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SURVEY ARTICLE: THE REAL NUMBERS– A SURVEY OF CONSTRUCTIONS

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ABSTRACT. We present a comprehensive survey of constructions of the real numbers (from either the rationals or the integers) in a unified fashion, thus providing an overview of most (if not all) known constructions ranging from the earliest attempts to recent results, and allowing for a simple comparison-at-a-glance between different constructions.

1. Introduction. The novice, through the standard elementary mathematics indoctrination, may fail to appreciate that, compared to the natural, integer and rational numbers, there is nothing simple about defining the real numbers. The gap, both conceptual and technical, that one must cross when passing from the former to the latter is substantial and perhaps best witnessed by history. The existence of line segments whose length cannot be measured by any rational number is well known to have been discovered many centuries ago (though the precise details are unknown). The simple problem of rigorously introducing mathematical entities that do suffice to measure the length of any line segment proved very challenging. Even relatively modern attempts due to such prominent figures as Bolzano, Hamilton and Weierstrass were only partially rigorous, and it was only with the work of Cantor and Dedekind in the early part of the 1870's that the reals finally came into existence. The interested reader may consult [**12**] for more on the historical developments and further details.

Two of the most famous constructions of the reals are *Cantor's construction* by means of Cauchy sequences of rational numbers and *Dedekind's construction* by means of cuts of rational numbers, named after him. Detailed accounts of these constructions and their ubiquity in textbooks, together with the well-known categoricity of the axioms of a complete ordered field, would have put an end to the quest for other

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constructions, and yet two phenomena persist. Firstly, it appears that human curiosity concerning the real numbers is not quite quenched with just these two constructions. Even though any two models of the axioms of a complete ordered field are isomorphic, so it really does not matter which model one works with, we still seem to be fascinated with finding more and more different models of the same abstract concept. Secondly, and more practically, from the constructive point of view, not all models of the real numbers are isomorphic. Fueled by applications in automated theorem proving and verification, where one must represent the real numbers in a computer, nuances of the differences between various constructions of the reals become very pronounced. We refer the reader to **[6, 17, 18]** for more details on the constructive reals and on theorem proving with the real numbers, respectively.

2. A survey of constructions.

In order to present a uniform survey of constructions of the real numbers we choose to adopt the following somewhat debatable point of view according to which every construction of the real numbers ultimately relies on an *observation* about the reals (treated axiomatically) leading to a bijective correspondence between the set of real numbers and a set defined in terms of simpler entities (often the rational or the integer numbers) upon which agreement of existence is present. That set is then taken to be the definition of the reals, with the *order* structure and the *arithmetical* operations defined, examined and eventually shown to form a complete ordered field.

We present what we hope is an exhaustive list of constructions of the reals one can find in the literature, all following the presentation style exposited above. Certainly, this restrictive decision sometimes necessitates a suboptimal presentation of a particular construction; however, the uniform style makes comparison between the definitions easier.

As a convention, let $\mathbb{N}^+ = \mathbb{N} \cup \{\omega\}$ be the set of all natural numbers augmented with the symbol ω which algebraically behaves like ∞. In particular, $x \leq \omega$ for all $x \in \mathbb{N}^+$, and we define $x + \omega = \omega = \omega + x$ and $x\omega = \omega = \omega x$ for all $x \in \mathbb{N}^+$, and $x/\omega = 0$ for all $x \in \mathbb{Z}$. The sole use of this convention is in treating finite sequences of integers as infinite ones ending with a constant stream of ω's.

Finally, we mention at this point, rather than at each construction surveyed below, that typically it makes little difference whether one constructs the positive (or nonnegative) reals $\mathbb{R}_+$ and then extend to all the reals by formally adding inverses (and a 0 if needed), or constructing all of $\mathbb{R}$ in one go. However, the former approach may sometimes be technically simpler than the latter. Consequently, below, a survey of a construction will be considered complete even if it only produces $\mathbb{R}_+$.

Remark 2.1. In those constructions below that refer to convergence of a sequence (q_n) of rationals to a rational number q, the precise meaning of such a statement is that, for every rational number $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ with $|q_n - q| < \varepsilon$ for all $n > n_0$. A sequence that converges to 0 is also known as a *null sequence*.

2.1. Stevin's "construction" (*De Thiende* and *L'Arithmetique*, 1585).

Remark 1. Stevin is credited with laying down the foundations of the decimal notation. Stevin did not produce a rigorous construction of the reals though he did present the then controversial point of view that there is nothing significantly different in nature between the rational numbers and the irrational ones. Constructing the reals as decimal expansions (or in any other base) is a popular approach by novices but is fraught with technical difficulties. The allure of this approach most likely lies in the emphasis decimal expansions receive in the current mathematical curriculum, where decimal expansions triumph over anything else. The details presented below are nowhere near what Stevin presented. Instead, we follow Gowers (**[15]**) but leave the details at a minimum. We also mention **[20]** for a more holistic view of Stevin's numbers.

Observation. Every real number a can be written as

$$a = \sum_{k=-\infty}^{\infty} \frac{a_k}{b^k},$$

where the *base* $b \geq 2$ is an integer and the $a_k \in \{0, 1, 2, \dots, b-1\}$ are digits, and there exists $k_0 \in \mathbb{Z}$ such that $a_k = 0$ for all $k < k_0$.

Moreover, the presentation is unique if we further demand that there does not exist $k_0 \in \mathbb{Z}$ with $a_k = b - 1$ for all $k > k_0$.

The reals. One may now take the formal b-base expansions as above to be the real numbers.

Order. Defining the order between b-base expansions presents no difficulties; $a < a'$ precisely when $a_{k_0} < a'_{k_0}$ for the largest index k_0 with $a_{k_0} \neq a'_{k_0}$.

Arithmetic. The algorithms for symbolically performing addition and multiplication of real numbers are cumbersome. Gowers suggests that the simplest approach to turn Stevin's b-base expansions into a construction of the reals is by employing limiting arguments to define addition and multiplication.

2.2. Weierstrass's construction (unpublished by Weierstrass, paraphrased following [31, ca. 1860]).

Observation. Every positive real number a can be written as

$$a = \sum_{s \in S} s,$$

where S is a multiset, i.e., a set where elements may be repeated more than once, whose elements consist of positive integers and positive rationals of the form $1/n$. Such a presentation is, of course, not unique.

