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ON A WARING-GOLDBACH PROBLEM FOR MIXED POWERS

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ABSTRACT. Let P_r denote an almost-prime with at most r prime factors, counted according to multiplicity. In this paper, it is proved, among other results, that, for every sufficiently large, even integer N satisfying the congruence condition $N \not\equiv 2 \pmod 3$, the equation

$$N = x^2 + p^2 + p_1^3 + p_2^4 + p_3^4 + p_4^4$$

is solvable with x being a P_5 and the other variable primes. This result constitutes an enhancement upon that of Vaughan **[10]** and Mu **[7]**.

1. Introduction. The Waring problem of mixed type concerns the representation of a natural number N as the form

$$N = x_1^{k_1} + \cdots + x_s^{k_s}, \quad k_1 \leq \cdots \leq k_s. \tag{1.1}$$

Little is known about results of this type. For references, we refer the reader to the bibliography in [13].

In principle, the Hardy-Littlewood method is applicable to problems of this type, but various difficulties not experienced in the pure Waring problem (1.1) with $k_1 = \cdots = k_s$ must be overcome. In particular, the choice of relevant parameters in the definitions of major and minor arcs tends to become complicated if a deeper representation problem (1.1) is under consideration.

Vaughan **[9, 10]** obtained the asymptotic formula for the number of representations of the equation

$$N = x_1^2 + x_2^2 + y_1^3 + y_2^4 + y_3^4 + y_4^4.$$

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Afterwards, motivated by **[2, 3, 4]**, Mu **[7]** proved that, for every sufficiently large, even integer N satisfying specific congruence conditions, the equation

$$N = x^2 + p^2 + p_1^3 + p_2^4 + p_3^4 + p_4^k, \quad k = 4, 5,$$

is solvable with x an almost-prime P_r and the other variables primes, where $r = 6$ for $k = 4$ and $r = 9$ for $k = 5$.

In this paper, P_r denotes an almost-prime with at most r prime factors, counted according to multiplicity. We obtain the next refinements of the result of Mu **[7]**.

Theorem 1.1. *For every sufficiently large, even integer N with $N \not\equiv 2 \pmod 3$, the number of solutions of the equation*

$$N = x^2 + p^2 + p_1^3 + p_2^4 + p_3^4 + p_4^4, \tag{1.2}$$

with x a P_5 and the other variables primes, is

$$\gg N^{13/12} \log^{-6} N.$$

Theorem 1.2. *For every sufficiently large, even integer N, the number of solutions of the equation*

$$N = x^2 + p^2 + p_1^3 + p_2^4 + p_3^4 + p_4^5, \tag{1.3}$$

with x a P_8 and the other variables primes, is

$$\gg N^{31/30} \log^{-6} N.$$

In this paper, we present a detailed proof of Theorem 1.1 only. By $Q_1 = N^{7/15+6\varepsilon}$, $Q_2 = N^{1/2}$ and $D = N^{1/40-7\varepsilon}$, Theorem 1.2 can be proved using a similar argument.

2. Notation and some preliminary lemmas. In this paper, $\varepsilon \in (0, 10^{-10})$ and N denotes a sufficiently large, even integer in terms of ε. The constants in O-term and $\ll$-symbol depend at most on ε. By $A \asymp B$, we mean that $A \ll B$ and $B \ll A$. The letter p, with or without subscript, is reserved for a prime number. We denote by (m, n) the greatest common divisor of m and n. As usual, $\varphi(n)$ and $\mu(n)$ denote Euler's function and the Mőbius function, respectively. By $\tau(n)$, we denote the divisor function, and, by $a(n)$, we denote an arithmetical function bounded above by $\tau(n)$. We use $e(\alpha)$ to denote $e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. We denote, by $\sum_{x(q)}$ and $\sum_{x(q)*}$, sums with x running over a complete system and a reduced system of residues modulo q, respectively. We always denote by χ a Dirichlet character $\pmod q$ and by χ^0 the principal Dirichlet character $\pmod q$. By $\sum_{\chi(q)}$, we denote a sum with χ running over the Dirichlet characters $\pmod q$. Let $A = 10^{10}$, $Q_0 = \log^{20A} N$, $Q_1 = N^{5/12+6\varepsilon}$, $Q_2 = N^{1/2}$, $D = N^{1/16-7\varepsilon}$, $z = D^{1/3}$, $U_k = 0.5N^{1/k}$,

