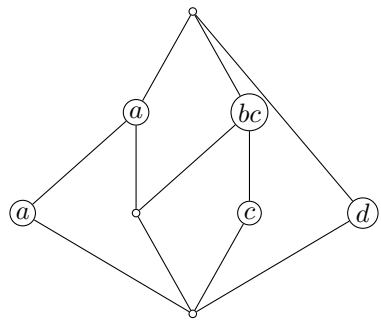

FIGURE 1. Lattice for Example 2.4.

3. Characterizing coordinatizations. A partial characterization of how to coordinatize a finite atomic lattice is given in [**11,** Theorem 3.2]. Further, [**11**] explains, given a monomial ideal M, how the coordinatization of L_M which produces M may be found. Here, we aim to use this process to characterize when a labeling is a coordinatization. With this in mind, we introduce a construction similar to $\mathcal{D}$, for a labeling $\mathcal{M}$ and a finite atomic lattice P. This object, which, as will be shown, agrees with $\mathcal{D}$ in the case where $\mathcal{M}$ is a coordinatization of P, is denoted $\mathcal{D}_{\mathcal{M}}$. Let P be a finite atomic lattice with n atoms and $\mathcal{M}$ any labeling of the lattice P. As in subsection 2.2, $M_{\mathcal{M}}$ will be the monomial ideal generated by the monomials $x(a_i)$, described in (2.1).

The construction of $\mathcal{D}_{\mathcal{M}}$ is similar to $\mathcal{D}$. However, rather than using the lcm lattice of $M_{\mathcal{M}}$, the original lattice of P itself is used. Here,

$$l_{a_i} = x(a_i) \tag{3.1}$$

for each atom $a_i \in P$, and

$$l_p = \operatorname{lcm}\{l_t \mid p > t\} \tag{3.2}$$

for each element $p \in P$. It is worth emphasizing that each of the $x(a_i)$ is used, appearing as l_{a_i} for an atom of P, not merely a minimal generating set for $M_{\mathcal{M}}$. The labeling $\mathcal{D}_{\mathcal{M}}$ is then defined as the set of monomials m_p described in equation (2.2), using the monomials l_p defined in (3.1) and (3.2). This means $\mathcal{D}_{\mathcal{M}}$ is a labeling of P (remember that P may not be the lcm lattice of $M_{\mathcal{M}}$).

Example 3.1. In Figure 2, we show a poset P and the vertices which are labeled using the variable x in a labeling $\mathcal{M}$. This example $\mathcal{M}$ violates condition (C2), since the variable x appears at non-comparable positions. The symbol X indicates which elements of P are labeled with x in $\mathcal{D}_{\mathcal{M}}$. Note that one of the x labels "moves" to the minimal element in P. This is due to the fact that there is at least one atom of P that is not less than any of the poset elements labeled with x in the original labeling $\mathcal{M}$. As a result, $x(a_i)$ will have a factor of x for each atom a_i.

We also note that, if this partial labeling $\mathcal{M}$ is completed such that the other variables satisfy (C2) and all the meet irreducibles are nontrivially labeled, thus satisfying (C1), the resulting labeling will be a coordinatization although it does not satisfy (C2).

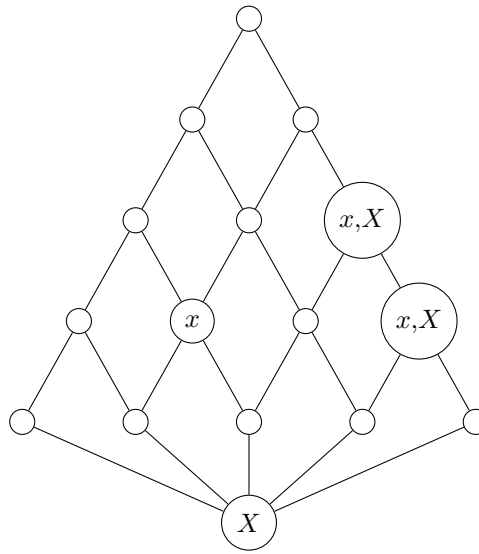

FIGURE 2. The partial labeling described in Example 3.1, showing one variable from $\mathcal{M}$ and the same variable in $\mathcal{D}_{\mathcal{M}}$.

The next proposition asserts that $\mathcal{D}_{\mathcal{M}}$ determines whether the original labeling $\mathcal{M}$ is in fact a coordinatization of P. An equivalent result appears independently in [**9,** Theorem 3.4] using different terminology. We include our proof here as it uses a continuation of the language and terminology of [**11,** Theorem 3.2]. In the proof, we will use $\deg_x m$ to denote the degree of a variable x in a given monomial m.

