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ELLIPTIC CURVES COMING FROM HERON TRIANGLES

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ABSTRACT. Triangles having rational sides a, b, c and rational area Q are called *Heron triangles*. Associated to each Heron triangle is the quartic

$$v^2 = u(u-a)(u-b)(u-c).$$

The Heron formula states that $Q = \sqrt{P(P-a)(P-b)(P-c)}$ where P is the semi-perimeter of the triangle, so the point $(u, v) = (P, Q)$ is a rational point on the quartic. Also, the point of infinity is on the quartic. By a standard construction, it can be proved that the quartic is equivalent to the elliptic curve

$$y^2 = (x + a\,b)(x + b\,c)(x + c\,a).$$

The point (P, Q) on the quartic transforms to

$$(x, y) = \left(\frac{-2abc}{a+b+c}, \frac{4Qabc}{(a+b+c)^2}\right)$$

on the cubic, and the point of infinity goes to $(0, abc)$. Both points are independent, so the family of curves induced by Heron triangles has rank ≥ 2. In this note we construct subfamilies of rank at least 3, 4 and 5. For the subfamily with rank ≥ 5, we show that its generic rank is exactly equal to 5, and we find free generators of the corresponding group. By specialization, we obtain examples of elliptic curves over $\mathbf{Q}$ with rank equal to 9 and 10. This is an improvement of results by Izadi et al., who found a subfamily with rank ≥ 3 and several examples of curves of rank 7 over $\mathbf{Q}$.

1. Triangles and elliptic curves.

1.1. Heron triangles.

Definition. A triangle with sides of rational lengths $\{a, b, c\}$ is called a Heron triangle if its area Q is also a rational number.

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The Heron formula states that the area Q of a triangle with sides $\{a, b, c\}$ is equal to

$$Q = \sqrt{P(P-a)(P-b)(P-c)}, \quad \text{where } P = \frac{a+b+c}{2}.$$

So, a triangle with sides a, b, c is a Heron triangle when

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$

is a rational square. The Indian mathematician, Brahmagupta, 598–668 A.D., showed that, for a triangle with integral sides $\{a, b, c\}$ and integral area Q, there are positive integers k, m, n, with $k^2 < m\,n$, such that

$$\begin{cases} a = n(k^2 + m^2), \\ b = m(k^2 + n^2), \\ c = (m+n)(mn - k^2), \\ Q = kmn(m+n)(mn - k^2). \end{cases}$$

Observe that the value of Q is a consequence of the Heron formula. A classical reference for Heron triangles is the second volume of the book by Dickson [**10**].

1.2. Elliptic curves associated to Heron triangles. Given a Heron triangle of sides $\{a, b, c\}$ and area Q, consider the quartic

$$v^2 = u(u-a)(u-b)(u-c). \tag{1}$$

The point $(u, v) = ((a+b+c)/2, Q)$ is on the quartic (1), due to the Heron formula. The point at infinity is also on the quartic. Now the change of coordinates

$$(u, v) \longrightarrow \left(-\, abc\frac{1}{x}, abc\frac{y}{x^2} \right)$$

transforms the quartic (1) into the cubic

$$y^2 = (x+ab)(x+bc)(x+ca), \tag{2}$$

and the two points mentioned above into

$$(x, y) = \left(\frac{-2abc}{a+b+c}, \frac{4Qabc}{(a+b+c)^2} \right)$$

and

$$(x, y) = (0, abc),$$

respectively.

The relation between elliptic curves and Heron triangles appears in the work of many authors in the mathematical literature. Let us mention some of them.

The quartics (1) have been used by Bremner in [**2**] in order to study the existence of sets of N Heron triangles with given perimeter and area, for a given positive integer N. Similar kind of problems have been treated in [**15, 16**].

The existence of infinitely many Heron triangles with a given area has been shown in [**20**]. This result is also obtained in [**12**], by exploiting properties of a family of elliptic curves which generalize the congruent number elliptic curves.

In [**3**] it is shown that there exists an infinite set of Heron triangles having two rational medians.

Elliptic curves of the shape (2) appear in a natural way in the study of elliptic curves induced by Diophantine m-tuples (see [**1, 6, 7, 8, 9**]), where the values a, b and c represent three components of a Diophantine triple, instead of the sides of a Heron triangle as in the present context.