The reals. Consider the set $\mathbb{S}$ of all non-empty multisets S of positive integers and positive rationals of the form $1/n$, which are *bounded* in the sense that there exists $M > 0$ with

$$\sum_{s \in S_0} s < M$$

for all finite submultisets $S_0 \subseteq S$ (being finite means that the total number of elements in S_0, counting multiplicities, is finite). Declare for two multisets $S, T \in \mathbb{S}$ that $S \leq T$ if for every finite submultiset $S_0 \subseteq S$ there exists a finite submultiset $T_0 \subseteq T$ with

$$\sum_{s \in S_0} s \leq \sum_{t \in T_0} t.$$

Declare $S \sim T$ if both $S \leq T$ and $T \leq S$ hold. The set $\mathbb{R}_+$ of positive real numbers is then defined to be $\mathbb{S}/\sim$.

Order. For two real numbers $a = [S]$ and $b = [T]$, the relation $a \leq b$ holds when $S \leq T$.

Arithmetic. Addition and multiplication of positive integers and of positive rationals of the form $1/n$ extends to $\mathbb{S}$ by $S + T = \{s + t \mid s \in S, t \in T\}$ and $ST = \{st \mid s \in S, t \in T\}$, subject to the convention that multiplicities are taken into consideration and that any sum or product which is not of the form of an integer or $1/n$ is replaced by a (necessarily) finite number of elements of this form (there is no canonical choice, and any would do).

The addition and multiplication of the real numbers $a = [S]$ and $b = [T]$ is given by

$$a + b = [S + T]$$

and

$$ab = [ST].$$

Remark 2.2. As noted, the construction lacks in rigor.

2.3. Dedekind's construction ([10, 1872]).

Observation. Any real number a determines a partition of $\mathbb{Q}$ into a pair (A, B) where $A = \{q \in \mathbb{Q} \mid q < a\}$ and $B = \{q \in \mathbb{Q} \mid a \leq q\}$. Obviously, A is non-empty and downward closed, B is non-empty and upward closed, and A has no greatest element. Any partition of $\mathbb{Q}$ satisfying these properties is called a *Dedekind cut*, and this construction is a bijection between the real numbers and the set of all cuts.

The reals. The set $\mathbb{R}$ of real numbers is defined to be the set of all Dedekind cuts.

Order. Given real numbers $a_1 = (A_1, B_1)$ and $a_2 = (A_2, B_2)$, the relation $a_1 \leq a_2$ holds precisely when $A_1 \subseteq A_2$.

Arithmetic. Obviously, in any Dedekind cut (A, B), any one of A or B determines the other and, if $A \subsetneq \mathbb{Q}$ satisfies the properties of the left 'half' of a Dedekind cut, then $(A, \mathbb{Q} \setminus A)$ is a Dedekind cut. It thus suffices to concentrate on A. Addition of real numbers given by Dedekind cuts represented by sets A_1 and A_2 is defined by

$$A_1 + A_2 = \{a_1 + a_2 \mid a_1 \in A_1, a_2 \in A_2\}.$$

If A_1 and A_2 represent non-negative reals, then their product is given by

$$A_1 A_2 = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{a_1 a_2 \mid a_1 \in A_1, a_2 \in A_2, a_1 \geq 0, a_2 \geq 0\}.$$

Multiplication is then extended by sign cases as usual.

2.4. Cantor's construction ([7, 1873]).

Observation. Every real number a is the limit of a sequence (q_n) of rationals. Moreover, any two convergent sequences (q_n) and (q'_n) converge to the same value a if, and only if, $|q_n - q'_n| \xrightarrow[n\to\infty]{} 0$.

The reals. Declare a sequence (q_n) of rationals to be a *Cauchy sequence* if, for all $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ with $|q_n - q_m| < \varepsilon$, provided that $n, m > k_0$. Let C be the set of all Cauchy sequences of rational numbers, and let $\sim$ be the equivalence relation on C given by $(q_n) \sim (q'_n)$ precisely when $|q_n - q'_n| \to 0$. The set of real numbers is then $\mathbb{R} = C/\!\sim$, the set of all equivalence classes of Cauchy sequences modulo $\sim$.

Order. Declare that two real numbers $a = [(q_n)]$ and $b = [(q'_n)]$ satisfy $a < b$ when $a \neq b$ and when there exists $k_0 \in \mathbb{N}$ with $q_n < q'_n$ for all $n > k_0$.

Arithmetic. Addition and multiplication are given by $a + b = [(q_n + q'_n)]$ and $ab = [(q_n q'_n)]$, respectively.

2.5. Bachmann's construction ([3, 1892]). The details below are essentially identical to those given by Bachmann, but the style is slightly adapted.

Observation. A sequence $\{I_n\}_{n\geq 1}$ of intervals $I_n = [a_n, c_n]$ in the real line is said to be a *nested family of intervals*, or more simply a *nest* if $I_{k+1} \subseteq I_k$ for all $k \geq 1$ and $c_n - a_n \xrightarrow[n\to\infty]{} 0$. For each such nest there is then a unique real number b satisfying $b \in I_k$ for all $k \geq 1$. Moreover, two nests determine in this way the same real number if, and only if, the nests admit a common refinement. In more detail, a nest $\{I_n\}$ is *finer* than a nest $\{J_n\}$ when $I_n \subseteq J_n$, for all $n \geq 1$. Two nests have a common refinement if there is a nest finer than each of them. Due to the density of the rational numbers in the real numbers the intervals above can be replaced by rational intervals consisting of rational numbers only, while retaining the correspondence with the reals.

The reals. Consider now *rational intervals* of the form $I = [a, c] = \{x \in \mathbb{Q} \mid a \leq x \leq c\}$, where $a, c \in \mathbb{Q}$. A *rational nest* is a family $\{I_n\}_{n\geq 1}$ of rational intervals $I_n = [a_n, c_n]$ satisfying $I_{k+1} \subseteq I_k$ for all $k \geq 1$ and $c_n - a_n \xrightarrow[n\to\infty]{} 0$. A rational nest $\{I_n\}$ is *finer* than a rational nest $\{J_n\}$ if $I_n \subseteq J_n$ for all $n \geq 1$. Consider now the set N of all rational nests, and define on it the relation $\sim$ whereby $\{I_n\} \sim \{I'_n\}$ precisely when there exists a common refinement of $\{I_n\}$ and $\{I'_n\}$. It follows easily that $\sim$ is an equivalence relation on N and the set $\mathbb{R}$ of real numbers is defined to be $N/{\sim}$, the set of equivalence classes of rational nests.