Proposition 3.2. *$\mathcal{M}$ is a coordinatization of P if and only if $\mathcal{D}_{\mathcal{M}}$ satisfies* (C1) *and* (C2). *In particular, this means $M_{\mathcal{M}}$ has an* lcm *lattice isomorphic to P.*

Proof. The forward direction follows from [**11,** Proposition 3.6] since, if $\mathcal{M}$ is a coordinatization of P, by definition, $\mathcal{D}_{\mathcal{M}} = \mathcal{D}$ (i.e., $P = P_{M_{\mathcal{M}}}$).

For the reverse direction, we assume that $\mathcal{D}_{\mathcal{M}}$ satisfies (C1) and (C2), meaning that, by Theorem 2.2, $\mathcal{D}_{\mathcal{M}}$ is a coordinatization. In particular, this tells us that P is the lcm lattice of the monomial ideal

$M_{\mathcal{D}}$ generated by the labeling $\mathcal{D}_{\mathcal{M}}$. If we can show that the monomial ideal $M_{\mathcal{D}} = M_{\mathcal{M}}$, then the lcm lattice of $M_{\mathcal{M}}$ will also be P, making $\mathcal{M}$ a coordinatization. It is enough to show that the monomial generators derived from $\mathcal{M}$ agree with those obtained from $\mathcal{D}_{\mathcal{M}}$, specifically, that the exponents on each variable agree.

For clarity in the proof we emphasize that, in what follows, the monomials l_p will always be least common multiples of the generators of $M_{\mathcal{M}}$, and the monomials m_p will always be the labelings found in $\mathcal{M}_{\mathcal{D}}$.

Let x be a variable appearing in some generator of $M_{\mathcal{M}}$, and let r be the highest power of x that divides any generator. There is a subset of atoms in P whose corresponding generators in $M_{\mathcal{M}}$ have x^r as a factor; call this set $\max(x)$. Define the set A to be the set of elements in $P \geq$ the elements in $\max(x)$, and define the set B to be the complement of A in P.

For each element p in A, $\deg_x l_p$ must be r; and, for each element p in B, $\deg_x l_p$ must be strictly less than r. In order to ensure that $\deg_x x(a_i)$ is r for each a_i in $\max(x)$, where $x(a_i)$ is a generator of $M_{\mathcal{D}}$, it is enough to show that

$$\sum_{p \in B} \deg_x m_p$$

is r.

Note that A must contain the maximal element $\widehat{1}$ in P, as it is the least common multiple of all generators. Also, the minimal element $\widehat{0}$, whose least common multiple is defined to be 1, must be in B.

We know that $\widehat{1}$ is in A and $\deg_x l_p$ for all p in A. Thus, x^r divides $\gcd\{l_t \mid t > b\}$, from equation (2.2), for every element b which is the maximal element of a chain in B. If

$$\sum_{p \in B - \{\hat{0}\}} \deg_x m_p = k < r,$$

then x^{r-k} will be a factor of $m_{\hat{0}}$. In order to see this, consider equation (2.2), which shows that $m_{\hat{0}} = \gcd\{l_t \mid t > \widehat{0}\}$. This means that

$$\sum_{p \in B} \deg_x m_p = r,$$

as required. Therefore, $\deg_x x(a_i)$ is r for each a_i in $\max(x)$, where $x(a_i)$ is the corresponding generator of $M_{\mathcal{D}}$.

It remains to show that exponents on x agree for generators of $M_{\mathcal{D}}$ corresponding to atoms in B. For this purpose, we consider the subposet B of P, which is itself a poset, and iteratively apply our previous procedure. Let $s < r$ be the highest power of x derived from a generator of $M_{\mathcal{M}}$ corresponding to an atom in B. The elements in B greater than the set of atoms for which x^s divides $x(a_i)$ in $M_{\mathcal{M}}$ will be A_s, and the set of elements in $B - A_s$ will be B_s.

The monomial l_p for each p in A_s has x^s as a factor. This means that, if A_s has a unique maximal element, it will be labeled with x^{r-s} in $\mathcal{D}_{\mathcal{M}}$ using equation (2.2), and no other element p in A_s can have x as a factor of the monomial m_p in $\mathcal{D}_{\mathcal{M}}$.

As in the previous case, there will be s copies of x remaining to label elements in B_s in the construction of $\mathcal{D}_{\mathcal{M}}$. Again, since $\widehat{0}$ is in B_s, the

$$\sum_{p \in B_s} \deg_x m_p$$

will be s. Therefore, s is the $\deg_x x(a_i)$ for a_i an atom in A_s, where the monomial $x(a_i)$ is a generator for $M_{\mathcal{D}}$.