In [**4**], the authors describe connections between the problem of finding Heron triangles with a given area and finding Diophantine quadruples, and they are led to study the relation of these problems with the elliptic curves over $\mathbf{Q}$ having rational torsion group equal to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$.

In [**14**] the authors show a family of Heron triangles whose associated elliptic curves have generic rank equal to 3, and they also exhibit particular examples of curves with rank equal to 7 over $\mathbf{Q}$. The purpose of this note is to improve their results. In fact, we have found families of elliptic curves induced by Heron triangles having rank ≥ 4 and ≥ 5. Using the algorithm from [**13**], we are able to show that the generic rank of the last family is equal to 5 and to find generators of its Mordell-Weil group. Furthermore, we have found particular examples of curves whose rank over $\mathbf{Q}$ is 9 and 10.

2. Search for higher rank.

2.1. Rank 2. As an initial step in our construction we transform the cubic (2) into the form

$$y^2 = x^3 + Ax^2 + Bx, \tag{3}$$

by $x \mapsto x - ab$. The coefficients A and B are

$$\begin{cases} A = -2ab + ac + bc, \\ B = ab(a-c)(b-c). \end{cases}$$

Now we insert the values of the Brahmagupta parametrization with $k = 1$. There is no loss of generality since k acts as a scaling factor. We get

$$\begin{aligned} A_2 &= -m^2 - 4mn - 2m^3n - n^2 + m^4n^2 - 2mn^3 + m^2n^4, \\ B_2 &= mn(1+m^2)(-2m - n + m^2n)(1+n^2)(-m - 2n + mn^2). \end{aligned}$$

The cubic $y^2 = x^3 + A_2x^2 + B_2x$ has rank ≥ 2 over $\mathbf{Q}(m, n)$. In terms of m and n, the X-coordinates of the two independent infinite order points are

$$\begin{aligned} X_1 &= mn(1+m^2)(1+n^2), \\ X_2 &= (1+m^2)(1+n^2). \end{aligned}$$

Observe that the condition $k^2 < m\,n$ becomes $1 < m\,n$.

Just a word of explanation on how we have found the conditions for new points. In every case we have curves with the shape $y^2 = x^3 + Ax^2 + Bx$, where the coefficient B is a polynomial expression in the parameters involved. Since B has several polynomial factors F_j, we look for new points in the homogenous spaces corresponding to each F_j, i.e., we search for conditions like

$$F_jU^4 + AU^2V^2 + \frac{B}{F_j}V^4$$

that can be converted into squares by an adequate choice of the parameters. In all the cases with $(U, V) = (1, 1)$, we have been able to impose a new point into the curve for the values of F_j quoted in each step below.

2.2. Rank 3. Now we impose that $(a+b-c)/2$ is the u-coordinate for a new point in the quartic, or equivalently that

$$-m\,n\,(1+m^2)(-2+mn)(1+n^2)$$

is the X-coordinate for a point on the cubic (3). This can be done with the substitution $m = 2/[n(1+w^2)]$. The subfamily corresponding to this specialization of the parameter has rank ≥ 3. After getting rid of denominators, the coefficients of the cubic are

$$\begin{aligned} A_3 &= -4 - 8n^2 - n^4 - 24w^2 - 24n^2w^2 - 8n^4w^2 - 4w^4 - 24n^2w^4 \\ &\quad - 14n^4w^4 - 8n^2w^6 - 8n^4w^6 - n^4w^8 \\ B_3 &= 4(1+n^2)(1+w^2)^2(1+n^2w^2)(4+n^2+2n^2w^2+n^2w^4) \\ &\quad (n^2+4w^2+2n^2w^2+n^2w^4). \end{aligned}$$

The X-coordinates of the three independent points are

$$\begin{aligned} X_1 &= 2(1+n^2)(1+w^2)(4+n^2+2n^2w^2+n^2w^4), \\ X_2 &= (1+n^2)(1+w^2)^2(4+n^2+2n^2w^2+n^2w^4) \\ X_3 &= 4(1+n^2)w^2(4+n^2+2n^2w^2+n^2w^4). \end{aligned}$$