Order. Two real numbers $x = [\{I_n\}]$ and $y = [\{J_n\}]$ satisfy $x < y$ precisely when there exists $n_0 \in \mathbb{N}$ with $I_{n_0} < J_{n_0}$ in the sense that $\alpha < \beta$ for all $\alpha \in I_{n_0}$ and all $\beta \in J_{n_0}$.

Arithmetic. Extending the arithmetic operations of addition and multiplication of rational numbers to subsets S, T of rational numbers by means of $S+T = \{s+t \mid s \in S, t \in T\}$ and $ST = \{st \mid s \in S, t \in T\}$, it is easily seen that, for all rational intervals I and J, both $I + J$ and IJ are again rational intervals. Addition and multiplication of the real numbers x and y is given by $x + y = [\{I_n + J_n\}]$ and $xy = [\{I_n J_n\}]$.

2.6. Bourbaki's approach to the reals ([5, ca. 1960]**).** Bourbaki develops the general machinery of uniform spaces and their completion, observes that the rationals admit a uniform structure, and takes $\mathbb{R}$ to be any completion of $\mathbb{Q}$. The structure of $\mathbb{R}$ as a complete ordered field is then deduced using the machinery of uniform spaces. Strictly speaking, then, Bourbaki does not construct the reals, and in fact stresses the point that no particular construction is required; the universal properties, provided by any completion, suffice. However, Bourbaki also (famously) discusses a particular completion process of any uniform space (initially by means of equivalence classes of Cauchy filters, with the canonical choice of minimal Cauchy filters later on). Bourbaki's constructions can be combined and expanded into a particular construction of the reals, which we thus refer to as the Bourbaki reals, and we cast them into the mold of the survey. The details of this construction will be given in a separate article.

Observation. Any real number x gives rise to two filters, namely, the principal filter $\langle x \rangle = \{S \subseteq \mathbb{R} \mid x \in S\}$ and the minimal Cauchy filter $\iota(x)$ generated by all intervals containing x. Each construction leads to a bijective correspondence between $\mathbb{R}$ and a certain set of filters on $\mathbb{R}$, but the latter can be used to obtain a bijection between $\mathbb{R}$ and the set of all minimal Cauchy filters on $\mathbb{Q}$ by means of simply intersecting each set in $\iota(x)$ with $\mathbb{Q}$.

The reals. The set $\mathbb{R}$ of real numbers is defined to be the set of all minimal Cauchy filters on $\mathbb{Q}$.

Order. The order relation on $\mathbb{Q}$ extends universally to $\mathcal{P}(\mathbb{Q})$ by declaring $A <_\forall B$ when for all $a \in A$ and $b \in B$ one has $a < b$. Similarly, the relation $<_\forall$ on $\mathcal{P}(\mathbb{Q})$ extends existentially to $\mathcal{P}(\mathcal{P}(\mathbb{Q}))$ by declaring $\mathcal{A} <_{\exists\forall} \mathcal{B}$ when there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $A <_\forall B$. None of these extensions is an order relation. However, since any filter on $\mathbb{Q}$ is an element in $\mathcal{P}(\mathcal{P}(\mathbb{Q}))$, the relation $<_{\exists\forall}$ restricts to a relation on $\mathbb{R}$, and this relation is an order relation.

Arithmetic. Addition and multiplication in $\mathbb{Q}$ extend element-wise to subsets $A, B \subseteq \mathbb{Q}$. Given real numbers x and y, i.e., minimal Cauchy filters on $\mathbb{Q}$, their sum is $x + y = \langle\{A + B \mid A \in a, B \in b\}\rangle$, and their product is $xy = \langle\{AB \mid A \in a, B \in b\}\rangle$. It should be noted that the fact that $x+y$ and xy are real numbers, i.e., that the defining generated filters are *minimal*, is not a triviality but rather a fact that encapsulates most of the technical details in the construction, rendering the rest of the proof quite straightforward.

Remark 2.3. This construction can be seen as Bachmann's construction (subsection 2.5) with a canonical choice of representatives. A direct comparison from Bachmann's reals to the Bourbaki reals is given by sending a nest of intervals to the roundification of the filter generated by the intervals.

2.7. Maier-Maier's construction by a variation on Dedekind cuts ([24, 1973]).

Observation. Every real number a occurs as the greatest lower bound of the set $\{q \in \mathbb{Q} \mid a < q\}$. Of course, the same real number is the greatest lower bound of many other subsets of $\mathbb{Q}$. However, two bounded below sets $T_1, T_2 \subseteq \mathbb{Q}$ have the same greatest lower bound provided that the set of lower bounds of T_1 coincides with the set of lower bounds of T_2.

The reals. Let B be the set of all subsets of $\mathbb{Q}$ which are bounded below and denote, for $T \in B$, by $b(T)$ the set of all lower bounds of T. Given $T_1, T_2 \in B$, declare that $T_1 \sim T_2$ precisely when $b(T_1) = b(T_2)$. It is easily seen that $\sim$ is an equivalence relation, and the real numbers are defined to be $B/{\sim}$, the set of equivalence classes.

Order. Given real numbers $x = [S]$ and $y = [T]$, the relation $x < y$ holds precisely when $b(S) \subset b(T)$.

Arithmetic. For real numbers x and y, their sum is given by $x + y = [\{s + t \mid s \in S, t \in T\}]$. The product of x and y, provided that all the elements in S and in T are positive, is given by $xy = [\{st \mid s \in S, t \in T\}]$. Multiplication is extended to all real numbers by sign considerations.

Remark 2.4. This construction is essentially Dedekind's construction without canonical choices of representatives. In more detail, given a real number $x = [S]$ the set $b(S)$, if it does not have a maximum determines a Dedekind cut, denoted by $b_0(x)$. If $b(S)$ does have a maximum, then $b(S) \setminus \{\max b(S)\}$ determines a Dedekind cut, again denoted by $b_0(x)$. The function $x \mapsto b_0(x)$ is then an isomorphism giving a direct comparison between Dedekind's construction and the present construction.

