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TOPOLOGICAL TRIVIALITY OF FAMILIES OF MAP GERMS FROM $\mathbb{R}^3$ TO $\mathbb{R}^3$

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ABSTRACT. We show that a one-parameter unfolding $F : (\mathbb{R}^3 \times \mathbb{R}, 0) \to (\mathbb{R}^3 \times \mathbb{R}, 0)$ of a finitely determined map germ f, with $S(f)$ regular, is topologically trivial if it is excellent in the sense of Gaffney, and the family of the double point curves and cuspidal edges $D(f_t) \cup C(f_t)$ is topologically trivial.

1. Introduction. In **[13]**, we defined a complete topological invariant for finitely determined map germs $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$, when their singular set $S(f)$ is regular, based on an adapted version of classical Gauss words, which we referred to as *Gauss paragraphs*. We produced this study using the topological type of the link of these germs. The link is obtained by taking a small enough representative $f : U \subset \mathbb{R}^3 \to \mathbb{R}^3$, and the restriction of f to $\widetilde{S}^2_\epsilon = f^{-1}(S^2_\epsilon)$, where S^2_ϵ is a small enough sphere centered at the origin. It follows that the link is a stable map $\gamma : S^2 \to S^2$, which is well defined up to $\mathcal{A}$-equivalence, and that f is topologically equivalent to the cone on its link. As an application of these techniques we gave a wide topological classification of these germs in the case of corank 1 **[14]**.

In this paper, we consider a one-parameter unfolding of f, that is, a map germ

$$F : (\mathbb{R}^3 \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}, 0)$$

of the form $F(x, t) = (f_t(x), t)$, and such that $f_0 = f$. We are interested in the topological triviality of F, which means that it is topologically equivalent as an unfolding to the constant unfolding. Our main result is that F is topologically trivial if it is excellent in the sense of Gaffney **[4]**.

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Moreover, the family of the double point curves and cuspidal edges $C(F) \cup D(F)$ is a topologically trivial deformation of $C(f) \cup D(f)$.

The techniques for proving this result have been used previously by the second author **[15]**, where a sufficient condition was obtained for topological triviality in the case of $\mathbb{R}^2$ to $\mathbb{R}^3$ and by both authors **[12]**, where the same was obtained for map germs from $\mathbb{R}^2$ to $\mathbb{R}^2$. For related results, we also refer to **[1, 7, 16]**.

For simplicity, all map germs considered are real analytic, unless otherwise stated; although, most of the results here are also valid for C^∞-map germs if they are finitely determined. We adopt the notation and basic definitions that are common in singularity theory (e.g., $\mathcal{A}$-equivalence, stability, finite determinacy, etc.), which the reader can find in Wall's survey paper **[18]**.

2. The link of a finitely determined map germ. In this section, we recall the basic definitions and results that we will utilize, including the characterization of stable maps from $\mathbb{R}^3$ to $\mathbb{R}^3$, the Mather-Gaffney finite determinacy criterion and the link of a map germ.

Two smooth map germs f and $g : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ are $\mathcal{A}$-equivalent if there exist diffeomorphism germs ϕ, $\psi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $g = \psi \circ f \circ \phi^{-1}$. If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then we say that f and g are topologically equivalent.

We say that $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ is k-determined if any map germ g with the same k-jet is $\mathcal{A}$-equivalent to f. We say that f is finitely determined if it is k-determined for some k.

Let $f : U \to V$ be a smooth proper map, where U, $V \subset \mathbb{R}^3$ are open subsets. We denote the singular set of f by

$$S(f) = \{p \in U : Jf_p = 0\},$$

where Jf is the Jacobian determinant. Following Mather's techniques of classification of stable maps, it is well known (see, for instance, **[5]**) that f is *stable* if and only if the following two conditions hold:

(i) its only singularities are folds (A_1), cusps (A_2) and swallowtails (A_3).

(ii) $f|_{S_{1,0,0}(f)}$ is an immersion with normal crossings: curves of double points (A_1^2) and isolated triple points (A_1^3), $f|_{S_{1,1,0}(f)}$ are injective immersions and the images of both restrictions transversally intersect (A_1A_2).

See Figure 1 for local pictures of the discriminant set of stable singularities.

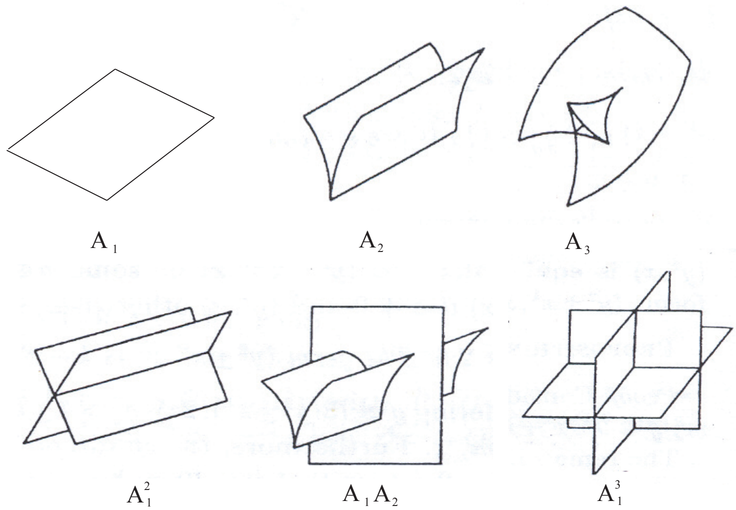

FIGURE 1.

Both the stability criterion and the classification of singular stable points are also true if we consider a holomorphic proper map $f : U \to V$, with U and V open subsets of $\mathbb{C}^3$. So, we now consider a holomorphic map germ

$$f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}^3, 0),$$

and we recall the Mather-Gaffney finite determinacy criterion **[18]**. Roughly speaking, f is finitely determined if and only if it has isolated instability at the origin. To simplify the notation, we state the Mather-Gaffney theorem only in the case of map germs from $(\mathbb{C}^3, 0)$ to $(\mathbb{C}^3, 0)$, although it is true in a more general form for map germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^p, 0)$.