This process may be repeated for the next highest power of x appearing in a generator of $M_{\mathcal{M}}$ derived from B_s.

If, however, A_s does not contain a unique maximal element, equation (2.2) shows that there will be at least two non-comparable elements in A_s that are labeled with x^{r-s} in $\mathcal{D}_{\mathcal{M}}$. This yields that $\mathcal{D}_{\mathcal{M}}$ contains copies of x at non-comparable elements, violating (C2), which we are assuming to be true.

Applying this procedure for each variable appearing in the generators of $M_{\mathcal{M}}$ shows that $M_{\mathcal{D}}$ and $M_{\mathcal{M}}$ have the same generators, proving $\mathcal{M}$ a coordinatization of P. □

Example 3.3. For clarity, we illustrate the sets A_s for our Example 3.1 in Figure 3. Since the greatest exponent of x among the generators $x(a_i)$ for the labeling $\mathcal{M}$ is 3, we use A_3 to denote the set A from the proof, distinguishing it from the subsequent sets A_2 and A_1. In this example, $B = A_2 \cup A_1 \cup \{\widehat{0}\}$, $B_2 = A_1 \cup \{\widehat{0}\}$ and $B_1 = \{\widehat{0}\}$.

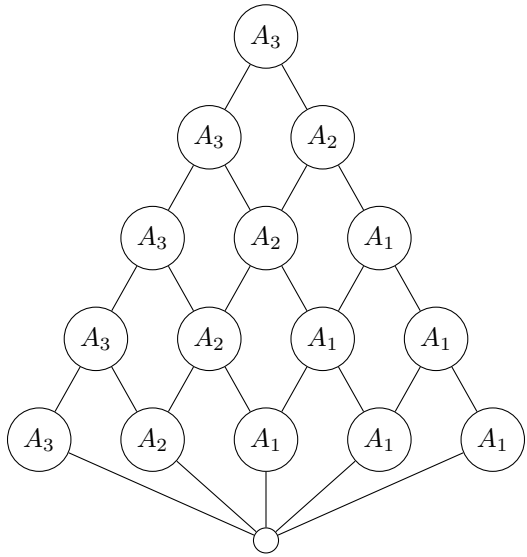

FIGURE 3. The sets A_s for variable x, as described in Example 3.3.

4. Nearly Scarf ideals and minimal monomial ideals. Coordinatizations of lattices have indirectly appeared in several other places as instances of associating monomial ideals to cell complexes which then support a minimal free resolution. One important example of coordinatizations are the "nearly Scarf" ideals introduced by Peeva and Velasco [**14, 17**]. The *nearly Scarf* construction is as follows. Given a simplical complex Δ, assign to each face σ of Δ the variable x_σ. For a fixed vertex v of Δ, let $A_\Delta(v)$ be the subcomplex of Δ contacting all the faces of Δ which do not contain the vertex v. The monomial ideal M_Δ is generated by the monomials

$$m_v = \prod_{\sigma \in A_\Delta(v)} x_\sigma$$

for each vertex v of Δ.

This construction may be seen as a coordinatization of the (augmented) face poset of the simplicial complex Δ (note that the face poset of a simplicial complex is a meet semi-lattice, and thus, with a maximal element becomes a finite atomic lattice). Since labeling ev-

ery face σ with a variable x_σ corresponds to labeling every element of the face poset of the simplicial complex with a different variable, the formula for m_v is identical to equation (2.1). Clearly, this satisfies the conditions of Proposition 2.2, since all faces are labeled with distinct variables.

Another important example of coordinatizations are Phan's "minimal monomial ideals" [**15**]. In some sense, these are the opposite of nearly Scarf ideals, as they use the smallest number of variables possible. The construction for *minimal squarefree ideals* is as follows. Given a finite atomic lattice P, let $\mathrm{mi}(P)$ denote the set of meet-irreducible elements in $P - \{\widehat{0}, \widehat{1}\}$. Then, label each element in $\mathrm{mi}(P)$ with a distinct variable. This labeling clearly satisfies conditions (C1) and (C2); thus, it is a coordinatization of P.

4.1. Resolutions supported on simplicial trees. In [**5**], Faridi addressed Scarf ideals corresponding to acyclic simplicial complexes, in particular, simplicial trees, offering an alternative Scarf ideal in fewer variables than those constructed by Peeva and Velasco. Given a simplicial complex Δ, a variable x_σ is still assigned to each face σ, but only variables for certain faces appear in the ideal. In [**5**], the monomial generators for each vertex v are defined as:

(4.1)
$$m'_v = \sqrt{\left(\prod_{\{G \in B_\Delta(v)\}} x_{G-\{v\}}\right)\left(\prod_{\{F \in A_\Delta(v)\}} x_F \left(\prod_{\{|\sigma|=|F|-1\}} x_\sigma\right)\right)}.$$

Here, $B_\Delta(v)$ and $A_\Delta(v)$ are the facets of Δ that do and do not contain v, respectively, and the square root indicates that m'_v is the square-free monomial containing all variables in the described product.