2.8. Shiu's construction by infinite series ([27, 1974]).

Observation. Since the harmonic series $\sum_n 1/n$ diverges, it follows that every positive real number x can be written as

$$x = \sum_{n \in A} \frac{1}{n}$$

for some (non-unique) infinite set $A \subseteq \mathbb{N}$.

The reals. Let α be the set of all infinite subsets of natural numbers, writing $A = (a_k)$ with $a_k < a_{k+1}$ for a typical element in α. For such an $A \in \alpha$, let

$$A_n = \sum_{k=1}^{n} \frac{1}{a_k}.$$

Let β be the subset of α consisting of those $A \in \alpha$ for which the sequence (A_n) is bounded. Introduce an equivalence relation on β by declaring $A \sim B$ precisely when $(A_n - B_n)$ is a null sequence. The positive real numbers are then defined to be $\mathbb{R}_+ = \beta/\sim$, the set of equivalence classes.

Order. Real numbers x and y satisfy $x \leq y$ precisely when $x = [A]$ and $y = [B]$ for some representatives satisfying $A \subseteq B$.

Arithmetic. Let $x = [A]$ and $y = [B]$ be positive real numbers. Consider the set $AB = \{ab \mid a \in A, b \in B\}$. Call the representing sets A and B *excellent* if $A \cap B = \emptyset$ and every $c \in AB$ can be written uniquely as $c = ab$ with $a \in A$ and $b \in B$. Heuristically,

$$x = \sum_{a \in A} \frac{1}{a}$$

and

$$y = \sum_{b \in B} \frac{1}{b}$$

so, since the representatives are excellent, it follows that both $A \cup B$ and AB represent real numbers, which intuitively are

$$\sum_{c \in A \cup B} \frac{1}{c} = \sum_{a \in A} \frac{1}{a} + \sum_{b \in B} \frac{1}{b} = x + y$$

and

$$\sum_{c\in AB}\frac{1}{c}=\Big(\sum_{a\in A}\frac{1}{a}\Big)\Big(\sum_{b\in B}\frac{1}{b}\Big)=xy.$$

This informal argument turns into a definition of addition and multiplication on representatives by the fact that excellent representatives can always be found.

Remark 2.5. With suitable adaptation the harmonic series can be replaced by other divergent series of positive rationals converging to 0.

Remark 2.6. This construction is very similar to Weierstrass's (subsection 2.2). Here repetitions are not allowed, resulting in a simpler definition of the set of real numbers at the cost of a slightly less immediate notion of addition and multiplication. Weierstrass allows repetitions, and thus arithmetic is immediate, but identifying the set of real numbers requires a delicate notion of equivalence.

2.9. Faltin-Metropolis-Ross-Rota's wreath construction ([13, 1975]).

Observation. The difficulty in defining the arithmetic operations when defining the reals as sequences of base b expansions lies in the need to keep track of carries. This necessity stems from the (almost) uniqueness of the digits of any given real number, resulting from the use of the base to limit the range of the digits. Instead, one may not place a limit on the digits, i.e., every real number a can be written in infinitely many ways as

$$a=\sum_{n\in\mathbb{Z}}\frac{a_n}{2^n},$$

where the a_n are integers, all of which are 0 for sufficiently small n (the base is taken to be $b=2$ only to conform with the construction in the mentioned article). The definition of addition and multiplication of such expansions is formally identical to the way one would add and multiply formal Laurent series, at the price of an algorithmically more intricate recovery of the order structure by manipulating carries. Interestingly, this exchange in algorithmic complexity between arithmetic and order results in a much simpler construction of the real numbers than Stevin's construction.

The reals. Let $\Sigma(\mathbb{Z})$ be the set of all formal expressions of the form

$$\sum_{n\in\mathbb{Z}} a_n x^n,$$

where x is an indeterminate, $a_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}$, and $a_k = 0$ for all $k < k_0$, for some $k_0 \in \mathbb{Z}$. With the formal operations of addition and multiplication of Laurent series the set $\Sigma(\mathbb{Z})$ becomes a ring, whose elements are also called *strings*. More explicitly,

$$\sum_{n\in\mathbb{Z}} a_n x^n + \sum_{n\in\mathbb{Z}} b_n x^n = \sum_{n\in\mathbb{Z}} (a_n + b_n) x^n$$

and

$$\left(\sum_{n\in\mathbb{Z}} a_n x^n\right)\left(\sum_{n\in\mathbb{Z}} b_n x^n\right) = \sum_{n\in\mathbb{Z}} c_n x^n,$$

where

$$c_k = \sum_{n\in\mathbb{Z}} a_n b_{k-n},$$

for all $k \in \mathbb{Z}$. The element $K \in \Sigma(\mathbb{Z})$ with $k_0 = 1$, $k_1 = -2$, and $k_i = 0$ for all $i \in \mathbb{Z} \setminus \{0, 1\}$ is called the *carry constant*. It is easily seen that two elements $A, B \in \Sigma(\mathbb{Z})$ are related by $A = B + KC$, where $C \in \Sigma(\mathbb{Z})$ has only finitely many non-zero coefficients, precisely when A can be obtained from B by formally performing carrying operations as indicated by C (in base 2). An element $A \in \Sigma(\mathbb{Z})$ is said to be *bounded* if there exists an integer $z \geq 1$ such that

$$\sum_{i\leq n} |a_i| 2^{n-i} \leq z 2^n.$$

for all non-negative n. The set of all bounded elements in $\Sigma(\mathbb{Z})$ is denoted by $\Sigma_2(\mathbb{Z})$. An element $C \in \Sigma(\mathbb{Z})$ is called a *carry string* if KC is bounded, and when for every positive integer z there exists $k \geq 0$ with

$$z|c_j| \leq 2^j$$

for all $j > k$. Finally, two bounded elements $A, B \in \Sigma_2(\mathbb{Z})$ are declared to be equivalent if there exists a carry string C with $A = B + KC$. The set $\mathbb{R}$ of real numbers is then $\Sigma_2(\mathbb{Z})/{\sim}$, the set of equivalence classes of formal carry-free binary expansions modulo the performance of carrying.

Arithmetic. The ring structure on $\Sigma(\mathbb{Z})$ restricts to one on $\Sigma_2(\mathbb{Z})$ and is compatible with the equivalence relation $A = B + KC$ and thus gives rise to addition and multiplication in $\mathbb{R}$, namely, the usual addition and multiplication of formal Laurent series performed on representatives.