Theorem 2.1. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a holomorphic map germ. Then, f is finitely determined if and only if there is a representative $f : U \to V$, where U and V are open subsets of $\mathbb{C}^3$ such that*:

(i) $f^{-1}(0) = \{0\}$,
(ii) $f : U \to V$ *is proper,*
(iii) *the restriction* $f|_{U\setminus\{0\}}$ *is stable.*

From condition (iii), the A_3, A_1A_2 and A_1^3 singularities are isolated points in $U \setminus \{0\}$. By the curve selection lemma **[11]**, we deduce that they are also isolated in U. Thus, we can shrink the neighborhood U, if necessary, and obtain a representative such that $f|_{U\setminus\{0\}}$ is stable only with folds, cuspidal edges and double fold point curves.

Returning to the real case, we now consider an analytic map germ

$$f : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^3, 0).$$

If we denote the complexification of f by

$$\widehat{f} : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}^3, 0),$$

it is well known that f is finitely determined if and only if $\widehat{f}$ is finitely determined. So, we have the following immediate consequence of the Mather-Gaffney finite determinacy criterion.

Corollary 2.2. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ. Then, there is a representative $f : U \to V$, with U and V open subsets of $\mathbb{R}^3$ such that*:

(i) $f^{-1}(0) = \{0\}$,
(ii) $f : U \to V$ *is proper,*
(iii) *the restriction* $f|_{U\setminus\{0\}}$ *is stable only with fold planes, cuspidal edges and double fold point curves.*

Definition 2.3. We say that $f : U \to V$ is a good representative for a finitely determined map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$, if Corollary 2.2 (i)–(iii) holds.

Definition 2.4. Let $f : U \to V$ be a good representative of a finitely determined map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$. Denote by:

(i) $C(f) = f(\sqrt{S_{1,1}(f)})$ the cuspidal edge curve of f;
(ii) $D(f) = \overline{\{y \in \Delta(f) : \# f^{-1}(y) \cap S(f) \geq 2\}}$ the double fold curve of f.

To conclude this section, we briefly recall the link of a finitely determined map germ. For more information and basic definitions, see [**13**].

First, we remember an important result [**3**], which demonstrates that any finitely determined map germ,

$$f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0), \quad \text{with } n \leq p,$$

has a conic structure over its link. In our case, we simply state the result as $n = p = 3$, and its proof can be found in [**13**].

Theorem 2.5. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ. Then, up to $\mathcal{A}$-equivalence, there are representatives $f : U \to V$ and $\epsilon_0 > 0$ such that, for any ϵ with $0 < \epsilon \leq \epsilon_0$, we have*:

(i) $\widetilde{S}^2_\epsilon = f^{-1}(S^2_\epsilon)$ *is diffeomorphic to* S^2.
(ii) *The map* $f|_{\widetilde{S}^2_\epsilon} : \widetilde{S}^2_\epsilon \to S^2_\epsilon$ *is stable.*
(iii) *Representative f is topologically equivalent to the cone on* $f|_{\widetilde{S}^2_\epsilon}$.

Definition 2.6. Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ. We say that the stable map

$$f|_{\widetilde{S}^2_\epsilon} : \widetilde{S}^2_\epsilon \longrightarrow S^2_\epsilon$$

is the *link* of f, where f is a representative such that (1)–(3) of Fukuda's theorem hold for any ϵ with $0 < \epsilon \leq \epsilon_0$. This link is well defined, up to $\mathcal{A}$-equivalence.

Since any finitely determined map germ is topologically equivalent to the cone on its link, we have the following immediate consequence.

Corollary 2.7. *Let $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be two finitely determined map germs such that their associated links are topologically equivalent. Then, f and g are topologically equivalent.*

We will show that the converse of this corollary is also true at the end of the following section, if we assume that the singular sets $S(f)$ and $S(g)$ are smooth.

3. Gauss paragraphs.

Now, we introduce the concept of the Gauss paragraph of a stable map with a connected singular set.

We recall that a Gauss word is a word which contains each letter exactly twice, one with exponent $+1$ and another with exponent -1. This was originally introduced by Gauss to describe the topology of closed curves in the plane $\mathbb{R}^2$, or in the sphere S^2 (see, for instance, [**17**]). Here, we use the same terminology to represent a different type of word, adapted to our particular case of stable maps $S^2 \to S^2$.

We assume that $\gamma : S^2 \to S^2$ is a stable map such that all of its singularities are folds and cusp points and that $\gamma|_{S(\gamma)}$ presents only simple cusps and double transverse points, where $S(\gamma)$ is the singular set of γ. Moreover, we assume that $S(\gamma)$ (and hence, its images $\Delta(\gamma)$ and $\gamma^{-1}(\Delta(\gamma))$, see [**13,** Lemma 3.1]) are connected.

By looking at the structure of the singular curves, we can split $\gamma^{-1}(\Delta(\gamma))$ into

$$\gamma^{-1}(\Delta(\gamma)) = S(\gamma) \cup X(\gamma),$$

where

$$X(\gamma) = \overline{\gamma^{-1}(\Delta(\gamma)) \setminus S(\gamma)}.$$

The local structure of these curves at a cusp or at a transverse double point is shown in Figure 2. In general, $X(\gamma)$ may have several components, that is, it is equal to a finite union of closed curves with cusps or transverse double points. We denote such components by $X_1(\gamma), \dots, X_k(\gamma)$.

We now choose orientations on the spheres S^2 (we may take different orientations on each S^2). Then, there are natural orientations induced on the singular curves:

- (i) $S(\gamma)$. On the left, we have the positive region (where γ preserves the orientation).
- (ii) $\Delta(\gamma)$. On the left, we have the region of bigger multiplicity (the number of inverse images of a value).
- (iii) $X_j(\gamma)$. On the left, we have the region of bigger multiplicity (here, the multiplicity of a point is the multiplicity of its image).