$$\mathcal{M}_r = \{m \mid U_2 < m \le 2U_2,\ m = p_1 \cdots p_r,\ z \le p_1 \le \cdots \le p_r\},$$

$$\mathcal{N}_r = \{n \mid n = p_1 \cdots p_{r-1},\ z \le p_1 \le \cdots \le p_{r-1},\ p_1 \cdots p_{r-2}p_{r-1}^2 \le 2U_2\},$$

$$\begin{aligned}
G_k(\chi, a) &= \sum_{r(q)} \chi(r) e_q(ar^k),\\
S_k^*(q, a) &= G_k(\chi^0, a), \qquad S_k(q, a) = \sum_{r(q)} e_q(ar^k),\\
B_d(q, N) &= \sum_{a(q)*} S_2(q, ad^2) S_2^*(q, a) S_3^*(q, a) S_4^{*3}(q, a) e_q(-aN),\\
A_d(q, N) &= \frac{B_d(q, N)}{q\varphi^5(q)}, \qquad A(q, N) = A_1(q, N),\\
\mathfrak{S}_d(N) &= \sum_{q=1}^{\infty} A_d(q, N), \qquad \mathfrak{S}(N) = \mathfrak{S}_1(N),\\
f_k(\alpha) &= \sum_{U_k < p \le 2U_k} (\log p) e(\alpha p^k),\\
g_r(\alpha) &= \sum_{\substack{n \in \mathcal{N}_r \\ U_2 < np \le 2U_2}} e(\alpha (np)^2) \frac{\log p}{\log(U_2/n)},\\
F_k(\alpha) &= \sum_{U_k < n \le 2U_k} e(\alpha n^k),\\
u_k(\lambda) &= \int_{U_k}^{2U_k} e(\lambda u^k)\, du,\\
\mathfrak{I}(N) &= \int_{-\infty}^{\infty} u_2^2(\lambda) u_3(\lambda) u_4^3(\lambda) e(-\lambda N)\, d\lambda.
\end{aligned}$$

Lemma 2.1 ([**1**])**.** *Let $2 \le k_1 \le k_2 \le \cdots \le k_s$ be natural numbers satisfying*

$$\sum_{i=j+1}^{s} \frac{1}{k_i} \le \frac{1}{k_j}, \quad 1 \le j \le s-1.$$

Then, we have

$$\int_0^1 \left| \prod_{i=1}^{s} F_{k_i}(\alpha) \right|^2 d\alpha \le N^{1/k_1+\cdots+1/k_s+\varepsilon}.$$

Lemma 2.2. *We have*

(i) $\displaystyle\int_0^1 |F_2(\alpha)F_3(\alpha)F_4^2(\alpha)|^2 d\alpha \ll N^{5/3},$

(ii) $\displaystyle\int_0^1 |f_2(\alpha)f_3(\alpha)f_4^2(\alpha)|^2 d\alpha \ll N^{5/3}\log^8 N.$

Proof. This is [**7,** Lemma 3]. □

Lemma 2.3. *For $\alpha = (a/q) + \beta$, let*

$$W(\alpha) = \sum_{d \le D} \frac{a(d)}{dq} S_2(q, ad^2) u_2(\beta), \tag{2.1}$$

$$\Delta_k(\alpha) = f_k(\alpha) - \frac{S_k^*(q,a)}{\varphi(q)} \sum_{U_k < n \le 2U_k} e(\beta n^k), \tag{2.2}$$

$$\mathcal{I}(q,a) = \left(\frac{a}{q} - \frac{1}{qQ_0}, \frac{a}{q} + \frac{1}{qQ_0} \right]. \tag{2.3}$$