The formal proof of independence will be given in subsection 2.5. The condition for the sides is $w^2 < 1$, since $mn > 1$ transforms into $mn = 2/(1+w^2) > 1$. The sides of the corresponding Heron triangle are:

$$\begin{aligned} a &= 4+n^2+2n^2w^2+n^2w^4, \\ b &= 2(1+n^2)(1+w^2), \\ c &= -(-1+w)(1+w)(2+n^2+n^2w^2). \end{aligned}$$

2.3. Another rank 3 family. We constructed the previous family of rank 3 by imposing $(a+b-c)/2$ as a new point on the quartic, which turns out to be equivalent to parametrizing a conic. In a similar way, we get the following family of rank at least 3. We impose $(a+b)/2$ as a new point on the quartic, which is equivalent to specializing $m = (-3-w^2)/[n(-1+w^2)]$. In this case the new family has the following coefficients:

$$\begin{aligned} A_{31} &= 2(9-6n^2+n^4+60w^2+16n^2w^2+4n^4w^2+46w^4 \\ &\quad -12n^2w^4-10n^4w^4+12w^6+4n^4w^6+w^8+2n^2w^8+n^4w^8), \\ B_{31} &= -(1+n^2)(-1+w)^2(1+w)^2(3+w^2)(3-n^2+k^2w^2-3n^2w^2) \\ &\quad \times (3-n^2+10w^2+2n^2w^2+3w^4-n^2w^4) \\ &\quad \times (9+n^2+6w^2-2n^2w^2+w^4+n^2w^4). \end{aligned}$$

The sides, for $w^2 < 1$, are

$$\begin{aligned} a &= 9+n^2+6w^2-2n^2w^2+w^4+n^2w^4, \\ b &= -(1+n^2)(-1+w)(1+w)(3+w^2), \\ c &= 2(1+w^2)(3+n^2+w^2-n^2w^2). \end{aligned}$$

2.4. Rank 4. We can force $(1+n^2)(1+w^2)^2(n^2+4w^2+2n^2w^2+n^2w^4)$ to became the X-coordinate for a new point on the cubic given by the coefficients $\{A_3, B_3\}$, solving $(3+w^2)(4-n^2+n^2w^2) =$ square. This can be realized by choosing

$$n = \frac{-3-6t+t^2+2w^2-2tw^2+w^4}{3+t^2-2w^2-w^4}.$$

The subfamily corresponding to this specialization of the parameters has rank ≥ 4. The X-coordinates of the four independent points are:

$$\begin{aligned} X_1 &= 4(1+w^2)(9+18t+18t^2-6t^3+t^4-12w^2-6tw^2+12t^2w^2 \\ &\quad -2t^3w^2-2w^4-10tw^4+2t^2w^4+4w^6-2tw^6+w^8) \\ &\qquad (45+36t+54t^2-12t^3+5t^4-42w^2+60tw^2+72t^2w^2-28t^3w^2 \\ &\quad +2t^4w^2-25w^4-8tw^4+84t^2w^4-20t^3w^4+t^4w^4+4w^6-56tw^6 \\ &\quad +40t^2w^6-4t^3w^6+11w^8-28tw^8+6t^2w^8+6w^{10}-4tw^{10}+w^{12}), \end{aligned}$$

$$\begin{aligned} X_2 &= 2(1+w^2)^2(9+18t+18t^2-6t^3+t^4-12w^2-6tw^2+12t^2w^2 \\ &\quad -2t^3w^2-2w^4-10tw^4+2t^2w^4+4w^6-2tw^6+w^8) \\ &\qquad (45+36t+54t^2-12t^3+5t^4-42w^2+60tw^2+72t^2w^2-28t^3w^2 \\ &\quad +2t^4w^2-25w^4-8tw^4+84t^2w^4-20t^3w^4+t^4w^4+4w^6-56tw^6 \\ &\quad +40t^2w^6-4t^3w^6+11w^8-28tw^8+6t^2w^8+6w^{10}-4tw^{10}+w^{12}), \end{aligned}$$