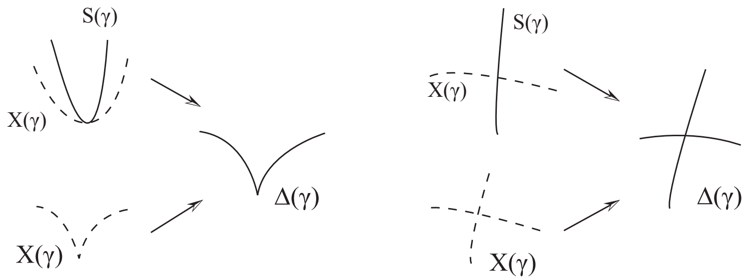

FIGURE 2.

At a transverse double point we have two oriented branches. One branch is called *positive* if the other branch crosses from right to left at the double point; otherwise, we call it *negative*. We always have a positive and a negative branch meeting at a double point (see Figure 3).

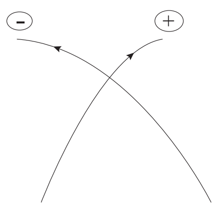

FIGURE 3.

The next step is to choose a base point on each curve $S(\gamma)$, $\Delta(\gamma)$ and $X_j(\gamma)$. We only need to choose one point in $S(\gamma)$. This point uniquely determines a base point on all of the other curves. Writing, for simplicity, $X_0(\gamma) = S(\gamma)$, we fix a point $z_0 \in X_0(\gamma)$, which determines a point $\gamma(z_0) \in \Delta(\gamma)$. By following the orientation in $X_0(\gamma)$, we consider the first point z_1, appearing in the curves $X_1(\gamma), \dots, X_k(\gamma)$, and we reorder the curves in such a way that $z_1 \in X_1(\gamma)$.

Now, we proceed by induction. Assume that a base point z_i is chosen on each curve $X_i(\gamma)$ for $i = 0, \dots, \ell$, with $\ell < k$ (after reordering the curves). Consider the first curve $X_i(\gamma)$, which intersects one of the remaining curves $X_{\ell+1}(\gamma), \dots, X_k(\gamma)$. Take $z_{\ell+1}$ as the first point of intersection, following the base point and the orientation of

$X_i(\gamma)$. Reorder the curves $X_{\ell+1}(\gamma), \dots, X_k(\gamma)$ in such a way that $z_{\ell+1} \in X_{\ell+1}(\gamma)$. Since $S(\gamma) \cup X(\gamma)$ is connected, this procedure will determine a unique base point z_i on each curve $X_i(\gamma)$ for $i = 1, \dots, k$. The algorithm used to choose the base points on the curves $X_1(\gamma), \dots, X_k(\gamma)$ is not unique.

Definition 3.1. Assume that $\Delta(\gamma)$ presents r double points and s simple cusps, which are labeled by $r+s$ letters $\{a_1, a_2, \dots, a_{r+s}\}$. The Gauss word of $\Delta(\gamma)$ is denoted by W_0, and it is the sequence of cusps and double points that appear when traveling around $\Delta(\gamma)$, starting from the base point and following the orientation. If we arrive at a point a_i, then we write a_i^2 if it is a cusp, a_i if it corresponds to the positive branch of a double point, or a_i^{-1} if it corresponds to the negative branch.

For each $j = 1, \dots, k$, the Gauss word of $X_j(\gamma)$ is denoted by W_j, and it is defined in an analogous way, but now we have more possibilities. Given a point which is an inverse image of a_i, if it belongs to $S(f)$, we use the same letter a_i to label the point; otherwise, we write $\overline{a_i}$, $\overline{\overline{a_i}}, \cdots$ (multiple bars are used in order to distinguish between different inverse images). The same convention is used with the exponents a_i^2, $\overline{a_i}^2$, $\overline{\overline{a_i}}^2, \dots$, for a cusp, a_i, $\overline{a_i}$, $\overline{\overline{a_i}}, \dots$, for a positive branch of a double point, or a_i^{-1}, $\overline{a_i}^{-1}$, $\overline{\overline{a_i}}^{-1}, \dots$, for a negative branch of a double point.

The list of Gauss words $\{W_0, W_1, \dots, W_k\}$ is called a *Gauss paragraph.*

Example 3.2.

(i) Let $\gamma : S^2 \to S^2$ be the link of the fold $f(x,y,z) = (x,y,z^2)$. Then, $\Delta(\gamma)$ does not present any simple cusp or double point. The Gauss paragraph is only $\{\emptyset\}$ (see Figure 4).

(ii) Let $\gamma : S^2 \to S^2$ be the link of the cuspidal edge $f(x,y,z) = (x,y,xz+z^3)$. Then, $\Delta(\gamma)$ presents two simple cusps, each one with a

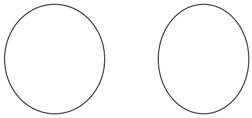

FIGURE 4.

single inverse image. The Gauss paragraph in this case is $\{a^2b^2, a^2b^2\}$ (see Figure 5).

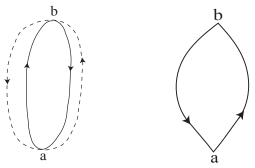

FIGURE 5.

(iii) Let $\gamma : S^2 \to S^2$ be the link of the swallowtail

$$f(x, y, z) = (x, y, z^4 + xz + yz^2).$$

Then, $\Delta(\gamma)$ present two simple cusps, each one with two inverse images, and a double fold point with two inverse images. The Gauss paragraph is

$$\{a^{-1}b^2c^2a, a\bar{c}^2b^2a^{-1}\bar{b}^2c^2\}$$

(see Figure 6).