The first product indicates that, for each facet G containing v, the variable for the facet of G not containing v is to be included. The second product indicates that, for each facet F not containing v, both the variable for that face and all of the facets of F are to be included. It was established in [**5,** Proposition 4.3] that, when Δ is acyclic (a simplicial tree), the ideal generated by the monomials m'_v has a minimal free resolution supported on Δ.

In order to see that this construction gives a coordinatization, we must check that the meet-irreducibles are still labeled (since each face

obtains its own variable, we do not need to check condition (C2)) and that the products obtained agree with the equations given in equation (2.1) for the given labeling.

In the $B_\Delta(v)$ term, since G will always be a facet and we are taking products of variables corresponding to $x_{G-\{v\}}$, as we let v vary we use the labels on all of the codimension 1 faces of each facet G. Moreover, since every vertex is contained in a facet, we are using the labels on every codimension 1 face of each of the facets of Δ.

In the $A_\Delta(v)$ term, we use the label on all of the facets F which do not contain v and then all of the codimension 1 faces of that facet F.

As we let v range over all vertices of Δ, we see that, if every facet contains every vertex v, i.e., if G is in $B_\Delta(v)$ for all v, $\Delta = G$ and Δ is a simplex. Otherwise, each facet G of Δ will be in $B_\Delta(v)$ for some v and then in $A_\Delta(v')$ for some v'. Thus, if Δ is not a simplex, we must use the label on every facet of Δ. If Δ is a simplex, the only facet corresponds to the maximal element of the face poset, and its labeling is irrelevant for the coordinatization since the element is greater than every atom (which is consistent with the fact that it will not appear as a monomial in equation (4.1)).

Therefore, we can describe the labeling of the (augmented) face poset P_Δ of Δ as labeling all of the elements corresponding to facets of Δ (unless Δ is a simplex) and all of the elements corresponding to codimension 1 faces of those facets. We call this labeling $\mathcal{F}_\Delta$.

Lemma 4.1. *The labeling $\mathcal{F}_\Delta$ of P_Δ is a coordinatization.*

Proof. Note that each element, or equivalently, each face in Δ, is labeled with a distinct variable. Therefore, in $\mathcal{F}_\Delta$, condition (C1) will automatically be satisfied. It remains to show that $\mathcal{F}_\Delta$ non trivially labels all of the meet irreducible elements of P_Δ.

If Δ is a simplex, $\mathcal{F}_\Delta$ labels all coatoms of the face poset, which coincide with the meet irreducibles.

If Δ is not a simplex, each facet F of Δ is a simplex. Therefore, the interval of P below F is a Boolean lattice. The meet irreducibles of a Boolean lattice are its coatoms, which, in this sublattice, correspond to codimension 1 faces of F. For each facet F, there is at least one vertex v in Δ not contained in F. For each vertex v, the codimension 1 faces

of facets not containing v are labeled; thus, the codimension 1 faces of every facet are labeled. The only other possible meet irreducibles of the lattice are elements corresponding to the facets themselves. Again, since, for each facet F, there is at least one vertex v of Δ such that F does not contain v, and, for each vertex v, the facets not containing v are labeled, all facets are labeled. Therefore, all meet irreducibles are labeled. □

Lemma 4.2. *The monomial ideal created by the labeling $\mathcal{F}_\Delta$ equals the ideal obtained using equation* (4.1).

Proof. Since both the ideal defined by $\mathcal{F}_\Delta$ and the ideal defined by equation (4.1) are squarefree, it suffices to show that the variable x divides m'_v if and only if x divides $x(a_v)$, where a_v is the atom in P_Δ corresponding to vertex v in Δ.

Unpacking equation (4.1), we see that, if x divides m'_v, then x is the variable associated to either a facet of Δ not containing v or a codimension 1 face of any facet of Δ which does not contain v. These simplices are precisely the elements P_Δ which are not greater than a_v and are labeled via $\mathcal{F}_\Delta$. Thus, by the definition of $x(a_v)$, x divides $x(a_v)$. This equality of sets also shows that, if x divides $x(a_v)$, then x will divide m'_v. □

In **[5]**, Faridi notes that her construction yields ideals using fewer variables than in the nearly Scarf construction. It should be noted that her ideals are not in general the "smallest possible." Phan's minimal monomial ideals produce the ideals using the fewest variables **[15]**. She also considers some "in-between" ideals, those where she adds back some of the variables found in the nearly Scarf ideal to some of the generators found using (4.1). The reason she finds these "in-between" ideals have different minimal resolutions is that, adding back variables to some, but not all, generators will typically cause the lcm lattice to change. In particular, we suggest that it could be interesting to characterize which "in-between" ideals have the same minimal resolution and study the relationship on the corresponding lcm-lattices. Understanding this type of behavior from either the cellular resolution or lcm-lattice perspective may shed light on topics related to generic deformation of monomial ideals, as described in **[1, 12]**.