$$
\begin{aligned}
X_3 = 8w^2(&9+18t+18t^2-6t^3+t^4-12w^2-6tw^2+12t^2w^2-2t^3w^2\\
&- 2w^4 - 10tw^4 + 2t^2w^4 + 4w^6 - 2tw^6 + w^8)\\
&\quad(45+36t+54t^2-12t^3+5t^4-42w^2+60tw^2+72t^2w^2-28t^3w^2\\
&+2t^4w^2-25w^4-8tw^4+84t^2w^4-20t^3w^4+t^4w^4+4w^6-56tw^6\\
&+40t^2w^6-4t^3w^6+11w^8-28tw^8+6t^2w^8+6w^{10}-4tw^{10}+w^{12}),
\end{aligned}
$$

$$
\begin{aligned}
X_4 = 2(1+w^2)^2(&9+18t+18t^2-6t^3+t^4-12w^2-6tw^2+12t^2w^2-2t^3w^2\\
&- 2w^4 - 10tw^4 + 2t^2w^4 + 4w^6 - 2tw^6 + w^8)\\
&(9+36t+30t^2-12t^3+t^4+42w^2+60tw^2+112t^2w^2-28t^3w^2\\
&+6t^4w^2-65w^4-8tw^4+76t^2w^4-20t^3w^4+t^4w^4-20w^6-56tw^6\\
&+32t^2w^6-4t^3w^6+23w^8-28tw^8+6t^2w^8+10w^{10}-4tw^{10} + w^{12}).
\end{aligned}
$$

The sides are, for $w^2 < 1$,

$$
\begin{aligned}
a &= 45+36t+54t^2-12t^3+5t^4-42w^2+60tw^2+72t^2w^2-28t^3w^2+2t^4w^2\\
&\quad -25w^4-8tw^4+84t^2w^4-20t^3w^4+t^4w^4+4w^6-56tw^6+40t^2w^6\\
&\quad - 4t^3w^6 + 11w^8 - 28tw^8 + 6t^2w^8 + 6w^{10} - 4tw^{10} + w^{12},\\
b &= 4(1+w^2)(9+18t+18t^2-6t^3+t^4-12w^2-6tw^2+12t^2w^2-2t^3w^2\\
&\quad - 2w^4 - 10tw^4 + 2t^2w^4 + 4w^6 - 2tw^6 + w^8),\\
c &= (1-w)(1+w)(3+w^2)(9+12t+14t^2-4t^3+t^4-12w^2+4tw^2\\
&\quad + 12t^2w^2 - 4t^3w^2 - 2w^4 - 12tw^4 + 6t^2w^4 + 4w^6 - 4tw^6 + w^8).
\end{aligned}
$$

2.5. Rank 5. In our last step, we impose

$$
\begin{aligned}
X = 8(&9 + 18t + 18t^2 - 6t^3 + t^4 - 12w^2 - 6tw^2 + 12t^2w^2 - 2t^3w^2\\
&- 2w^4 - 10tw^4 + 2t^2w^4 + 4w^6 - 2tw^6 + w^8)\\
&\quad(9+36t+30t^2-12t^3+t^4+42w^2+60tw^2+112t^2w^2-28t^3w^2+6t^4w^2\\
&-65w^4-8tw^4+76t^2w^4-20t^3w^4+t^4w^4 20w^6-56tw^6+32t^2w^6\\
&- 4t^3w^6 + 23w^8 - 28tw^8 + 6t^2w^8 + 10w^{10} - 4tw^{10} + w^{12})
\end{aligned}
$$

as X-coordinate of a new point on the cubic of rank 4. This is equivalent to solving

$$
\text{square} = 12t + 8t^2 - 4t^3 + 9w^2 + 16tw^2
$$

$$
\begin{aligned}
&+ 30t^2w^2 - 8t^3w^2 + t^4w^2 - 12w^4 - 8tw^4 \\
&+ 20t^2w^4 - 4t^3w^4 - 2w^6 - 16tw^6 \\
&+ 6t^2w^6 + 4w^8 - 4tw^8 + w^{10}.
\end{aligned}
$$