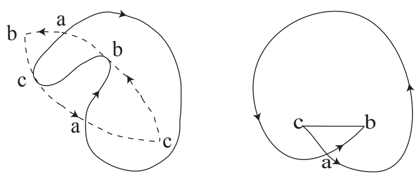

FIGURE 6.

(iv) Let $\gamma : S^2 \to S^2$ be the link of the germ $f(x, y, z) = (x, y, z^4 + xz - y^2z^2)$. Then, $\Delta(\gamma)$ presents four simple cusps, each with two inverse images and two double fold points, each having two inverse images. See Figure 7 for this case, where the Gauss paragraph is:

$$\begin{cases} a^{-1}b^2c^2ad^{-1}e^2f^2d, \\ a\bar{c}^2b^2a^{-1}\bar{b}^2c^2, \\ d\bar{f}^2e^2d^{-1}\bar{e}^2f^2. \end{cases}$$

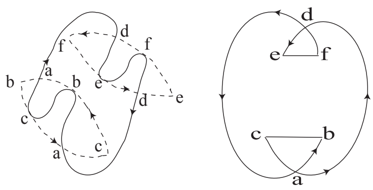

FIGURE 7.

It is obvious that the Gauss paragraph is not uniquely determined since it depends on the labels $a_1, \dots, a_{r+s}$, the chosen orientations in each S^2, and the base point $z_0 \in S(\gamma)$. Different choices will produce the following changes in the Gauss paragraph:

(i) a permutation in the set of letters $a_1, \dots, a_{r+s}$,
(ii) a reversion in the Gauss words together with a change in the exponents $+1$ to -1, and vice versa,
(iii) a cyclic permutation in the Gauss words.

We say that two Gauss paragraphs are *equivalent* if they are related through these three operations. Under this equivalence, the Gauss paragraph is now well defined.

In order to simplify the notation, given a stable map $\gamma : S^2 \to S^2$, we denote the associated Gauss paragraph by $w(\gamma)$, and the equivalence relation between Gauss paragraphs by $\simeq$.

As a consequence of this definition and previous remarks, we have the following important result.

Theorem 3.3. [**12**]. *Let γ, $\delta : S^2 \to S^2$ be two stable maps such that $S(\gamma)$ and $S(\delta)$ are connected and non empty. Then, γ and δ are topologically equivalent if and only if $w(\gamma) \simeq w(\delta)$.*

Note that Theorem 3.3 is not true if $S(\gamma)$ is not connected. An example of two stable maps from S^2 to S^2, both with empty Gauss words, which are not topologically equivalent, is found in [**6,** Figure 6].

On the other hand, the equivalence between the Gauss words of $\Delta(\gamma)$ and $\Delta(\delta)$ is not a sufficient condition to guarantee the topological equivalence between γ and δ. In fact, if γ and δ have isomorphic discriminants $\Delta(\gamma)$ and $\Delta(\delta)$, then they are not topologically equivalent in general (see **[2]**).

Using this last theorem, we can conclude that, if we consider two map germs f and g, such that their respective singular sets are smooth and non empty outside of the origin, then the converse of Corollary 2.7 is also true.

Theorem 3.4. [**12**]. *Let f and $g : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be two finitely determined map germs such that $S(f)$ and $S(g)$ are smooth and non empty outside of the origin. Then, if f and g are topologically equivalent, their respective links are topologically equivalent.*

Applying Fukuda's results, our topological classification will be based on the following corollary.

Corollary 3.5. *Let $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be two finitely determined map germs such that $S(f)$ and $S(g)$ are smooth and non empty outside of the origin. Then, f and g are topologically equivalent if and only if $w(f) \simeq w(g)$.*

In the case that $S(f)$ is empty outside of the origin, its associated link $\gamma : S^2 \to S^2$ becomes a regular map, and hence, a diffeomorphism, by [**13,** Lemma 3.1]. Hence, in this case, we have only one topological class, namely, the regular map $f(x, y, z) = (x, y, z)$.

Remark 3.6. By following the proof of [**13,** Theorem 3.3] step by step, we can observe the following fact. If $\gamma, \delta : S^2 \to S^2$ are stable maps with $S(\gamma)$ and $S(\delta)$ connected, $w(\gamma) \simeq w(\delta)$ and, if we fix any homeomorphism in the target $\psi : S^2 \to S^2$ such that $\psi(\Delta(\gamma)) = \Delta(\delta)$, then there is a unique homeomorphism in the source $\phi : S^2 \to S^2$ such that $\psi \circ \gamma \circ \phi^{-1} = \delta$.

By combining this observation with Corollary 2.7 and Theorem 3.4, we have an analogous result for map germs. Let $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be two finitely determined map germs, with $S(f)$ and $S(g)$ smooth and non-empty outside of the origin, which are topologically equivalent. If we fix any homeomorphism in the target $\psi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $\psi(\Delta(f)) = \Delta(g)$, then there is a unique homeomorphism in the source $\phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $\psi \circ f \circ \phi^{-1} = g$.

4. Cobordism of links. We recall that a cobordism between two smooth manifolds M_0 and M_1 is a smooth manifold with boundary W such that $\partial W = M_0 \sqcup M_1$. Analogously, a cobordism between smooth maps $f_0 : M_0 \to N_0$ and $f_1 : M_1 \to N_1$ is another smooth map $F : W \to Q$ such that W and Q are cobordisms between M_0, M_1 and N_0, N_1, respectively, and for each $i = 0, 1$, $F^{-1}(N_i) = M_i$ and the restriction $F|_{M_i} : M_i \to N_i$ equal to f_i. In the case that f_0 and f_1 belong to some special class of maps (for instance, immersions, embeddings, stable maps, etc.), then we also require that the cobordism F belongs to the same class.

