ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 2, 2017

ON SEMINORMAL SUBGROUPS OF FINITE GROUPS

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ABSTRACT. All groups considered in this paper are finite. A subgroup H of a group G is said to *seminormal* in G if H is normalized by all subgroups K of G such that $\gcd(|H|, |K|) = 1$. We call a group G an MSN-*group* if the maximal subgroups of all the Sylow subgroups of G are seminormal in G. In this paper, we classify all MSN-groups.

1. Introduction. In the following, G always denotes a finite group. Recall that a subgroup H of a group G is said to *permute* with a subgroup K of G if HK is a subgroup of G. The subgroup H is said to be *permutable* in G if H permutes with all subgroups of G.

There are many articles in the literature (for instance, **[6, 11, 13]**, to name just three) where global information about a group G is obtained by assuming that all p-subgroups H, p a prime, of a given order, satisfy a sufficiently strong embedding property extending permutability. In many cases, the subgroups H are the maximal subgroups of the Sylow p-subgroups of G, and the embedding assumption is that they are S-semipermutable in G.

Following **[7]**, we say that a subgroup X of a group G is said to be S-*semipermutable* in G provided that it permutes with every Sylow q-subgroup of G for all primes q not dividing $|H|$. We define the class of MS-*groups* to be the class of groups G in which the maximal subgroups of all the Sylow subgroups of G are S-semipermutable in G. This class was studied in **[1, 5, 10]**.

2010 AMS *Mathematics subject classification.* Primary 20D10, Secondary 20D15, 20D20.

Keywords and phrases. Finite group, soluble PST-group, T_0-group, MS-group, MSN-group.

The work of the first and fourth authors has been supported by the Ministerio de Economía y Competitividad, Spain and FEDER, European Union, grant No. MTM2014-54707-C3-1-P. The first author has also been supported by the National Natural Science Foundation of China, grant No. 11271085.

Received by the editors on June 12, 2015.

DOI:10.1216/RMJ-2017-47-2-419

Suppose that X is a subnormal S-semipermutable subgroup of a group G. If P is a subgroup, respectively, Sylow subgroup, of G with $\gcd(|X|, |P|) = 1$, then X is a subnormal Hall subgroup of XP, and so X is normalized by P. This observation motivates the following.

Definition 1.1 ([**4**])**.** A subgroup X of a group G is said to be *seminormal*, respectively, S-*seminormal*[1], in G if it is normalized by every subgroup, respectively, Sylow subgroup, K of G such that $\gcd(|X|, |K|) = 1$.

By [**4,** Theorem 1.2], a subgroup of a group is seminormal if and only if it is S-seminormal. Furthermore, a Sylow 2-subgroup of the symmetric group of degree 3 is an example of an S-semipermutable subgroup which is not seminormal.

We say that a group G is an MSN-*group* if the maximal subgroups of all the Sylow subgroups of G are seminormal in G. It is clear that the class of all MSN-groups is a subclass of the class of all MS-groups. To show that this inclusion is proper is the aim of the next example.

Example 1.2. Let $A = \langle y\rangle \times \langle z\rangle$ be a cyclic group of order 18 with y an element of order 9 and z an element of order 2. Let V be an irreducible A-module over the field of 19 elements such that $C_A(V) = \langle z\rangle$. Then V is a cyclic group of order 19. The maximal subgroups of the Sylow subgroups are either trivial or cyclic of order 3. Since V and $\langle z\rangle$ are normal Sylow subgroups of G, it follows that the maximal subgroups of the Sylow 3-subgroups are S-permutable. Hence, G is an MS-group. However, the cyclic subgroups of order 3 are not normalized by V and so G is not an MSN-group.

The main purpose of this paper is to characterize the class of all MSN-groups.

2. Preliminary results. In this section, we collect the definitions and results which are used to prove our theorems.

The book [**2**] will be the main reference for terminology and results on permutability.

S-semipermutability and seminormality are closely related to the following subgroup embedding property introduced by Kegel [**8**].

Definition 2.1. A subgroup H of G is said to be *S-permutable* in G if H permutes with every Sylow p-subgroup of G for every prime p.

The following classes of groups have been extensively studied in recent years. They play an important role in the structural study of groups.

Definition 2.2.

(1) A group G is a T-*group* if normality is a transitive relation in G, that is, if every subnormal subgroup of G is normal in G.
(2) A group G is a PT-*group* if permutability is a transitive relation in G, that is, if H is permutable in K and K is permutable in G, then H is permutable in G.
(3) A group G is a PST-*group* if S-permutability is a transitive relation in G, that is, if H is S-permutable in K and K is S-permutable in G, then H is S-permutable in G.

A classical result of Kegel shows that every S-permutable subgroup must be subnormal ([**2,** Theorem 1.2.14(3)]). Therefore, a group G is a PST-group (respectively a PT-group) if and only if every subnormal subgroup is S-permutable (respectively permutable) in G.

Note that a T-group is a PT-group and a PT-group is a PST-group. On the other hand, a PT-group is not necessarily a T-group (non-Dedekind modular p-groups) and a PST-group is not necessarily a PT-group (non-modular p-groups).