Order. For the definition of a *clear string* refer to [**13,** Section 6]. For every real number $[A]$ there exists a unique clear string B with $[A] = [B]$. Declare $[A]$ to be *positive* if, when cleared, the leading digit (i.e., leading non-zero coefficient) is 1. Then the set of positive reals defines an ordering in the usual manner. Equivalently, $[A] \le [B]$ is the lexicographic order on the cleared strings representing $[A]$ and $[B]$.

2.10. De Bruijn's construction by additive expansions ([9, 1976]).

Observation. As noted in subsection 2.1, the set of real numbers can be identified with formal decimal expansions (or other bases), i.e., as certain strings of digits indexed by the integers. The difficulty of performing the arithmetical operations (and even just addition) directly on the strings of digits stems in some sense from the expansions arising in complete disregard of the arithmetical operations; the expansions are analytic, not algebraic. If, instead, one considered the set of formal expansions with the aim of focusing on easily defining addition, then one is led to interpret the expansions differently. This is the approach taken in this construction.

The reals. Fix an integer $b > 1$, and let Σ be the set of all functions $f : \mathbb{Z} \to \{0, 1, 2, \ldots b-1\}$ which satisfy the condition that, for all $i \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ with $k > i$ and $f(k) < b-1$. Given any two functions $f, g \colon \mathbb{Z} \to \{0, 1, 2, \ldots, b-1\}$, define two other such functions, denoted by $\operatorname{difcar}(f, g)$ (standing for the difference carry of f and g) and $f - g$, as follows. For $k \in \mathbb{Z}$, define $\operatorname{difcar}(f, g)(k) = 1$ if there exists $x \in \mathbb{Z}$ with $x > k$, $f(x) < g(x)$, and such that $f(y) \le g(y)$ for all $k < y < x$. In all other cases, $\operatorname{difcar}(f, g)(k) = 0$. The value of $f - g$ at $k \in \mathbb{Z}$ is given by

$$(f - g)(k) = f(x) - g(x) - \operatorname{difcar}(f, g),$$

computed $\mod b$. Following this procedure leads to defining $f \in \Sigma$ to be positive if $f \ne 0$ and if some $k \in \mathbb{Z}$ exists with $f(y) = 0$ for all $y < k$. Similarly, declare $f \in \Sigma$ to be negative if there exists $k \in \mathbb{Z}$

with $f(y) = b-1$ for all $y < k$. Then the set of real numbers is defined to be the set of all $f \in \Sigma$ such that either $f = 0$, f is positive, or f is negative.

Order. For real numbers f and g, the relation $f < g$ holds precisely when $g - f$ is positive. The greatest lower bound property is then verified, allowing for limit-like arguments used only when defining the product of real numbers.

Arithmetic. Addition of real numbers is given by $f + g = f - (0 - g)$. Multiplication is defined as a supremum over suitably constructed approximations.

2.11. Rieger's construction by continued fractions ([26, 1982]).

Observation. Every irrational real number a can be written uniquely as a continued fraction

$$a = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} = [a_0; a_1, a_2, \ldots, a_k, \ldots]$$

where $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N}$ with $a_k \geq 1$ for all $k \geq 1$. When a is rational, the continued fraction terminates at some $k_0 \geq 0$, and if one further demands that if $k_0 > 0$, then $a_{k_0} > 1$, then the presentation of rational numbers is also unique.

The reals. Let $\mathbb{R}$ be the set of all sequences $[a_0; a_1, a_2, \ldots, a_k, \ldots]$ where $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N}^+$ with $a_k \geq 1$ for all $k \geq 1$, subject to the demand that, if $a_k = \omega$, then $a_t = \omega$ for all $t > k$ and if k_0 is the last index where $a_{k_0} \neq \omega$ and $k_0 > 0$, then $a_{k_0} > 1$.

Order. Given real numbers $a = [a_0; a_1, a_2, \ldots, a_k, \ldots]$ and $b = [b_0; b_1, b_2, \ldots, b_k, \ldots]$ the relation $a < b$ holds precisely when $a \neq b$ and when for the smallest index k_0 with $a_{k_0} \neq b_{k_0}$ one has

- $a_{k_0} < b_{k_0}$, if k is even;
- $a_{k_0} > b_{k_0}$, if k is odd.

The least upper bound property of $\mathbb{R}$ is then established and the proof of the Euclidean algorithm produces an order embedding $\mathbb{Q} \to \mathbb{R}$, which thus serves to identify the rationals in $\mathbb{R}$ as precisely those real numbers in which ω appears. It then follows that every real number $a = [a_0; a_1, a_2, \ldots, a_k, \ldots]$ can be approximated by suitably constructed rationals to obtain

$$a^{(0)} < a^{(2)} < a^{(4)} < \cdots < a < \cdots < a^{(5)} < a^{(3)} < a^{(1)}.$$

Arithmetic. The sum of a and b is defined to be

$$a + b = \sup\{a^{(2n)} + b^{(2n)} \mid n \geq 0\}.$$

Multiplication of positive real numbers is given by

$$ab = \sup\{a^{(2n)} b^{(2n)} \mid n \geq 0\}$$

and extended to all of $\mathbb{R}$ by the usual sign conventions. The proofs of the algebraic properties utilize the rational approximations using limit-like arguments.

2.12. Schanuel (et al.)'s construction using approximate endomorphisms of $\mathbb{Z}$ ([2, 11, 16, 29, 30, 1985]).

Observation. Given a real number a, the function $f_a \colon \mathbb{R} \to \mathbb{R}$ given by $f_a(x) = ax$ is a linear function whose slope is a, and the assignment $a \mapsto f_a$ thus sets up a bijection between the real numbers and linear operators $\mathbb{R} \to \mathbb{R}$. Under this bijection, addition in $\mathbb{R}$ corresponds to the point-wise addition of functions, while multiplication in $\mathbb{R}$ corresponds to composition of functions.