Then, we have

$$\sum_{q \le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathcal{I}(q,a)} |W^2(\alpha)\Delta_k^2(\alpha)|\, d\alpha \ll N^{2/k} \log^{-100A} N.$$

Proof. This is [**7,** Lemma 4]. □

Lemma 2.4. *For $\alpha = (a/q) + \beta \in \mathcal{I}(q,a)$, let*

$$U_k(\alpha) = \frac{S_k^*(q,a)}{\varphi(q)} u_k(\beta). \tag{2.4}$$

Then, we have

$$\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)}|U_k(\alpha)|^2d\alpha \ll N^{2/k-1}\log^{21A}N, \tag{2.5}$$

$$\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)}|W(\alpha)|^2d\alpha \ll \log^{21A}N, \tag{2.6}$$

where $W(\alpha)$ *and* $\mathcal{I}(q,a)$ *are defined by* (2.1) *and* (2.3), *respectively.*

Proof. This is [**7,** Lemma 5]. □

For $(a,q)=1$, $1\le a\le q$, set

$$\mathfrak{M}_0(q,a)=\left(\frac{a}{q}-\frac{Q_0}{N},\frac{a}{q}+\frac{Q_0}{N}\right], \qquad \mathfrak{M}_0=\bigcup_{1\le q\le Q_0^5}\bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathfrak{M}_0(q,a),$$

$$\mathfrak{M}(q,a)=\left(\frac{a}{q}-\frac{1}{qQ_2},\frac{a}{q}+\frac{1}{qQ_2}\right], \quad \mathfrak{M}=\bigcup_{1\le q\le Q_0^5}\bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathfrak{M}(q,a),$$

$$\mathfrak{M}_1(q,a)=\left(\frac{a}{q}-\frac{1}{qN^{7/12-6\varepsilon}},\frac{a}{q}+\frac{1}{qN^{7/12-6\varepsilon}}\right],$$

$$\mathfrak{m}_1=\bigcup_{Q_0^5<q\le Q_1}\bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathfrak{M}_1(q,a), \quad \mathfrak{m}=\bigcup_{Q_0^5<q\le Q_1}\bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathfrak{M}(q,a),$$

$$\mathfrak{J}_0=\left(\frac{1}{Q_2},1+\frac{1}{Q_2}\right],$$

$$\mathfrak{m}_0=\mathfrak{M}\setminus\mathfrak{M}_0, \qquad \mathfrak{m}_2=\mathfrak{m}\setminus\mathfrak{m}_1, \qquad \mathfrak{m}_3=\mathfrak{J}_0\setminus\left(\mathfrak{M}\bigcup\mathfrak{m}\right).$$

Then, we have the Farey dissection

$$\mathfrak{J}_0=\mathfrak{M}_0\bigcup\mathfrak{m}_0\bigcup\mathfrak{m}_1\bigcup\mathfrak{m}_2\bigcup\mathfrak{m}_3. \tag{2.7}$$

Lemma 2.5. *For* $\alpha=(a/q)+\beta\in\mathfrak{M}_0$, *we have*

(i) $f_k(\alpha)=U_k(\alpha)+O(U_k\exp(-\log^{1/3}N))$,
(ii) $g_r(\alpha)=(c_rU_2(\alpha))/(\log U_2)+O(U_2\exp(-\log^{1/3}N))$,

where $U_k(\alpha)$ *is defined by* (2.4), *and*

$$c_r = (1+O(\varepsilon))\int_{r-1}^{23}\frac{dt_1}{t_1}\int_{r-2}^{t_1-1}\frac{dt_2}{t_2}\cdots\int_{3}^{t_{r-4}-1}\frac{dt_{r-3}}{t_{r-3}}\int_{2}^{t_{r-3}-1}\frac{\log(t_{r-2}-1)\,dt_{r-2}}{t_{r-2}}.$$