Another interesting class of groups in this context is the class of T_0-groups studied in [**3, 9, 12**].

Definition 2.3. A group G is called a T_0-*group* if the Frattini factor group $G/\Phi(G)$ is a T-group.

The next example shows that the class of all T_0-groups properly contains the class of all T-groups.

Example 2.4. Let $E = \langle x, y \rangle$ be an extraspecial group of order 27 and exponent 3. Let a be an automorphism of order 2 of G given by $x^a = x^{-1}$, $y^a = y^{-1}$. Let $G = E \rtimes \langle a \rangle$ be the corresponding semidirect product. Clearly, G is a T_0-group. The subgroup $H = \langle x \rangle$ is a subnormal subgroup of G which does not permute with the Sylow 2-subgroup $\langle ay \rangle$. Therefore, H is not S-permutable. Hence, G is not a PST-group and so is not a T-group either.

The next theorem shows that soluble T_0-groups are closely related to PST-groups.

Theorem 2.5 ([**9,** Theorems 5, 7 and Corollary 3])**.** *Let G be a soluble T_0-group with nilpotent residual $L = \gamma_\infty(G)$. Then*:

(i) *G is supersoluble.*
(ii) *L is a nilpotent Hall subgroup of G.*
(iii) *If L is abelian, then G is a* PST-*group.*

Here the *nilpotent residual* $\gamma_\infty(G)$ of a group G is the smallest normal subgroup N of G such that G/N is nilpotent, that is, the limit of the lower central series of G defined by $\gamma_1(G) = G$, $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for $i \geq 1$.

Let G be a group whose nilpotent residual $L = \gamma_\infty(G)$ is a Hall subgroup of G. Let $\pi = \pi(L)$ and let $\theta = \pi'$ be the complement of π in the set of all prime numbers. Let θ_N denote the set of all primes p in θ such that, if P is a Sylow p-subgroup of G, then P has at least two maximal subgroups. Further, let θ_C denote the set of all primes q in θ such that, if Q is a Sylow q-subgroup of G, then Q has only one maximal subgroup, or equivalently, Q is cyclic.

Throughout this paper we will use the notation presented above concerning π, $\theta = \pi'$, θ_N and θ_C.

We bring the section to a close with a characterization theorem proved in [**1,** Theorem A].

Theorem 2.6. *Let G be a group with nilpotent residual $L = \gamma_\infty(G)$. Then G is an* MS-*group if and only if G satisfies the following*:

(i) *G is a T_0-group.*

(ii) *L is a nilpotent Hall subgroup of G.*
(iii) *If $p \in \pi$ and $P \in \mathrm{Syl}_p(G)$, then a maximal subgroup of P is normal in G.*
(iv) *Let p and q be distinct primes with $p \in \theta_N$ and $q \in \theta$. If $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, then $[P, Q] = 1$.*
(v) *Let p and q be distinct primes with $p \in \theta_C$ and $q \in \theta$. If $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ and M is the maximal subgroup of P, then $QM = MQ$ is a nilpotent subgroup of G.*

3. Main results. Our first theorem gives precise conditions for an MS-group to be an MSN-group. It is, therefore, a characterization theorem.

Theorem A. *A group G is an* MSN-*group if and only if G satisfies the following conditions*:

(i) *G is an* MS-*group.*
(ii) *Let p and q be distinct primes with $p \in \pi$ and $q \in \theta_N$. If $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, then $[P, Q] = 1$.*
(iii) *Let p and q be distinct primes with $p \in \pi$ and $q \in \theta_C$. If $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ and T is a maximal subgroup of Q, then $[P, T] = 1$.*

Proof. Let G be an MSN-group. Then G is an MS-group.

Let $p \in \pi$ and $q \in \theta_N$. In addition, let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. Further, let T_1 and T_2 be maximal subgroups of Q. Now P is the Sylow p-subgroup of L and P is normal in G since L is a nilpotent Hall π-subgroup of G by Theorem 2.6 (ii). Since G is an MSN-group, P normalizes T_1 and T_2. Hence, P normalizes $Q = \langle T_1, T_2 \rangle$, and so, $[P, Q] = 1$. Thus, statement (ii) is true.

Assume now that p and q are distinct primes with $p \in \pi$ and $q \in \theta_C$. Let $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ and T a maximal subgroup of Q. Since P is a normal subgroup of G normalizing T, it follows that $[P, T] = 1$. Therefore, statement (iii) holds.

Conversely, assume that G is an MS-group satisfying assertions (ii) and (iii). We shall show that G is an MSN-group. By Theorem 2.6, G is a soluble T_0-group and the nilpotent residual L of G is a nilpotent Hall π-subgroup of G. Let $p \in \pi$, and let $P \in \mathrm{Syl}_p(G)$. Then P is a

Sylow p-subgroup of L, and it is normal in G. Let M be a maximal subgroup of P. By Theorem 2.6 (iii), M is normal in G and so it is seminormal in G. Moreover, by assertions (ii) and (iii), P normalizes every maximal subgroup of every Sylow r-subgroup of G for all $r \in \theta$.