We believe the perspective presented here, using lcm lattices and coordinatizations, can provide insights for the questions posed towards the end of [**5**] concerning localization, removal of facets and other operations that preserve forests.

5. Maximal ideals with resolutions supported on trees. In [**6**], Fløystad defines the category of monomial ideals M in a polynomial ring S where the quotients S/M are Cohen-Macaulay, and he defines maximal elements in this category. He then gives constructions which associate maximal elements in this category to certain regular cell complexes (trees, and some polytopes), when minimal resolutions are supported on the cell complexes. As in [**15**], the focus is on constructing monomial ideals with a specific cellular resolution. We will discuss the relationship between the two works in subsection 5.1, but first we provide a summary of the main points from [**6**] that will be used.

In [**6**], the set $CM(n, c)$ is defined as the set of ordered sets of n monomials generating a monomial ideal M such that the quotient ring is Cohen-Macaulay of codimension c. This is a category but the added structure is not necessary for our work here. The set $CM_*(n, c)$ is the subset (subcategory) of $CM(n, c)$ consisting of monomial ideals which are squarefree and for which the sets

$$V_t = \{i \mid x_t \text{ divides } m_i\} \subseteq [n]$$

are distinct.

In [**6**], Fløystad initially defines what it means for a monomial ideal to be *maximal* using the maps in the category $CM(n, c)$. However, the maps in this category are heavily dependent upon the choice of coordinatization for each monomial ideal. Thus, for our work, his characterization identifying objects in $CM(n, c)$ with families $\mathcal{F}$ consisting of subsets of $[n]$ is more useful. These sets $\mathcal{F}$ correspond to the set of all sets V_t described above. In [**6**], he also gives a description of what properties a family of sets $\mathcal{F}$ must have in order to correspond to an element in $CM_*(n, c)$.

The following are presented as [**6,** Propositions 1.7, 1.10]. Since they are equivalence statements, we state them here as definitions to simplify notation.

Definition 5.1. A family of subsets of $[n]$, denoted $\mathcal{F}$, as described above, is *reduced* if it consists of elements which are not the disjoint union of other elements in $\mathcal{F}$. Such a family $\mathcal{F}$, which corresponds to an element in $CM_*(n,c)$, is *maximal* if it is reduced and is maximal among reduced families corresponding to the elements in $CM_*(n,c)$ for the refinement order. The *refinement order* states that, for two families of subsets, $\mathcal{F} > \mathcal{G}$ if and only if $\mathcal{F}$ consists of refinements of elements of $\mathcal{G}$ together with additional subsets of $[n]$.

In general, characterizing families $\mathcal{F}$ which are also in $CM_*(n,c)$ seems to be a nontrivial task. Fløystad restricts to families whose minimal resolution is supported on a specific regular CW-complex. These sets are defined as follows.

Definition 5.2. Given a regular d-dimensional cell complex X, $CM_*(X)$ is the subset of $CM_*(n,c)$ whose minimal resolution is supported on X.

A family $\mathcal{F}$ is an object in $CM_*(X)$ if the following conditions hold.

(1) No d of the subsets in $\mathcal{F}$ cover $[n]$.

(2) If W is a union of subsets $\mathcal{F}$, the restriction of X to the complement of W is acyclic.

(3) For every pair $F \subsetneq G$ of faces of X, there is a set $S \in \mathcal{F}$ such that $S \cap F$ is empty, but $S \cap G$ is not empty.

Definition 5.2 describes the method of "labeling" a regular cell complex X so that we can construct an appropriate monomial ideal whose resolution is supported on X. In particular, condition (1) shows that the corresponding ideal has codimension at least $d+1$, condition (2) guarantees that X supports a cellular resolution and condition (3) ensures this resolution is minimal.

The next lemma, **[6,** Lemma 1.13], gives a necessary condition for when a family $\mathcal{F}$ satisfying Definition 5.1 is maximal.