This condition can be achieved with $t = (w^4 - 1)/2w^2$. The subfamily corresponding to this specialization of the parameter has rank equal to 5 over $\mathbf{Q}(w)$, as shown below. The X-coordinates of the five independent points are:

$$
\begin{aligned}
X_1 = {} & 4(1 + w^2)(1 + 14w^2 + 99w^4 + 52w^6 + 55w^8 + 30w^{10} + 5w^{12}) \\
& (5 + 36w^2 + 320w^4 + 564w^6 + 818w^8 \\
& + 300w^{10} + 8w^{12} - 4w^{14} + w^{16}),
\end{aligned}
$$

$$
\begin{aligned}
X_2 = {} & 2(1 + w^2)^2(1 + 14w^2 + 99w^4 + 52w^6 + 55w^8 + 30w^{10} + 5w^{12}) \\
& (5 + 36w^2 + 320w^4 + 564w^6 + 818w^8 \\
& + 300w^{10} + 8w^{12} - 4w^{14} + w^{16}),
\end{aligned}
$$

$$
\begin{aligned}
X_3 = {} & 8w^2(1 + 14w^2 + 99w^4 + 52w^6 + 55w^8 + 30w^{10} + 5w^{12}) \\
& (5 + 36w^2 + 320w^4 + 564w^6 + 818w^8 + 300w^{10} \\
& + 8w^{12} - 4w^{14} + w^{16}),
\end{aligned}
$$

$$
\begin{aligned}
X_4 = {} & 2(1 + w^2)^2(1 + 6w^2 + w^4) \\
& \times (1 + 26w^2 + 79w^4 + 44w^6 + 79w^8 + 26w^{10} + w^{12}) \\
& \times (1 + 14w^2 + 99w^4 + 52w^6 + 55w^8 + 30w^{10} + 5w^{12})
\end{aligned}
$$

$$
\begin{aligned}
X_5 = {} & 8(1 + 6w^2 + w^4) \\
& \times (1 + 26w^2 + 79w^4 + 44w^6 + 79w^8 + 26w^{10} + w^{12}) \\
& \times (1 + 14w^2 + 99w^4 + 52w^6 + 55w^8 + 30w^{10} + 5w^{12}).
\end{aligned}
$$

The coefficients of the cubic are:

$$
\begin{aligned}
A_5 = {} & -13 - 348w^2 - 4452w^4 - 35100w^6 - 202264w^8 \\
& - 697036w^{10} - 1414884w^{12} - 1913548w^{14} - 1779178w^{16} \\
& - 1349396w^{18} - 721420w^{20} - 227540w^{22} - 38768w^{24}
\end{aligned}
$$

$$
\begin{aligned}
&- 4036w^{26} - 556w^{28} - 68w^{30} - w^{32},\\
B_5 &= 8(1+w^2)^3(1+6w^2+w^4)\\
&\quad (1+2w^2+47w^4+156w^6+47w^8+2w^{10}+w^{12})\\
&\quad (1+26w^2+79w^4+44w^6+79w^8+26w^{10}+w^{12})\\
&\quad (1+14w^2+99w^4+52w^6+55w^8+30w^{10}+5w^{12})\\
&\quad (5+36w^2+320w^4+564w^6+818w^8+300w^{10}\\
&\qquad\qquad + 8w^{12} - 4w^{14} + w^{16}).
\end{aligned}
$$

The sides are, for $w^2 < 1$, as follows,

$$
\begin{aligned}
a &= 5+36w^2+320w^4+564w^6+818w^8\\
&\quad + 300w^{10} + 8w^{12} - 4w^{14} + w^{16},\\
b &= 4(1+w^2)(1+14w^2+99w^4\\
&\quad + 52w^6+55w^8+30w^{10}+5w^{12}),\\
c &= -(-1+w)(1+w)(3+w^2)\\
&\qquad (1-2w+7w^2+4w^3+7w^4-2w^5+w^6)\\
&\qquad (1+2w+7w^2-4w^3+7w^4+2w^5+w^6).
\end{aligned}
$$

By the Silverman specialization theorem **[21,** Theorem 11.4], in order to prove that the family of elliptic curves

$$
E : y^2 = x^3 + A_5(w)x^2 + B_5(w)x \tag{4}
$$

has rank ≥ 5 over $\mathbf{Q}(w)$, it suffices to find a specialization $w = w_0$ such that the points with X-coordinates $X_1(w_0), \ldots, X_5(w_0)$ are independent points on the specialized curve over $\mathbf{Q}$. Let us take $w = 2$. Then the points

$$
\begin{aligned}
&(829979290180, 709888756704565620),\\
&\quad (2074948225450, 2577272441349190 50),\\
&\quad (1327966864288, 824727181231740 96),\\
&\quad (7939152098050, 175990280826792589 50),\\
&\quad (2135602557625, 421478249567754750)
\end{aligned}
$$

are independent points of infinite order on the elliptic curve

$$(5) \qquad y^2 = x^3 - 3366916713149x^2 + 2712779764155114364021000x.$$

Indeed, the value of the discriminant of the canonical height matrix of these five points is $\approx 115940.98 \neq 0$. Let us mention that the rank of curve (5) is equal to 8.