Let q and r be distinct primes from θ, and let $Q \in \mathrm{Syl}_q(G)$ and $R \in \mathrm{Syl}_r(G)$. Consider a maximal subgroup M of R. If $r \in \theta_N$, then by Theorem 2.6 (iv), $[R, Q] = 1$ and so Q normalizes M. Hence, assume $r \in \theta_C$. Then, by Theorem 2.6 (v), MQ is a nilpotent subgroup of G and Q normalizes M.

Therefore, every maximal subgroup of every Sylow subgroup of G is seminormal in G. This means G is an MSN-group. $\square$

The second main result tells us how an MSN-group looks.

Theorem B. *Let G be an* MSN*-group. Then G is a split extension of a nilpotent Hall subgroup by a cyclic group.*

Proof. Let G be an MSN-group with nilpotent residual L. By Theorem 2.6 (ii), the nilpotent residual L of G is a nilpotent Hall π-subgroup of G. Let X be a Hall θ-subgroup of G, and note that $G = L \rtimes X$, the semidirect product of L by X. Since X is nilpotent, it follows that $X = Y \times T$, where Y is the Hall θ_N-subgroup of X and T is the Hall θ_C-subgroup of X. Note that T is cyclic. By Theorem A (ii), L centralizes Y and so Y is a normal nilpotent Hall subgroup of G. Therefore, $G = (L \times Y) \rtimes T$ is the semidirect product of the nilpotent Hall subgroup $L \times Y$ by the cyclic group T. This completes the proof. $\square$

Applying Theorems 2.5 and 2.6, if the nilpotent residual of an MSN-group G is abelian, then G is a PST-group. We should mention, however, that not every soluble PST-group is an MSN-group (see [**1**, Example 9]); those that can be MSN-groups are characterized in the next theorem.

Theorem C. *Let G be a soluble* PST*-group. Then G is an* MSN*-group if and only if G satisfies Theorem* 2.6 (iv) *and* (v) *and Theorem* A (ii) *and* (iii).

Proof. Let G be a soluble PST-group. By [**1,** Theorem B], G satisfies Theorem 2.6 (iv) and (v). Moreover, by Theorem A, properties (ii) and (iii) are satisfied by G.

Conversely, assume that Theorem 2.6 (iv), (v) and Theorem A (ii), (iii) are satisfied by G. By [**1,** Theorem B], G is an MS-group. Applying Theorem A, G is an MSN-group. □

Soluble PST-groups which are also MSN-groups are analyzed in our next result. It shows that they have a very restricted structure.

Theorem D. *Let G be a soluble* PST-*group, and let L be the nilpotent residual of G. Assume that G is an* MSN-*group. Then the following statements hold*:

(i) *every Hall θ_N-subgroup of G is contained in the hypercenter of G.*
(ii) *$G = (L \times X) \rtimes Y$, where Y is a cyclic Hall θ_C-subgroup of G.*
(iii) *If G is non-nilpotent, then Y is of square-free order.*

Proof. Applying a theorem of Agarwal ([**2,** Theorem 2.1.8]), L is an abelian Hall subgroup of G on which G acts by conjugation as a group of power automorphisms. By Theorem B, G is a semidirect product of the nilpotent Hall $(\pi \cup \theta_N)$-subgroup by a cyclic Hall θ_C-subgroup. Note that a Hall θ_N-subgroup E of G is a normal subgroup of G. Moreover, E normalizes every Sylow r-subgroup of G for all primes $r \in \pi \cup \theta_C$. This means that E is contained in the intersection of the normalizers of all Sylow subgroups of G, that is, E is contained in the hypercenter of G. Therefore, assertions (i) and (ii) hold.

Suppose that G is non-nilpotent. Then $L \neq 1$ and $\mathrm{Z}(G) = 1$. Let $r \in \theta_C$ and $R \in \mathrm{Syl}_r(G)$. If M is a maximal subgroup of R, then, by Theorem A, $[M, L] = 1$. Since M is central in a Hall θ-subgroup of G, it follows that $M \neq \mathrm{Z}(G) = 1$. Therefore, R is cyclic of order r. Hence, Y is cyclic of square-free order and assertion (iii) holds. □

Example 3.1. Let L be a cyclic group of order 19, let E be an extraspecial 3-group of order 27 and exponent 3, and let $C = \langle c \rangle$ be a cyclic group of order 2. Put $X = E \times C$, and let X act on L as follows: E centralizes L and, if $l \in L$, then $l^c = l^{-1}$. Let $G = L \rtimes X$ be the semidirect product of L by X. Then, G is a PST-group with nilpotent residual L. Note that G is an MSN-group with $\pi = \{19\}$, $\theta_N = \{3\}$ and $\theta_C = \{2\}$. We also note that $G = (L \times E) \rtimes C$.

ENDNOTES

1. Note that the term *seminormal* has different meanings in the literature.

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