Definition 4.1. Given two stable maps $\gamma_0, \gamma_1 : S^2 \to S^2$, a cobordism between γ_0 and γ_1 is a stable map

$$\Gamma : S^2 \times I \to S^2 \times I,$$

where $I = [0, 1]$ and such that, for $i = 0$, 1,

$$\Gamma^{-1}(S^2 \times \{i\}) = S^2 \times \{i\} \quad \text{and} \quad \Gamma|_{S^2 \times \{i\}} = \gamma_i \times \{i\}.$$

The first condition implies that

$$\Gamma(S^2 \times \{0\}) \subset S^2 \times \{0\},$$
$$\Gamma(S^2 \times \{1\}) \subset S^2 \times \{1\},$$

and

$$\Gamma(S^2 \times (0, 1)) \subset S^2 \times (0, 1),$$

but, in general, Γ is not level preserving.

Theorem 4.2. *Let Γ be a cobordism between γ_0 and γ_1, with $S(\gamma_0)$ connected. If $S(\Gamma)$ is homeomorphic to $S(\gamma_0) \times I$ and $C(\Gamma) \cup D(\Gamma)$ is homeomorphic to $(C(\gamma_0) \cup D(\gamma_0)) \times I$, then Γ is topologically trivial. In particular, γ_0 and γ_1 are topologically equivalent.*

Proof. In order to show that Γ is topologically trivial, we will first prove that $\Delta(\Gamma)$ and $\Gamma^{-1}(\Delta(\Gamma))$ are homeomorphic to $\Delta(\gamma_0) \times$

I and $\gamma_0^{-1}(\Delta(\gamma_0)) \times I$, respectively, and that the restriction of Γ to the complementary sets is a homeomorphism on each connected component.

Since $S(\Gamma)$ is homeomorphic to $S(\gamma_0) \times I$, we have that

$$S(\Gamma) \setminus (\Gamma^{-1}(C(\Gamma) \cup D(\Gamma)) \cap S(\Gamma)$$

is a disjoint union of open discs. If we consider the restricted map

$$\Gamma| : S(\Gamma) \setminus (\Gamma^{-1}(C(\Gamma) \cup D(\Gamma)) \cap S(\Gamma)) \longrightarrow \Delta(\Gamma) \setminus (C(\Gamma) \cup D(\Gamma)),$$

we get a homeomorphism and, as consequence,

$$\Delta(\Gamma) \setminus (C(\Gamma) \cup D(\Gamma))$$

is also a disjoint union of open discs. Now, the homeomorphism between $C(\Gamma) \cup D(\Gamma)$ and $(C(\gamma_0) \cup D(\gamma_0)) \times I$ extends to a homeomorphism between $\Delta(\Gamma)$ and $\Delta(\gamma_0) \times I$ (see Figure 8).

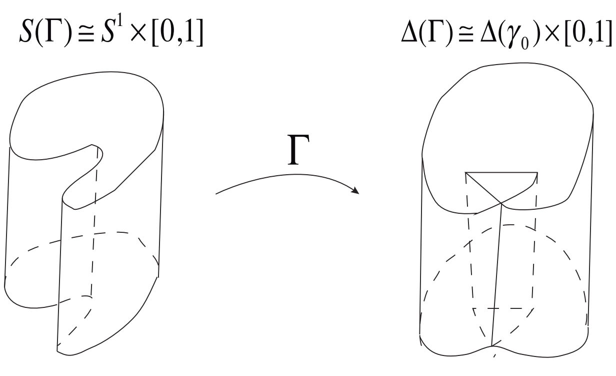

FIGURE 8.

Now, let us show that $\Gamma^{-1}(\Delta(\Gamma))$ is homeomorphic to $\gamma_0^{-1}(\Delta(\gamma_0)) \times I$. We consider the restricted map:

$$\Gamma| : \Gamma^{-1}(\Delta(\Gamma)) \setminus \Gamma^{-1}(C(\Gamma) \cup D(\Gamma)) \longrightarrow \Delta(\Gamma) \setminus (C(\Gamma) \cup D(\Gamma)),$$

and we denote each one of the connected components of $\Gamma^{-1}(\Delta(\Gamma)) \setminus \Gamma^{-1}(C(\Gamma) \cup D(\Gamma))$ by C_i. We have that the restriction:

$$\Gamma|_{C_i} : C_i \longrightarrow D_j$$

is a d-fold covering for some $d \geq 1$, where we denote each one of the open discs of $\Delta(\Gamma) \setminus (C(\Gamma) \cup D(\Gamma))$ by D_j. Therefore,

$$1 - \beta_1(C_i) = \chi(C_i) = d\chi(D_j) = d \geq 1,$$

where $\beta_1(C_i)$ is the first Betti number of C_i. Hence, $\beta_1(C_i) = 0$ and $d = 1$. We deduce that each C_i is an open disc and $\Gamma|_{C_i} : C_i \to D_j$ is a homeomorphism. Then,

$$\Gamma^{-1}(\Delta(\Gamma)) \setminus \Gamma^{-1}(C(\Gamma) \cup D(\Gamma))$$

is a disjoint union of open discs. On the other hand, by hypothesis, $C(\Gamma) \cup D(\Gamma)$ is homeomorphic to $(C(\gamma_0) \cup D(\gamma_0)) \times I$. Therefore, the curves cannot intersect each other, that is, Γ cannot have triple points, swallowtails or simple cusps. Thus, Γ restricted to $\Gamma^{-1}(C(\Gamma) \cup D(\Gamma))$ is a local homeomorphism, and hence, the restriction to each one of the curves is a homeomorphism. It follows that $\Gamma^{-1}(C(\Gamma) \cup D(\Gamma))$ is homeomorphic to $\gamma_0^{-1}(C(\gamma_0 \cup D(\gamma_0)) \times I$ and, as a consequence, $\Gamma^{-1}(\Delta(\Gamma))$ is homeomorphic to $\gamma_0^{-1}(\Delta(\gamma_0)) \times I$ (see Figure 9).