Proof. This is [**7,** Lemma 6]. □

Lemma 2.6. *Let*

$$h(\alpha) = \sum_{\substack{m\le D^{2/3}\\ n\le D^{1/3}}} u(m)v(n)\sum_{U_2/mn<l\le 2U_2/mn} e(\alpha(mnl)^2),$$

where $|u(m)|\le 1, |v(n)|\le 1$. *Then, for* $\alpha = (a/q)+\beta$, $(a,q)=1$, $q\le N^{1/2}$, $|\beta|\le 1/qN^{1/2}$, *we have*

$$h(\alpha) \ll \frac{N^{1/2+\varepsilon}}{q^{1/2}(1+N|\beta|)^{1/2}} + N^{(1/4)+\varepsilon}D^{2/3}.$$

Proof. This is [**4,** (4.6)]. □

3. Mean value theorems. In this section, we prove two mean value theorems for the proof of Theorem 1.1.

Proposition 3.1. *Let*

$$J_d(N) = \sum_{\substack{(dl)^2+p^2+p_1^3+p_2^4+p_3^4+p_4^4=N\\ U_2<dl,\, p\le 2U_2\\ U_3<p_1\le 2U_3\\ U_4<p_2,\, p_3,\, p_4\le 2U_4}} (\log p)(\log p_1)\cdots(\log p_4).$$

Then, for $|u(m)|\le 1$, $|v(n)|\le 1$, *we have*

$$\sum_{\substack{m\le D^{2/3}\\ n\le D^{1/3}}} u(m)v(n)\left(J_{mn}(N) - \frac{\mathfrak{S}_{mn}(N)}{mn}\,\mathfrak{J}(N)\right) \ll N^{13/12}\log^{-A} N.$$

Proof. Let

$$K(\alpha) = h(\alpha) f_2(\alpha) f_3(\alpha) f_4^3(\alpha) e(-\alpha N).$$

Then, by Farey dissection (2.7), we have

$$\sum_{\substack{m \le D^{2/3} \\ n \le D^{1/3}}} u(m) v(n) J_{mn}(N) = \int_{\mathfrak{I}_0} K(\alpha)\, d\alpha = \left(\int_{\mathfrak{M}_0} + \int_{\mathfrak{m}_0} + \int_{\mathfrak{m}_1} + \int_{\mathfrak{m}_2} + \int_{\mathfrak{m}_3} \right) K(\alpha)\, d\alpha. \tag{3.1}$$

From Schwartz's inequality and Lemma 2.1, we obtain

$$\int_0^1 |f_2(\alpha) f_3(\alpha) f_4^3(\alpha)|\, d\alpha \ll \left(\int_0^1 |f_2(\alpha) f_4^2(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_0^1 |f_3(\alpha) f_4(\alpha)|^2 d\alpha \right)^{1/2} \ll N^{(19/24)+\varepsilon}. \tag{3.2}$$

By (3.2), we obtain

$$\int_{\mathfrak{m}_3} K(\alpha)\, d\alpha \ll \max_{\alpha \in \mathfrak{m}_3} |h(\alpha)| \left(\int_0^1 |f_2(\alpha) f_3(\alpha) f_4^3(\alpha)|\, d\alpha \right) \ll N^{13/12-\varepsilon}, \tag{3.3}$$

where the bound $h(\alpha) \ll N^{7/24-2\varepsilon}$ for $\alpha \in \mathfrak{m}_3$ is used, which follows from Lemma 2.6.

Similarly, applying Lemma 2.6 and (3.2) again, we obtain

$$\int_{\mathfrak{m}_2} K(\alpha)\, d\alpha \ll N^{13/12-\varepsilon}. \tag{3.4}$$

Write

$$a(d) = \sum_{\substack{m \le D^{2/3} \\ n \le D^{1/3} \\ mn = d}} u(m) v(n), \qquad h(\alpha) = \sum_{d \le D} a(d) \sum_{U_2/d \le l \le 2U_2/d} e(\alpha (dl)^2).$$