Lemma 5.3. *If a family of subsets $\mathcal{F}$ of $[n]$ corresponds to a maximal object in $CM_*(X)$, then, for every $S \in \mathcal{F}$, the restriction of X to S is connected.*

The families $\mathcal{F}$, used to describe ideals whose resolutions are supported on cell complexes **[6]**, can be viewed as subsets of the lcm lat-

tices of these monomial ideals. Next is a description of the connections between the constructions appearing in **[6, 15]**.

5.1. Dictionary between labeling regular cell complexes and coordinatizing lattices. We begin by addressing how to translate between the families $\mathcal{F}$ in **[6]** and the lcm lattice associated to the ideal they represent. First, consider the sets $V_t = \{i \mid x_t \text{ divides } m_i\}$. Let M be the squarefree monomial ideal in $CM(n, c)$ corresponding to a family $\mathcal{F} = \{V_1, \ldots, V_s\}$. For each variable x_t, there is a point in the deficit labeling $p \in P = \mathrm{LCM}(M)$ such that p is labeled with the variable x_t. Thus, by the construction in **[15]**, x_t will precisely divide the monomials that correspond to

$$V_t = \lfloor p \rfloor^c \cap \mathrm{atoms}(P),$$

where $\lfloor p \rfloor^c$ is the complement of the set of elements in P which are less than or equal to p. Reversing this, we can determine which element p must be labeled with x_t by taking the join of the elements in the complement of V_t among atoms(P). By definition, this labeling should yield the original ideal M.

5.2. Codimension 2 Cohen-Macaulay monomial ideals. The Auslander-Buchsbaum formula makes it clear that the projective dimension of Cohen-Macaulay monomial ideals of codimension 2 must be two.

In terms of cellular resolutions, this implies their resolutions are supported on trees. For this special case, **[6]** gives a very specific construction which associates a maximal monomial ideal in $CM(T)$ to every tree T using any given orientations of the edges of T.

First, let us establish the notation that a tree $T = (V, E)$ is a pair of sets: a set of vertices, denoted V, and a set of edges, denoted E. The construction assigns to each vertex in V a monomial in the following way. Given an edge $e_i \in E$, deleting the edge e_i produces two connected components of T, $T_{i,1}$ and $T_{i,2}$. The monomial associated to each vertex $v \in T$ is

$$m_v = \left(\prod_{\{i | v \in T_{i,1}\}} x_i \right) \left(\prod_{\{i | v \in T_{i,2}\}} y_i \right).$$

The squarefree monomial ideal $M_T = (m_{v_1}, \ldots, m_{v_{n+1}})$ is maximal Cohen-Macaulay, and its minimal resolution is supported on T.

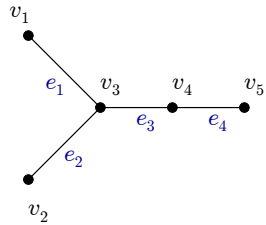

FIGURE 4. The tree T described in Example 5.4.

Example 5.4. For each edge e_i between two vertices v_j and v_k, assume that $j < k$. Let $T_{i,1}$ be the component of T containing vertex j and $T_{i,2}$ the component of T containing vertex k. With this convention, the ideal obtained via the construction in **[6]** from the tree in Figure 4 is:

$$M_T = (x_1y_2x_3x_4, y_1x_2x_3x_4, y_1y_2x_3x_4, y_1y_2y_3x_4, y_1y_2y_3y_4)$$

in the ring $k[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4]$.

We will show that, for an appropriate choice of finite atomic lattice P, this construction coincides with the minimal squarefree coordinatization in **[15]**. Given a tree T with n edges and $n+1$ vertices, we define P_T to be the set of all subtrees of T ordered by inclusion (note that we include vertices and the empty set as subtrees).

Lemma 5.5. *A poset P_T defined as above is an element of $\mathcal{L}(n+1)$, where $\mathcal{L}(n)$ is the set of all finite atomic lattices with n ordered atoms.*

Proof. We will show that P_T is a meet-semilattice with a maximal element making it a finite lattice by **[16,** Proposition 3.3.1]. In order to show that P_T is a meet-semilattice, we must show that, for every pair of elements $a, b \in P_T$, there exists a meet or least upper bound. Since a and b are subtrees of T, define $a \wedge b$ to be the intersection of a and b. Since $\emptyset \in P_T$, $a \cap b$ will be a subtree of T; thus, P_T is a meet-semilattice.

It remains to show that P_T is atomic with $n+1$ atoms. This follows from the fact that T has $n+1$ vertices and that every subtree can be realized as an induced graph on a subset of the vertices. □

The next lemma demonstrates that any coordinatization of the lattice P_T will yield a monomial ideal in $\mathrm{Mon}(T)$, which is the set of monomial ideals whose resolution is supported on T.