Of course, this point of view of the real numbers as linear operators (thought of as slopes) requires the existence of the real numbers for the operators to operate on. Thus, in order to obtain a construction of the reals, one seeks to modify f_a to a linear operator on $\mathbb{Z}$ instead of on $\mathbb{R}$. Restricting the domain of f_a to $\mathbb{Z}$ does not produce a function $f_a \colon \mathbb{Z} \to \mathbb{Z}$ (unless a is an integer), and it is tempting to simply adjust $f_a(x)$ to an integer near ax so as to obtain a function $g_a \colon \mathbb{Z} \to \mathbb{Z}$. Of course, this new function need not be linear anymore, but it is approximately so in the sense that, though the choice of g_a may be somewhat arbitrary, as long as the adjustment to an integer was not too out of hand, the set

$$\{g_a(x + y) - g_a(x) - g_a(y) \mid x, y \in \mathbb{Z}\},$$

which measures how non-linear g_a is, is finite. Furthermore, while it is obvious that different g_a, g'_a may arise from the same f_a, for sensible processes leading to a g_a and g'_a, the set

$$\{g_a(x) - g'_a(x) \mid x \in \mathbb{Z}\},$$

measuring how different g_a and g'_a are, is finite. Further motivation is gathered from the observation that defining $g_a(x) = [ax]$, the integer part of ax, is a function with the property that $g_a(x)/x \xrightarrow[x\to\infty]{} a$, so there is at least one obvious way of adjusting the linear function f_a to an approximately linear function from which a can be extracted.

The reals. Let $\mathbb{Z}$ be the integers considered as a group under addition. Call a function $f\colon \mathbb{Z} \to \mathbb{Z}$ a *quasihomomorphism* if the set

$$\{f(x+y) - f(x) - f(y) \mid x, y \in \mathbb{Z}\}$$

is finite. Introduce an equivalence relation on the set H of all quasihomomorphisms whereby $f \sim g$ precisely when the set

$$\{f(x) - g(x) \mid x \in \mathbb{Z}\}$$

is finite. The real numbers are then defined to be $H/{\sim}$.

Arithmetic. Given real numbers $a = [f]$ and $b = [g]$, their sum is represented by $f + g\colon \mathbb{Z} \to \mathbb{Z}$, where $(f + g)(x) = f(x) + g(x)$. The product ab is represented by $f \circ g\colon \mathbb{Z} \to \mathbb{Z}$, the composition of f and g.

Order. It can be shown that, for any quasihomomorphism $f\colon \mathbb{Z} \to \mathbb{Z}$, precisely one of the conditions

- f has bounded range;
- for all $C > 0$, there exists $n_0 \in \mathbb{N}$ with $f(x) > C$ for all $x > n_0$;
- for all $C > 0$, there exists $n_0 \in \mathbb{N}$ with $f(x) < -C$ for all $x > n_0$

holds. A real number $a = [f]$ is said to be positive if the second condition holds for f. For all real numbers b and c it is said that $b < c$ precisely when $c - b$ is positive.

2.13. Knopfmacher-Knopfmacher's construction using Cantor's theorem ([21, 1987]).

Observation (Cantor). Every real number $a > 1$ can be written uniquely as

$$a = \prod_{k=1}^{\infty} \left(1 + \frac{1}{a_k}\right) = [a_1, \ldots, a_k, \ldots],$$

where $a_k > 0$ is an integer for all $k \geq 1$ with $a_k > 1$ from some point onwards, and further $a_{k+1} \geq a_k^2$ for all $k \geq 1$. The number a is rational if, and only if, $a_{k+1} = a_k^2$ for all $k \geq k_0$, for some $k_0 \geq 1$. Every real number $0 < b < 1$ can be written uniquely as

$$b = \prod_{k=1}^{\infty} \left(1 - \frac{1}{b_k}\right) = [-b_1, \ldots, -b_k, \ldots],$$

where $b_k > 1$ is an integer for all $k \geq 1$ with $b_k > 2$ from some point onwards, and further $b_{k+1} > (b_k - 1)^2$. The real number b is rational if, and only if, $b_{k+1} = 1 + (b_k - 1)^2$ for all $k \geq k_0$, for some $k_0 \geq 1$.

The reals. Let S_0 be the set of all sequences $[-b_1, \ldots, -b_k, \ldots]$ of negative integers $-b_k < -1$, not all equal to -2, and such that $b_{k+1} > (b_k - 1)^2$ for all $k \geq 1$. Let S_1 be the set of all sequences $[a_1, \ldots, a_k, \ldots]$ of positive integers $a_k \geq 1$, not all equal to 1, and such that $a_{k+1} \geq a_k^2$ for all $k \geq 1$. The set of non-negative real numbers is then $\mathbb{R}_+ = S_0 \cup \{1\} \cup S_1$.

Order. For real numbers $a = [a_1, \ldots, a_k, \ldots]$, $c = [c_1, \ldots, c_k, \ldots] \in S_1$ the relation $a < c$ holds precisely when for the first index k_0 where $a_{k_0} \neq c_{k_0}$ one has $a_{k_0} > b_{k_0}$. The ordering among real numbers in S_0 is best seen by first introducing the bijection $[a_1, \ldots, a_k, \ldots] \leftrightarrow [-b_1, \ldots, -b_k, \ldots]$, where $b_k = a_k + 1$, which is in fact the reciprocal correspondence $x \mapsto x^{-1}$. Then $a < c$ holds for real numbers $a, c \in S_0$ precisely when $c^{-1} > a^{-1}$ holds in S_1. Lastly, $a > 1 > b$ holds for all $a \in S_1$ and $b \in S_0$. The least upper bound property for $\mathbb{R}$ is then proven.

Arithmetic. The proof of Cantor's theorem yields an embedding of $\mathbb{Q}_+$ in $\mathbb{R}_+$ and further one obtains for every positive real number a, by properly truncating it, sequences $a_{(n)}$ and $a^{(n)}$ in $\mathbb{R}_+$ of rational numbers which approximate a from below and from above, respectively. Addition of positive real numbers a and b is then given by

$$a + b = \sup\{a_{(n)} + b_{(n)} \mid n \geq 1\},$$

and their product is given by

$$ab = \sup\{a_{(n)}b_{(n)} \mid n \geq 1\}.$$

2.14. Pintilie's construction by infinite series ([25, 1988]).

Observation. With the same starting point as in Shiu's construction described in subsection 2.8, any sequence (a_n) of rational numbers with the properties that $a_n \to 0$ and $\sum a_n = \infty$ gives rise to a presentation of the positive reals, i.e., for every real number $a > 0$ there exists a subsequence (a_{n_k}) with

$$a = \sum_{k=1}^{\infty} a_{n_k},$$

albeit non-uniquely. In other words, if A is the set of all subsequences (a_{n_k}) such that

$$\sum_{k=1}^{\infty} a_{n_k} < \infty,$$

then $\mathbb{R}_+ \cong A$ as sets. To recover uniqueness, one may normalize the sequence (a_n) by demanding that $a_0 = 0$ and only consider those subsequences leading to a bounded series which further satisfy

$$n_{k+1} = \min\left\{p > n_k \mid \text{ there exists } m\colon \sum_{i=1}^{k} a_{n_i} + a_p < \sum_{i=1}^{m} a_{n_i}\right\}$$

for all $k \geq 0$.