Then, from [**11,** Theorem 4.1], for $\alpha \in \mathfrak{m}_1$, we obtain

$$h(\alpha) = W(\alpha) + O(DN^{5/24+4\varepsilon}) = W(\alpha) + O(N^{13/48-2\varepsilon}), \tag{3.5}$$

where $W(\alpha)$ is defined by (2.1). Let

$$K_1(\alpha) = W(\alpha) f_2(\alpha) f_3(\alpha) f_4^3(\alpha) e(-\alpha N).$$

Then, by (3.2) and (3.5), we have

$$\int_{\mathfrak{m}_1} K(\alpha)\, d\alpha = \int_{\mathfrak{m}_1} K_1(\alpha)\, d\alpha + O(N^{13/12-\varepsilon}). \tag{3.6}$$

Let

$$\begin{aligned} \mathcal{I}_0(q,a) &= \left(\frac{a}{q} - \frac{1}{N^{37/48}}, \frac{a}{q} + \frac{1}{N^{37/48}} \right], \\ \mathcal{I}_1(q,a) &= \mathcal{I}(q,a) \setminus \mathcal{I}_0(q,a), \end{aligned}$$

where $\mathcal{I}(q,a)$ is defined by (2.3). Then, we have

$$\begin{aligned} \int_{\mathfrak{m}_1} K_1(\alpha)\, d\alpha \le & \sum_{q \le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_0(q,a)} \mid K_1(\alpha) \mid d\alpha \\ & + \sum_{q \le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_1(q,a)} \mid K_1(\alpha) \mid d\alpha. \end{aligned} \tag{3.7}$$

From (3.2), we have

$$\begin{aligned} & \sum_{q \le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_1(q,a)} |K_1(\alpha)|\, d\alpha \\ & \quad \ll N^{(7/24)-2\varepsilon} \int_0^1 |f_2(\alpha) f_3(\alpha) f_4^3(\alpha)|\, d\alpha \ll N^{(13/12)-\varepsilon}, \end{aligned} \tag{3.8}$$

where the bound $W(\alpha) \ll N^{(7/24)-2\varepsilon}$ for $\alpha \in \mathcal{I}_1(q,a)$ is used, which follows from [**7,** (3.6), (3.7)].

By [**8,** Lemma 4.8], we obtain

(3.9)

$$\begin{aligned} \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_0(q,a)} |K_1(\alpha)|\, d\alpha = & \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_0(q,a)} |W(\alpha) U_4(\alpha) f_2(\alpha) f_3(\alpha) f_4^2(\alpha)|\, d\alpha \\ & + \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_0(q,a)} |W(\alpha) \Delta_4(\alpha) f_2(\alpha) f_3(\alpha) f_4^2(\alpha)|\, d\alpha \end{aligned}$$

$$+O\Big(\int_{\mathfrak{m}_1\bigcap\mathcal{I}_0(q,a)}|W(\alpha)f_2(\alpha)f_3(\alpha)f_4^2(\alpha)|\,d\alpha\Big),$$

where $\Delta_4(\alpha)$ and $U_4(\alpha)$ are defined by (2.2) and (2.4), respectively.

From Schwartz's inequality and Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned}
&\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_1\bigcap\mathcal{I}_0(q,a)}|W(\alpha)\Delta_4(\alpha)f_2(\alpha)f_3(\alpha)f_4^2(\alpha)|\,d\alpha\\
&\ll\Big(\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)}|W(\alpha)\Delta_4(\alpha)|^2d\alpha\Big)^{1/2}\\
&\quad\times\Big(\int_0^1|f_2(\alpha)f_3(\alpha)f_4^2(\alpha)|^2d\alpha\Big)^{1/2}\ll N^{13/12}\log^{-10A}N.
\end{aligned}\tag{3.10}$$

It follows from Schwartz's inequality and Lemmas 2.2 and 2.4 (i) that

$$\begin{aligned}
&\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_1\bigcap\mathcal{I}_0(q,a)}|W(\alpha)U_4(\alpha)f_2(\alpha)f_3(\alpha)f_4^2(\alpha)|\,d\alpha\\
&\ll N^{1/2}\log^{-49A}N\Big(\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}_0(q,a)}|U_4(\alpha)|^2d\alpha\Big)^{1/2}\\
&\quad\times\Big(\int_0^1|f_2(\alpha)f_3(\alpha)f_4^2(\alpha)|^2d\alpha\Big)^{1/2}\ll N^{13/12}\log^{-10A}N,
\end{aligned}\tag{3.11}$$

where the bound $W(\alpha)\ll N^{1/2}\log^{-49A}N$ is used for $\alpha\in\mathfrak{m}_1$, which follows from [**7,** (3.6), (3.7)].