Lemma 5.6. *The minimal resolution of any coordinatization of P_T is supported on T.*

Proof. Since P_T will be the lcm lattice for any coordinatization of P_T, by **[2,** Proposition 1.2], it is enough to show that $T_{\leqslant p}$ is acyclic for each $p \in P_T$.

Each $p \in P_T$ corresponds to a subtree of P_T ordered by inclusion; thus, by construction, $T_{\leqslant p}$ is the subtree corresponding to p. Since they are themselves trees, each subtree is acyclic; therefore, T supports the minimal resolution of any coordinatization of P_T. □

Finally, we show that minimal squarefree coordinatization of P_T in **[15]** always agrees with the ideals constructed in **[6]**.

Theorem 5.7. *If $\mathcal{M}$ is the minimal squarefree coordinatization of P_T, then $M_{\mathcal{M}} \cong M_T$.*

Proof. Recall that, in the construction of M_T, we assigned a variable to each subtree $T_{i,1}$ or $T_{i,2}$ of T obtained by deleting an edge e_i of T. Hence, each component was assigned a variable, and each vertex v was assigned a monomial (the product of the variables corresponding to the trees $T_{i,j}$ containing v).

We must show that the trees $T_{i,j}$ obtained by deleting edges are precisely the meet-irreducibles of P_T and explain how to coordinatize P_T to obtain M_T.

Clearly, the meet-irreducibles of P_T will be the subtrees of $T' \subset T$ that have only one subtree $T'' \subset T$ containing them, which satisfy

$$|\{e_i \in T'\}| + 1 = |\{e_i \in T''\}|, \tag{5.1}$$

where the e_i are the edges of a tree.

If T' is obtained as above by deleting an edge e_i, i.e., $T' = T_{i,1}$, then the only subtree T'' satisfying equation (5.1) is

$$T'' = T' \cup e_i,$$

i.e., T'' is obtained by adding edge e_i to T'. Since the only other edges which could be added are in the other connected component, to add an edge, we would be forced to add e_i as well, which would violate equation (5.1). Therefore, the meet-irreducibles are precisely the subtrees $T_{i,j}$ obtained by deleting edge e_i.

As stated above, we want to use a minimal squarefree coordinatization of P_T. If we carefully place variables, it will be clear that

$$M_{P_T,\mathcal{M}} \cong M_T.$$

Recall that, for M_T, the variables x_i were assigned to the trees $T_{i,1}$ and y_is to the trees $T_{i,2}$. Moreover, note that, if $v \in T_{i,1}$, it is necessarily not in $T_{i,2}$, and conversely. Thus, the trees $T_{i,1}$ and $T_{i,2}$ partition the vertices into two disjoint sets. The monomial label for the construction of M_T assigns to each vertex the product of the variables corresponding to the subtrees containing v. In lattice language, the subtrees containing v will be in $\lceil a_v \rceil$, where a_v is the atom corresponding to the vertex v. For our coordinatization construction, this is not what we want since we take the product over the complement of the filter. However, the complement of the filter consists precisely of the subtrees not containing v; thus, we can proclaim the following coordinatization.

Let $\mathcal{M}$ label P_T as follows. If $p \in P_T$ is a meet-irreducible corresponding to a $T_{i,1}$, denote it as p_{i1}, and label it with y_i. Similarly, if p corresponds to a $T_{i,2}$, denote it as p_{i2}, and label it with x_i. This yields:

$$\begin{aligned} x(a_v) &= \prod_{p \in \lceil a_v \rceil^c} m_p \\ &= \left(\prod_{p_{i1} \in \lceil a_v \rceil^c} y_i \right) \left(\prod_{p_{i2} \in \lceil a_v \rceil^c} x_i \right) \\ &= \left(\prod_{\{i \mid v \in T_{i,2}\}} y_i \right) \left(\prod_{\{i \mid v \in T_{i,1}\}} x_i \right) = m_v. \end{aligned}$$ □

Example 5.8. Figure 5 depicts the lattice P_T and the minimal squarefree coordinatization used in the proof of Theorem 5.7 for the tree in Figure 4, as described in Example 5.4.

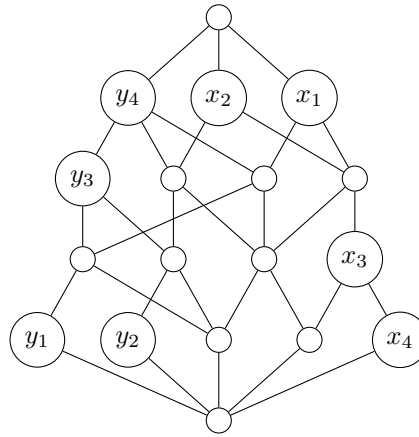

FIGURE 5. P_T with the minimal squarefree labeling outlined in Example 5.8.