The reals. Fix a sequence $(a_n)_{n\geq 1}$ of positive rational numbers, and set $a_0 = 0$. Let A be the set of all subsequences (a_{n_k}) leading to a convergent series $\sum a_{n_k}$. Define the positive real numbers to be the subset $\mathbb{R}_+ \subseteq A$ consisting only of those subsequences satisfying

$$n_{k+1} = \min\left\{p > n_k \mid \text{ there exists } m\colon \sum_{i=1}^{k} a_{n_i} + a_p < \sum_{i=1}^{m} a_{n_i}\right\}$$

for all $k \geq 0$.

Arithmetic. Given positive real numbers $b = (b_n)$ and $c = (c_n)$, their sum is given by the subsequence (a_{n_k}) determined, for all $k > 0$, by

$$n_{k+1} = \min\left\{ n > n_k \mid \text{ there exists } m \in \mathbb{N}\colon \sum_{i=1}^{k} a_{n_i} + a_n < \sum_{i=1}^{m} (b_i + c_i) \right\},$$

and similarly their product is the subsequence determined by the conditions

$$n_{k+1} = \min\left\{ n > n_k \mid \text{there exist } m \in \mathbb{N}\colon \sum_{i=1}^{k} a_{n_i} + a_n < \left(\sum_{i=1}^{m} b_i\right)\left(\sum_{i=1}^{m} c_i\right)\right\}$$

for all $k > 0$.

Order. For positive real numbers $b = (a_{b_k})$ and $c = (a_{c_k})$, the meaning of $b < c$ is that there exists $k \in \mathbb{N}$ with $b_k > c_k$ and $b_i = c_i$, for all $i < k$.

2.15. Knopfmacher-Knopfmacher's construction using Engel's theorem ([22, 1988]).

Observation (Engel). Every real number a can be written uniquely as

$$a = a_0 + \frac{1}{a_1} + \frac{1}{a_1 a_2} + \cdots + \frac{1}{a_1 a_2 \cdots a_n} + \cdots = (a_0, a_1, a_2, \ldots),$$

where the a_i are integers satisfying $a_{i+1} \geq a_i \geq 2$ for all $i \geq 1$.

The reals. Let $\mathbb{R}$ be the set of all infinite sequences $(a_0, a_1, a_2, \ldots)$ of integers satisfying $a_{k+1} \geq a_k \geq 2$, for all $k \geq 1$.

Order. Given real numbers $A = (a_0, a_1, a_2, \ldots)$ and $B = (b_0, b_1, b_2, \ldots)$, declare that $A < B$ precisely when

- $a_0 < b_0$, if $a_0 \neq b_0$, or
- $a_k > b_k$ for the first index $k \geq 1$ with $a_k \neq b_k$ otherwise.

The least upper bound property of $\mathbb{R}$ is then established, and the proof of Engel's theorem produces an order embedding $\mathbb{Q} \to \mathbb{R}$, which thus serves to identify the rationals in $\mathbb{R}$. It then follows that every real number can be approximated from above and from below by, respectively, sequences $A_{(n)}$ and $A^{(n)}$ of rationals.

Arithmetic. Addition and multiplication of real numbers A and B is given by exploiting the upper bound property of $\mathbb{R}$ and the rational approximations above. That is, the sum $A+B$ is given by

$$A+B=\sup\{A_{(n)}+B_{(n)}\},$$

and the product of positive reals is given by

$$AB=\sup\{A_{(n)}B_{(n)}\}$$

and extended to all of $\mathbb{R}$ as usual. The proofs of the algebraic properties utilize the rational approximations using limit-like arguments.

2.16. Knopfmacher-Knopfmacher's construction using Sylvester's theorem ([22, 1988]). The construction is formally identical to the one given in subsection 2.15 and will thus be presented quite briefly.

Observation (Sylvester). Every real number a can be written uniquely as

$$a=a_0+\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}+\cdots=((a_0,a_1,a_2,\ldots))$$

where the a_i are integers satisfying $a_1\geq 2$ and $a_{i+1}\geq a_i(a_i-1)+1$ for all $i\geq 1$.

The reals. Let $\mathbb{R}$ be the set of all infinite sequences $((a_0,a_1,a_2,\ldots))$ of integers satisfying $a_k\geq 2$ and $a_{k+1}\geq a_k(a_k-1)+1$, for all $k\geq 1$.

Order. Given real numbers $A=((a_0,a_1,a_2,\ldots))$ and $B=((b_0,b_1,b_2,\ldots))$, declare that $A<B$ precisely when

- $a_0<b_0$, if $a_0\neq b_0$, or
- $a_k>b_k$ for the first index $k\geq 1$ with $a_k\neq b_k$, otherwise.

The least upper bound property of $\mathbb{R}$ is then established and the proof of Sylvester's theorem produces an order embedding $\mathbb{Q}\to\mathbb{R}$, identifying the rationals in $\mathbb{R}$. It then follows that every real number can be approximated from above and from below by, respectively, sequences $A_{(n)}$ and $A^{(n)}$ of rationals.

Arithmetic. Formally identical to subsection 2.15.

Remark 2.7. A generalization of Sylvester's theorem, and consequently a generalization of this construction of the real numbers, is given, along essentially the same lines, in [**28**].

2.17. Knopfmacher-Knopfmacher's construction using the alternating Engel theorem ([23, 1989]).

Observation (alternating Engel). Every real number A can be written uniquely as

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \cdots + \frac{(-1)^{n+1}}{a_1 a_2 \cdots a_n} + \cdots ,$$

where the a_k are integers satisfying $a_{k+1} \geq a_k + 1 \geq 2$ for all $k \geq 1$. Furthermore, this representation terminates after a finite number of summands if, and only if, A is rational.