By Schwartz's inequality and Lemmas 2.2 and 2.4 (ii), we have

$$\begin{aligned}
&\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_1\bigcap\mathcal{I}_0(q,a)}|W(\alpha)f_2(\alpha)f_3(\alpha)f_4^2(\alpha)|\,d\alpha\\
&\ll\Big(\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}_0(q,a)}|W(\alpha)|^2d\alpha\Big)^{1/2}
\end{aligned}\tag{3.12}$$

$$\times \left(\int_0^1 |f_2(\alpha) f_3(\alpha) f_4^2(\alpha)|^2 d\alpha \right)^{1/2} \ll N^{13/12} \log^{-10A} N.$$

It follows from (3.9)–(3.12) that

$$\sum_{q \le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_0(q,a)} K_1(\alpha)\, d\alpha \ll N^{13/12} \log^{-10A} N. \tag{3.13}$$

From (3.6)–(3.8) and (3.13), we obtain

$$\int_{\mathfrak{m}_1} K(\alpha)\, d\alpha \ll N^{13/12} \log^{-10A} N. \tag{3.14}$$

By arguments similar to, but simpler than, those leading to (3.14), we obtain

$$\int_{\mathfrak{m}_0} K(\alpha)\, d\alpha \ll N^{13/12} \log^{-10A} N. \tag{3.15}$$

For $\alpha \in \mathfrak{M}_0$, let

$$K_0(\alpha) = W(\alpha) U_2(\alpha) U_3(\alpha) U_4^3(\alpha) e(-\alpha N). \tag{3.16}$$

Then, it follows from Lemma 2.5 and (3.5), which holds for $\alpha \in \mathfrak{M}_0$, that we have

$$K(\alpha) - K_0(\alpha) \ll N^{25/12} \exp(-\log^{1/4} N). \tag{3.17}$$

By (3.17), we obtain

$$\int_{\mathfrak{M}_0} K(\alpha)\, d\alpha = \int_{\mathfrak{M}_0} K_0(\alpha)\, d\alpha + O(N^{13/12} \log^{-A} N). \tag{3.18}$$

Now, the well-known standard endgame technique in the Hardy-Littlewood method establishes that

$$\int_{\mathfrak{M}_0} K_0(\alpha)\, d\alpha = \sum_{\substack{m \le D^{2/3} \\ n \le D^{1/3}}} u(m) v(n) \frac{\mathfrak{S}_{mn}(N)}{mn} \mathfrak{J}(N) + O(N^{13/12} \log^{-A} N), \tag{3.19}$$

$$\mathfrak{J}(N) \asymp N^{13/12}. \tag{3.20}$$

Now, upon combining (3.1), (3.3), (3.4), (3.14), (3.15), (3.18) and (3.19), the proof of Proposition 3.1 is complete. □

By the same method, we have:

Proposition 3.2. *For* $6 \le r \le 23$, *let*

$$J_d^{(r)}(N) = \sum_{\substack{(dl)^2+(np)^2+p_1^3+p_2^4+p_3^4+p_4^4=N \\ U_2<dl,\, np\le 2U_2 \\ U_3<p_1\le 2U_3,\, n\in\mathcal{N}_r \\ U_4<p_2,p_3,p_4\le 2U_4}} (\log p_1)\cdots(\log p_4)\left(\frac{\log p}{\log(U_2/n)}\right).$$

Then, for $|u(m)| \le 1$, $|v(n)| \le 1$, *we have*

$$\sum_{\substack{m\le D^{2/3} \\ n\le D^{1/3}}} u(m)v(n)\left(J_{mn}^{(r)}(N) - c_r\frac{\mathfrak{S}_{mn}(N)}{mn\log U_2}\,\mathfrak{I}(N)\right) \ll N^{13/12}\log^{-A} N,$$

where c_r *is defined in Lemma* 2.5.