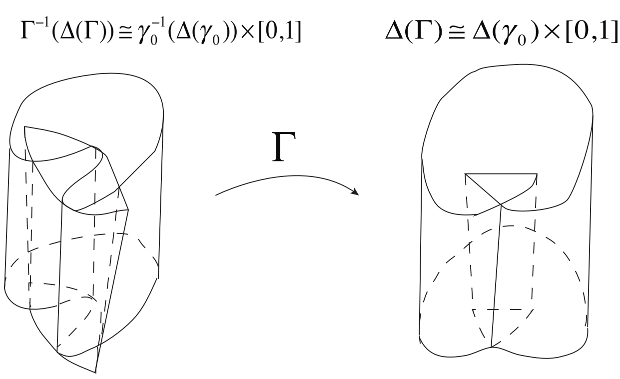

FIGURE 9.

Now, we assign E_i to each of the connected components of $S^2 \times I \setminus \Gamma^{-1}(\Delta(\Gamma))$, and F_j to each of the connected components of $S^2 \times I \setminus \Delta(\Gamma)$. We have that $\Gamma|_{E_i} : E_i \to F_j$ is a d-fold covering. Let us prove that $d = 1$ and, as a consequence, $\Gamma|_{E_i}$ is a homeomorphism.

Since $\Delta(\gamma_0)$ is connected, we know that $S^2 \setminus \Delta(\gamma_0)$ is a disjoint union of open discs D_i and, as a consequence, $\pi_1(D_i) = 0$ for all i. Since $\Delta(\Gamma)$ is homeomorphic to $\Delta(\gamma_0) \times I$, we have that $S^2 \times I \setminus \Delta(\Gamma)$ is homeomorphic to $(S^2 \setminus \Delta(\gamma_0)) \times I$ and, as a direct consequence, that, for each j, $\pi_1(F_j) = \pi_1(D_j) = 0$. Thus, F_j is simply connected for all j. By applying the universal covering theorem, we have that $d = 1$ and, for each i, $\Gamma|_{E_i}$ is a homeomorphism.

Finally, we construct homeomorphisms $\Phi, \Psi : S^2 \times I \to S^2 \times I$ such that the following diagram commutes.

$$\begin{array}{ccc} S^2 \times I & \xrightarrow{\Gamma} & S^2 \times I \\ \Phi \downarrow & & \downarrow \Psi \\ S^2 \times I & \xrightarrow[\gamma_0 \times \mathrm{id}]{} & S^2 \times I \end{array}$$

In the proof of Theorem 3.4 (see [**13**]), we see that $\gamma_0 : S^2 \to S^2$ has a cellular structure induced by its Gauss paragraph in such a way that γ_0 is a homeomorphism restricted to each cell. Indeed, the 0-cells are the cusps and the double points and their inverse images, which are labeled by the letters in the Gauss word. The 1-skeleton is $\Delta(\gamma_0)$ in the target and its inverse image in the source, and each 1-cell is labeled by a pair of letters in the Gauss words. Finally, each 2-cell (in the source or target) is determined by a closed sequence of oriented edges. It follows from the first part of the proof that both

$$\Gamma : S^2 \times I \longrightarrow S^2 \times I$$

and

$$\gamma_0 \times \mathrm{id} : S^2 \times I \longrightarrow S^2 \times I$$

have a cellular structure induced by the product of γ_0 and the identity map.

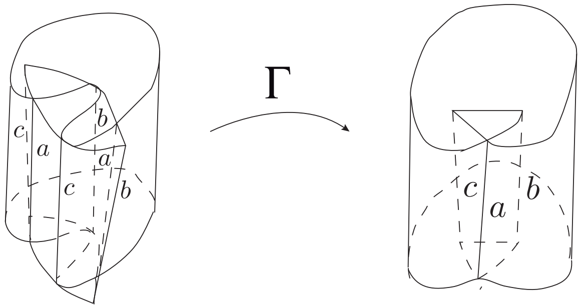

FIGURE 10.

We write

$$\Gamma : M_1 \longrightarrow P_1$$

and

$$\gamma_0 \times \mathrm{id} : M_2 \longrightarrow P_2,$$

where M_i and P_i denote $S^2 \times I$ with the associated cellular structure in the source or the target, respectively. We have that P_1 and P_2 are isomorphic as CW-complexes, and we choose any cellular homeomorphism $\Psi : P_1 \to P_2$. Then, we construct another cellular homeomorphism $\Phi : M_1 \to M_2$ such that $(\gamma_0 \times \mathrm{id}) \circ \Phi = \Psi \circ \Gamma$. Given a cell E in M_1, there is a unique cell E' in M_2 corresponding to the same label in the Gauss word such that $\Psi(\Gamma(E)) = \gamma_0 \times \mathrm{id}(E')$. We define $\Phi|_E : E \to E'$ as

$$\Phi|_E = (\gamma_0 \times \mathrm{id}\,|_{E'})^{-1} \circ \Psi|_{\Gamma(E)} \circ \Gamma|_E.$$

Therefore, the diagram is commutative, and as a consequence, Γ is topologically trivial. $\square$

5. An extension of the cone structure. Let $f : U \to V$ be a good representative of a finitely determined map germ

$$f : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^3, 0),$$

with $S(f) \subset U$ diffeomorphic to the open disc B^2. Since $C(f) \cup D(f)$ is a one-dimensional analytic subset, we can also shrink the neighborhoods U and V so that this set is contractible. Then, $(C(f) \cup D(f)) \setminus \{0\}$ has a finite number of connected components, each an edge which joins the origin with the boundary of V. We orient each from 0 to ∂V.