The reals. The set $\mathbb{R}$ of real numbers is defined to be the set of all infinite sequences $(a_0, a_1, a_2, \ldots)$ of elements in $\mathbb{N}^+$ which satisfy $a_0 \in \mathbb{N}$ and $a_{k+1} \geq a_k + 1 \geq 2$, for all $k \geq 1$.

Order. Two real numbers $A = (a_0, a_1, a_2, \ldots)$ and $B = (b_0, b_1, b_2, \ldots)$ satisfy $A < B$ precisely when $a_{2n} < b_{2n}$ or $a_{2n+1} > b_{2n+1}$, where the index $i = 2n$ or $i = 2n + 1$ is the first index with $a_i \neq b_i$. It is then shown that $\mathbb{R}$ satisfies the least upper bound property. The proof of the alternating Engel theorem produces an order embedding $\mathbb{Q} \to \mathbb{R}$, which thus serves to identify the rationals in $\mathbb{R}$. It then follows that every real number can be approximated from above and from below by, respectively, sequences $A_{(n)}$ and $A^{(n)}$ of rationals.

Arithmetic. Addition is defined by

$$A + B = \sup\{A^{(2n)} + B^{(2n)} \mid n \geq 0\},$$

and multiplication of positive reals is given by

$$AB = \sup\{A^{(2n)} B^{(2n)} \mid n \geq 0\}.$$

The rest of the construction is formally very similar to the one presented in subsection 2.15.

2.18. Knopfmacher-Knopfmacher's construction using the alternating Sylvester theorem ([23, 1989]).

Observation (alternating Sylvester). Every real number A can be written uniquely as

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3} - \cdots + \frac{(-1)^n}{a_n} + \cdots = ((a_0, a_1, a_2, \ldots)),$$

where the a_k are integers satisfying $a_1 \geq 1$ and $a_{k+1} \geq a_k(a_k + 1)$ for all $k \geq 1$. Furthermore, this representation terminates after a finite number of summands if, and only if, A is rational.

The reals. The set $\mathbb{R}$ of real numbers is defined to be the set of all infinite sequences $((a_0, a_1, a_2, \ldots))$ of elements in $\mathbb{N}^+$, which satisfy $a_0 \in \mathbb{N}$, $a_1 \geq 1$ and $a_{k+1} \geq a_k(a_k + 1)$, for all $k \geq 1$.

Order. Formally identical to subsection 2.17, except that it is the proof of the alternating Sylvester theorem that gives the identification of the rationals in $\mathbb{R}$.

Arithmetic. Formally identical to subsection 2.17.

Remark 2.8. In [**19**], a generalization of the alternating Sylvester theorem is presented along with a corresponding construction of the real numbers which is formally identical to this one.

2.19. Arthan's irrational construction ([1, 2001]).

Observation. A closer look at the construction of $\mathbb{R}$ as a completion of $\mathbb{Q}$ by means of Dedekind cuts (cf., 2.3) reveals what the crucial ingredients present in $\mathbb{Q}$ actually are that lead to the real numbers. In more detail, Dedekind's construction is well known to be a special case of the Dedekind-MacNeille completion of an ordered set, and a famous theorem of Cantor shows that the completion of any countable, unbounded and densely ordered set is order isomorphic to $\mathbb{R}$. Further, it is the archimedean property of $\mathbb{Q}$ that assures the additive structure on the completion has the right properties, and finally, by a theorem of Hölder, the completion of any ordered group which is dense and archimedean must be isomorphic to an additive subgroup of $\mathbb{R}$, and thus admits a multiplication.

It follows then that any countable, unbounded, archimedean and densely ordered group admits a completion isomorphic to $\mathbb{R}$ as a field. If the multiplicative structure can effectively be defined in terms of the given ordered group, then a construction of the reals emerges.

The reals. Given a dense, archimedean ordered commutative group, the reals are constructed as its Dedekind-MacNeille completion. Once such an ordered group is chosen, the details are essentially identical to those of Dedekind's construction, and thus we further concentrate on the presentation of the ordered group, namely, $\mathbb{Z}[\sqrt{2}]$, which formally we view as the set $\mathbb{Z} \times \mathbb{Z}$. Other rational numbers may be chosen, with more or less adverse effects on the desired properties of the group, the ease of establishing those properties and the implementability on a computer.

Arithmetic. Addition and multiplication in $\mathbb{Z}[\sqrt{2}]$ are easily established to be given by $(a,b)+(c,d) = (a+c, b+d)$ and $(a,b)(c,d) = (ac+2bd, ad+bc)$.

Order. Recovering the ordering on $\mathbb{Z}[\sqrt{2}]$ solely in terms of integers uses the fact that $\sqrt{2}$ is approximated by certain rational numbers whose numerators and denominators admit an efficient recursive formula. For details, see [**1,** subsection 5.2].

2.20. Notes on Conway's surreal numbers and nonstandard models. We conclude the survey by briefly mentioning two other venues leading to the real numbers which, however, do not quite fall into the same category as the constructions surveyed above.

2.20.1. Conway's surreal numbers ([8, 1976]**).** Conway's famous construction of the surreal numbers is a construction of a proper class in which every ordered field embeds. It thus follows that the surreal numbers contain a copy of the real numbers, and thus one may view the surreal number system as providing yet another construction of the real numbers. However, when one distills just the real numbers from the entire array of surreal ones, the construction basically collapses to the Dedekind cuts construction. The interest in the surreal numbers is not so much for the reals embedded in them, but rather for the far reaching extra numbers beyond the reals that occupy most of the surreal realm.

2.20.2. Nonstandard constructions. Since $\mathbb{R}$ is a completion of $\mathbb{Q}$, and since it is well known that the techniques of nonstandard analysis yield completions, any path towards nonstandard analysis is also a path to a definition of the real numbers. However, to what extent can these nonstandard definitions be seen as *constructions* of the real numbers is a delicate issue. Inseparable to the technique of enlargement, which is at the heart of nonstandard analysis, is the axiom of choice (or some slightly weaker variant) and therefore the objects produced are not particularly tangible. For that reason, we avoided including any details of nonstandard definitions of the real numbers. The interested reader is referred to [**14**] for a very detailed and carefully motivated exposition of one nonstandard model of the reals (the hyperreals) and to [**4**] for seven other possibilities.

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