4. On the function $\omega(d)$. In this section, we investigate the function $\omega(d)$ which is defined in (4.1) and is required in the proof of Theorem 1.1.

Lemma 4.1. *The series* $\mathfrak{S}(N)$ *is convergent, and* $\mathfrak{S}(N) > 0$.

Proof. This is [**7,** Lemma 9]. □

In view of Lemma 4.1, for square-free natural number d, we define

$$\omega(d) = \frac{\mathfrak{S}_d(N)}{\mathfrak{S}(N)}. \tag{4.1}$$

Lemma 4.2. *For every sufficiently large, even integer* N *with* $N \not\equiv 2 \pmod 3$, *the function* $\omega(d)$ *is multiplicative, and*

$$0 \le \omega(p) < p, \quad \omega(p) = 1 + O(p^{-1}),$$

for each prime p.

Proof. This is [**7,** Lemma 10]. □

5. Proof of Theorem 1.1. In this section, $f(s)$ and $F(s)$ denote the classical functions in linear sieve theory, and $\gamma = 0.577\ldots$ denotes Euler's constant. It is well known that

$$f(s) = \frac{2e^{\gamma}\log(s-1)}{s}, \quad 2 \le s \le 4;$$
$$F(s) = \frac{2e^{\gamma}}{s}, \quad 1 \le s \le 3.$$

In the proof of Theorem 1.1 we adopt the following notation:

$$\mathfrak{P} = \prod_{2<p<z} p, \qquad \log 2\mathbf{U} = (\log 2U_2)(\log 2U_3)(\log^3 2U_4),$$

$$\log \mathbf{U} = (\log U_2)(\log U_3)(\log^3 U_4).$$

Let

$$\mathfrak{N}(z) = \prod_{2<p<z}\left(1 - \frac{\omega(p)}{p}\right).$$

Then, by Lemma 4.2 and Merten's prime number theorem, we obtain

$$\mathfrak{N}(z) \asymp \frac{1}{\log N}. \tag{5.1}$$

Let $R(N)$ denote the number of solutions of equation (1.2) with x a P_5 and the other variables primes. Then, we have

(5.2)

$$\begin{aligned} R(N) &\ge \sum_{\substack{l^2+p^2+p_1^3+p_2^4+p_3^4+p_4^4=N \\ U_2<l,\, p\le 2U_2 \\ U_3<p_1\le 2U_3 \\ (l,\mathfrak{P})=1 \\ U_4<p_2,p_3,p_4\le 2U_4}} 1 - \sum_{r=6}^{23} \sum_{\substack{h^2+p_1^2+p_2^3+p_3^4+p_4^4+p_5^4=N \\ U_2<p_1\le 2U_2 \\ U_3<p_2\le 2U_3 \\ h\in\mathcal{M}_r \\ U_4<p_3,p_4,p_5\le 2U_4}} 1 \\ &\ge \sum_{\substack{l^2+p^2+p_1^3+p_2^4+p_3^4+p_4^4=N \\ U_2<l,\, p\le 2U_2 \\ U_3<p_1\le 2U_3 \\ (l,\mathfrak{P})=1 \\ U_4<p_2,p_3,p_4\le 2U_4}} 1 - \sum_{r=6}^{23} \sum_{\substack{(np)^2+p_1^2+p_2^3+p_3^4+p_4^4+p_5^4=N \\ U_2<np,\, p_1\le 2U_2 \\ U_3<p_2\le 2U_3 \\ n\in\mathcal{N}_r \\ U_4<p_3,p_4,p_5\le 2U_4}} 1 \end{aligned}$$

$$= \mathcal{R}(N) - \sum_{r=6}^{23} \mathcal{R}_r(N).$$

Next, we shall give a non-trivial lower bound for $R(N)$ by the linear sieve theory with the assistance of the bilinear error term in [**5**].