5.3. Connection to Betti strata. The sets $\mathcal{L}(n)$, introduced in Lemma 5.5, have a rich structure [**11**]. Here, we will highlight the important features necessary for our discussion. The most important is that $\mathcal{L}(n)$ is itself a finite atomic lattice (as shown in [**15,** Theorem 4.2]) under the partial order given by $P > Q$, if there is a join preserving map which is a bijection on atoms from P to Q. Covering relations in $\mathcal{L}(n)$ may be described as: P covers Q if, as sets, $P = Q \cup \{p\}$, where p is a new element introduced to Q with appropriate relations [**11,** Proposition 4.2]. Since each element in $\mathcal{L}(n)$ is a finite atomic lattice, and therefore can be associated to a monomial ideal, we will discuss the Betti numbers of these lattices as those of the associated monomial ideals. Theorem 3.3 of [**8**] guarantees that $\mathcal{L}(n)$ is stratified by total Betti numbers, which prompts the next definition.

Definition 5.9. Given a vector $\beta \in \mathbb{N}^{n+1}$, we call the subset $\mathcal{L}(n)_\beta$ of $\mathcal{L}(n)$, where all of the finite atomic lattices have total Betti numbers equal to β, the *Betti stratum* of $\mathcal{L}(n)$ associated to β.

Understanding the boundaries of these Betti strata in $\mathcal{L}(n)$ can provide insights for how to move from one monomial ideal M to another, whose minimal resolution is simple to determine, in a manner that produces a minimal resolution for M.

The next proposition is a special case of Conjecture 5.11 in the case of trees.

Proposition 5.10. *If P is a lattice in $\mathcal{L}(n+1)$ satisfying $P > P_T$, then the total Betti numbers of P are greater than that of P_T, i.e., P is in a different Betti stratum than P_T.*

Before providing the proof, we introduce the following useful formulae from [**8**]. One can compute the "multigraded" Betti numbers for monomial ideals (or equivalently finite atomic lattices) using intervals in the corresponding lcm lattice P. Since the multidegree of a monomial will always correspond to an element in the finite atomic lattice, we abuse notation and say that the *multidegree* is an element p in the lattice P. Thus, the computations for graded and total Betti numbers are, respectively, as follows:

$$b_{i,p} = \widetilde{H}_{i-2}(\Gamma(\widehat{0}, p), k)$$

and

$$b_i = \sum_{p \in P} \widetilde{H}_{i-2}(\Gamma(\widehat{0}, p), k).$$

Proof. We only need consider the lattices P in $\mathcal{L}(n+1)$ which cover P_T. If we can show that, for each of these lattices the total Betti numbers are greater than that of P_T, we are finished. We know that these lattices P differ from the lattice P_T by only one element, denoted p; thus, we need only consider how that one element affects the Betti number computations.

First, we note that, since T supports the minimal free resolution of any monomial ideal with P_T as the lcm lattice, the only elements in P_T for which $b_{i,p}$ are nonzero are the atoms and the elements covering the atoms.

Now, in P, which covers P_T, we know that all of the elements q from P_T where $b_{i,q}$ were nonzero will continue to be nonzero since they correspond to the face poset of T which is simplicial; thus, they are undisturbed by the addition of the element p. This means that the total Betti numbers of P are at least the total Betti numbers of P_T.

An obvious candidate for where a new nonzero Betti number might exist is the element p which has been added to P_T to create P.

Consider $\widetilde{H}_0(\Gamma(\widehat{0}, p), k)$. If this is zero, the order complex of the interval $(\widehat{0}, p)$ in P is contractible. However, since p is the element we added to P_T, all of the elements in $(\widehat{0}, p)$ correspond to subtrees of T. Therefore, $\Gamma(\widehat{0}, p)$ will be the union of the subtrees of T corresponding to the elements covered by p. If this is contractible, it should also be a subtree of T, and p would already be an element of P_T. Thus, $\Gamma(\widehat{0}, p)$ is not contractible, and $\widetilde{H}_0(\Gamma(\widehat{0}, p), k)$ is nonzero. This shows that the total Betti numbers, namely b_2, of P are greater than those of P_T. □

We conjecture that this should be true more generally. Moreover, if the next conjecture is true, it offers an alternate (and perhaps more useful) description of elements on the boundary of these Betti strata.

Conjecture 5.11. *Let X be a regular cell complex. The* lcm *lattice P of a maximal monomial ideal $M \in CM_*(X)$ satisfies the property that, if $Q > P$ in $\mathcal{L}(n)$, then the minimal resolution of Q has total Betti numbers greater than that of P. In other words, P is maximal in its Betti stratum.*

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