Definition 5.1. Let $f : U \to V$ be a good representative of a finitely determined map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $S(f)$ is diffeomorphic to B^2 and $C(f) \cup D(f)$ is contractible. We say that $\epsilon > 0$ is a *convenient radius* for f if the following conditions hold:

- (i) $\widetilde{S}^2_\epsilon$ is diffeomorphic to S^2;
- (ii) $\widetilde{S}^2_\epsilon$ intersects $S(f)$ properly, that is, this intersection is transversal and diffeomorphic to S^1;
- (iii) S^2_ϵ intersects $C(f) \cup D(f)$ properly, that is, S^2_ϵ is transverse to $C(f) \cup D(f)$ and intersects each connected component of $(C(f) \cup D(f)) \setminus \{0\}$ at exactly one point.

Remark 5.2. Let us denote $g = \|f\|^2$, J the Jacobian determinant of f and T the unit tangent vector of $C(f) \cup D(f) \setminus \{0\}$. Then, $\epsilon > 0$ is a convenient radius if and only if the following conditions hold:

- (a) $\widetilde{S}^2_\epsilon$ is diffeomorphic to S^2;
- (b) $\widetilde{S}^2_\epsilon \cap S(f)$ is diffeomorphic to S^1;
- (c) $\nabla g(x) \wedge \nabla J(x) \neq 0$ for all $x \in \widetilde{S}^2_\epsilon \cap S(f)$;
- (d) $\langle y, T(y) \rangle > 0$ for all $y \in S^2_\epsilon \cap (C(f) \cup D(f))$.

If ϵ_0 is a Milnor-Fukuda radius, then the three conditions of the definition are verified for any $0 < \epsilon \leq \epsilon_0$; but, this may not be true in general (see Figure 11).

Theorem 5.3. *Let $f : U \to V$ be a good representative of a finitely determined map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $S(f)$ is diffeomorphic to B^2 and $(C(f) \cup D(f)) \subset V$ is contractible, and let $\epsilon > 0$ be a convenient radius for f. Then*:

- (i) *$f|_{\widetilde{S}^2_\epsilon} : \widetilde{S}^2_\epsilon \to S^2_\epsilon$ is topologically equivalent to the link of f.*
- (ii) *$f|_{\widetilde{D}^3_\epsilon} : \widetilde{D}^3_\epsilon \to D^3_\epsilon$ is topologically equivalent to the cone of $f|_{\widetilde{S}^2_\epsilon}$.*

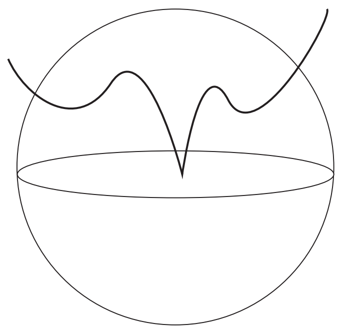

FIGURE 11.

Proof. Let $\epsilon_0 > 0$ be a Milnor-Fukuda radius for f. If $\epsilon \leq \epsilon_0$, the result follows from Theorem 2.5. If $\epsilon > \epsilon_0$, let us take δ such that $0 < \delta \leq \epsilon_0 < \epsilon$. We consider the two associated stable maps $\gamma_0 = f|_{\tilde{S}^2_\delta}$ and $\gamma_1 = f|_{\tilde{S}^2_\epsilon}$, and we denote them by

$$T^3_{\delta,\epsilon} = D^3_\epsilon \setminus B^3_\delta = \{y \in \mathbb{R}^3 : \delta \leq \|y\|^2 \leq \epsilon\},$$
$$\widetilde{T}^3_{\delta,\epsilon} = f^{-1}(T^3_{\delta,\epsilon}),$$

and

$$\Gamma = f|_{\tilde{T}^3_{\delta,\epsilon}} : \widetilde{T}^3_{\delta,\epsilon} \longrightarrow T^3_{\delta,\epsilon},$$

which defines a cobordism between γ_0 and γ_1. If we prove that Γ is topologically trivial, then we are finished because this directly implies that the cone structure of $f|_{\tilde{D}^3_\delta}$ can be extended to $f|_{\tilde{D}^3_\epsilon}$.

Since $S(f)$ is diffeomorphic to B^2 and since $\widetilde{S}^2_\epsilon$ and $\widetilde{S}^2_\delta$ intersect $S(f)$ properly, we have that $S(\Gamma)$ is diffeomorphic to $S^1 \times [\delta, \epsilon]$.

On the other hand, let $C_1, \dots, C_r$ and $D_1, \dots, D_s$ be the connected components of $(C(f) \cup D(f)) \setminus \{0\}$. Since $(C(f) \cup D(f))$ is closed, contractible and regular outside of the origin, we have that each C_i and D_j is diffeomorphic to an open interval of ends 0 and ∂V.

Since S^2_δ and S^2_ϵ intersect $C(f) \cup D(f)$ properly,

$$S^2_\delta \cap C_i = \{x_i\}, \qquad S^2_\epsilon \cap C_i = \{x'_i\},$$
$$S^2_\delta \cap D_j = \{x_{r+j}\} \quad \text{and} \quad S^2_\epsilon \cap D_j = \{x'_{r+j}\}$$

for each $i = 1, \ldots, r$ and $j = 1, \ldots, s$, it follows that

$$C(\Gamma) \cup D(\Gamma) = \overline{x_1 x'_1} \cup \cdots \cup \overline{x_{r+s} x'_{r+s}},$$

where $\overline{x_i x'_i}$ is the closed interval in C_i, joining the points x_i and x'_i, and $\overline{x_{r+j} x'_{r+j}}$ is the closed interval in D_j, joining the points x_{r+j} and x'_{r+j} (see Figure 12).

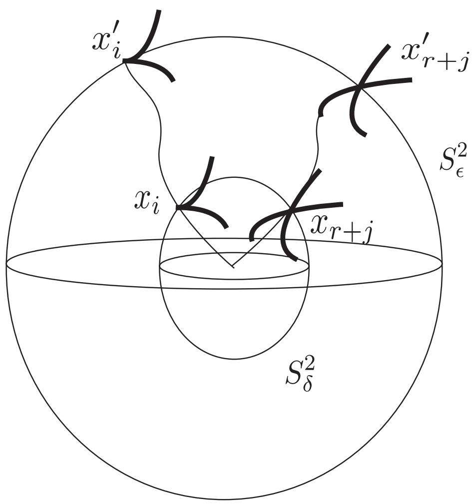

FIGURE 12.