(1) The lower bound for $\mathcal{R}(N)$. Let

$$\mathcal{N}(l) = \sum_{\substack{l^2+p^2+p_1^3+p_2^4+p_3^4+p_4^4=N \\ U_2<p\leq 2U_2 \\ U_3<p_1\leq 2U_3 \\ U_4<p_2,p_3,p_4\leq 2U_4}} (\log p)(\log p_1)\cdots(\log p_4),$$

$$\mathcal{E}(d) = \sum_{\substack{U_2<l\leq 2U_2 \\ l\equiv 0 \ (\mathrm{mod}\ d)}} \mathcal{N}(l) - \frac{\omega(d)}{d}\mathfrak{S}(N)\mathfrak{J}(N).$$

Then, by [**5,** Theorem 1] (see also [**6,** Lemma 9.1]) and Proposition 3.1, we obtain

(5.3)

$$\mathcal{R}(N) \geq \frac{1}{\log 2\mathbf{U}} \sum_{\substack{U_2<l\leq 2U_2 \\ (l,\mathfrak{P})=1}} \mathcal{N}(l)$$

$$\geq (1+O(\log^{-1/3} D))\frac{f(3)\mathfrak{S}(N)\mathfrak{J}(N)\mathfrak{N}(z)}{\log \mathbf{U}} + O(N^{13/12}\log^{-100} N).$$

(2) The upper bound for $\mathcal{R}_r(N)$. Let

$$\mathcal{N}_r(l) = \sum_{\substack{(np)^2+l^2+p_1^3+p_2^4+p_3^4+p_4^4=N \\ n\in\mathcal{N}_r \\ U_3<p_1\leq 2U_3 \\ U_2<np\leq 2U_2 \\ U_4<p_2,p_3,p_4\leq 2U_4}} (\log p_1)\cdots(\log p_4)\left(\frac{\log p}{\log U_2/n}\right)$$

$$\mathcal{E}_r(d) = \sum_{\substack{U_2<l\leq 2U_2 \\ l\equiv 0 \ (\mathrm{mod}\ d)}} \mathcal{N}_r(l) - \frac{c_r\omega(d)}{d\log U_2}\mathfrak{S}(N)\mathfrak{J}(N),$$

where c_r is defined in Lemma 2.5. Then, by [**5,** Theorem 1] (see also, [**6,** Lemma 9.1]) and Proposition 3.2, we have

(5.4)
$$\begin{aligned}\mathcal{R}_r(N) &\le \frac{\log U_2}{\log \mathbf{U}} \sum_{\substack{U_2<l\le 2U_2\\ (l,\mathfrak{P})=1}} \mathcal{N}_r(l)\\ &\le (1+O(\log^{-1/3}D))\frac{F(3)c_r\mathfrak{S}(N)\mathfrak{I}(N)\mathfrak{N}(z)}{\log \mathbf{U}}+O(N^{13/12}\log^{-100}N).\end{aligned}$$

Proof of Theorem 1.1. By numerical integration, we obtain

(5.5)
$$c_6<0.487,\quad c_7<0.1134,\quad c_8<0.02,\quad c_r<0.0024\quad \text{for } 9\le r\le 23,$$
$$\sum_{r=6}^{23} c_r<0.6564,\quad \log 2>0.6931.$$

From (5.1)–(5.5), we have

$$R(N)>0.0367\,\frac{2e^{\gamma}}{3}\,\frac{\mathfrak{S}(N)\mathfrak{I}(N)\mathfrak{N}(z)}{\log \mathbf{U}}+O(N^{13/12}\log^{-100}N)\gg \frac{N^{13/12}}{\log^6 N},$$

where (3.20) and Lemma 4.1 are employed. The proof of Theorem 1.1 is complete. $\square$

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