Therefore, $C(\Gamma) \cup D(\Gamma)$ is diffeomorphic to $\{x_1, \ldots, x_{r+s}\} \times [\delta, \epsilon]$, and Γ is topologically trivial by Lemma 4.2. $\square$

6. Topological triviality of families. Given a map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$, a one-parameter unfolding is a map germ

$$F : (\mathbb{R}^3 \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}, 0)$$

of the form $F(x,t) = (f_t(x), t)$ such that $f_0 = f$. Here, we assume that the unfolding is origin preserving, that is, $f_t(0) = 0$ for any t. Hence, we have a one-parameter family of map germs $f_t : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$.

Definition 6.1. Let F be a one-parameter unfolding of a finitely determined map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$.

(i) We say that F is *excellent* if there is a representative $F : U \to V \times I$, where U, V and I are open neighborhoods of the origin in $\mathbb{R}^3 \times \mathbb{R}$, $\mathbb{R}^3$ and $\mathbb{R}$, respectively, such that, for any $t \in I$, $f_t : U_t \to V$ is a good representative in the sense of Definition 2.3.

(ii) We say that F is *topologically trivial* if there are homeomorphism germs Ψ and $\Phi : (\mathbb{R}^3 \times \mathbb{R}, 0) \to (\mathbb{R}^3 \times \mathbb{R}, 0)$ such that they are unfoldings of the identity and $F = \Psi \circ (f \times \mathrm{id}) \circ \Phi$.

Theorem 6.2. *Let F be an excellent unfolding of a finitely determined map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $S(f)$ is regular. If $C(F) \cup D(F)$ is topologically trivial, then F is topologically trivial.*

Proof. Let $F : U \to V \times I$ be a representative of the unfolding F, where U, V and I are open neighborhoods of the origin in $\mathbb{R}^3 \times \mathbb{R}$, $\mathbb{R}^3$ and $\mathbb{R}$, respectively, such that $f_t : U_t \to V$ is a good representative of the map germ f_t for any $t \in I$. If necessary, we can shrink the neighborhoods and assume that $S(f_t)$ is diffeomorphic to B^2 and $C(f_0) \cup D(f_0) \subset V$ is contractible.

On the other hand, since $C(F) \cup D(F)$ is topologically trivial; again, shrinking the neighborhoods, if necessary, there is a homeomorphism $\Psi : V \times I \to V \times I$ of the form $\Psi = (\psi_t, t)$ such that $\psi_0 = \mathrm{id}$ and

$$\psi_t(C(f_t) \cup D(f_t)) = C(f) \cup D(f) \quad \text{for any } t \in I.$$

In particular, $C(f_t) \cup D(f_t)$ is homeomorphic to $C(f_0) \cup D(f_0)$, and it is also contractible.

For each $t \in I$, we fix the following notation for f_t as in Remark 5.2: $g_t = \|f_t\|^2$, J_t is the Jacobian determinant of f_t and T_t is the unit tangent vector of $C(f_t) \cup D(f_t) \setminus \{0\}$.

Let $\epsilon_0 > 0$ be a Milnor-Fukuda radius for f, and let $0 < \epsilon \leq \epsilon_0$. Then, Remark 5.2 (a)–(d) holds. Once ϵ is fixed, we can choose $\delta > 0$ such that, for any $t \in (-\delta, \delta)$:

(c') $\nabla g_t(x) \wedge \nabla J_t(x) \neq 0$ for all $x \in \widetilde{S}^2_{\epsilon,t} \cap S(f_t)$,
(d') $\langle y, T_t(y)\rangle > 0$ for all $y \in S^2_\epsilon \cap (C(f_t) \cup D(f_t))$,

where $\widetilde{S}^2_{\epsilon,t} = g_t^{-1}(\epsilon)$. By the fibration theorem, $\widetilde{S}^2_{\epsilon,t}$ is diffeomorphic to $\widetilde{S}^2_\epsilon$, and hence, to S^2. Analogously, $\widetilde{S}^2_{\epsilon,t} \cap S(f_t)$ is diffeomorphic to $\widetilde{S}^2_\epsilon \cap S(f)$, and hence, to S^1. In conclusion, we have shown that ϵ is a convenient radius for f_t, for any $t \in (-\delta, \delta)$. By Theorem 5.3, $\gamma_{\epsilon,t} = f_t|_{\tilde{S}^2_{\epsilon,t}}$ is the link of f_t and $f_t|_{\tilde{D}^3_{\epsilon,t}}$ is topologically equivalent to the cone of $\gamma_{\epsilon,t}$.

Since $\gamma_{\epsilon,t} : \widetilde{S}^2_{\epsilon,t} \to S^2_\epsilon$, with $t \in (-\delta, \delta)$, is stable, we have that this family of links is trivial. Hence, each $f_t|_{\tilde{D}^3_{\epsilon,t}}$ is topologically equivalent to $f|_{\tilde{D}^3_\epsilon}$. By Remark 3.6, there is a unique homeomorphism in the source ϕ_t such that $\psi_t \circ f_t \circ \phi_t^{-1} = f$. Note that the unicity of ϕ_t implies that it depends continuously on t. Now, we consider

$$\Phi = (\phi_t, t) : F^{-1}(D^3_\epsilon \times (-\delta, \delta)) \longrightarrow \widetilde{D}^3_\epsilon \times (-\delta, \delta).$$

Then, Φ is a homeomorphism, and it is an unfolding of the identity and $\Psi \circ F \circ \Phi^{-1} = f \times \mathrm{id}$. □

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