Our next goal is to prove that the curve E given by (4) has rank over $\mathbf{Q}(w)$ exactly equal to 5, and moreover to find free generators of the group $E(\mathbf{Q}(w))$. We have noted experimentally that the points with the X-coordinates $X_1, \ldots, X_5$ do not generate the full group, but its subgroup of index 2. So we searched for other points on E such that X-coordinate divides B_5. In that way, we find the point on E with X-coordinate

$$X_5' = 2(w-1)^2(w^2+1)(w^4+6w^2+1)^3$$
$$(w^{16}-4w^{14}+8w^{12}+300w^{10}+818w^8+564w^6+320w^4+36w^2+5).$$

Now we use the algorithm by Gusić and Tadić [**13,** Theorem 3.1 and Corollary 3.2]. It is applicable to our situation since the curve E has three nontrivial 2-torsion points, i.e., the equation for E can be written in the form

$$y^2 = (x-e_1)(x-e_2)(x-e_3),$$

with $(e_1, e_2, e_3 \in \mathbf{Z}[w])$. Indeed,

$$\begin{aligned}
e_1 &= 0,\\
e_2 &= 40w^{30}+440w^{28}+3880w^{26}+28216w^{24}+126792w^{22}+345368w^{20}\\
&\quad +630984w^{18}+885528w^{16}+976760w^{14}+749608w^{12}+345464w^{10}\\
&\quad + 86184w^8 + 13080w^6 + 1800w^4 + 152w^2 + 8,\\
e_3 &= w^{32}+28w^{30}+116w^{28}+156w^{26}+10552w^{24}+100748w^{22}+376052w^{20}\\
&\quad +718412w^{18}+893650w^{16}+936788w^{14}+665276w^{12}+351572w^{10}\\
&\quad + 116080w^8 + 22020w^6 + 2652w^4 + 196w^2 + 5.
\end{aligned}$$

Write

$$(e_1-e_2)\cdot(e_1-e_3)\cdot(e_2-e_3) = \alpha \cdot f_1^{\alpha_1}(w)\cdots f_l^{\alpha_l}(w),$$

where $\alpha \in \mathbf{Z}$ and $f_i \in \mathbf{Z}[w]$ are irreducible (of positive degree) and $\alpha_i \geq 1$. Let $w_0 \in \mathbf{Q}$. Assume that, for each $i = 1, \dots, l$, the integer square-free part of each of $f_i(w_0)$ has at least one prime factor that does not appear in the integer square-free part of any of $f_j(t_0)$ (for $j \neq i$) and does not appear in the factorization of α. Then the specialization homomorphism $E(\mathbf{Q}(w)) \to E(w_0)(\mathbf{Q})$ is injective ([**13,** Theorem 3.1]). Furthermore, if $|E(w_0)(\mathbf{Q})_{\text{tors}}| = 4$ and there exist points $P_1, \dots, P_r \in E(\mathbf{Q}(w))$ such that $P_1(w_0), \dots, P_r(w_0)$ are the free generators of $E(w_0)(\mathbf{Q})$, then the specialization homomorphism $E(\mathbf{Q}(w)) \to E(w_0)(\mathbf{Q})$ is an isomorphism. Thus, $E(\mathbf{Q}(w))$ and $E(w_0)(\mathbf{Q})$ have the same rank r, and $P_1, \dots, P_r$ are the free generators of $E(\mathbf{Q}(w))$ ([**13,** Corollary 3.2]).

We have $\alpha = 8$ and $l = 13$, with

$$
\begin{aligned}
f_1 &= w - 1, \\
f_2 &= w + 1, \\
f_3 &= w^2 + 1, \\
f_4 &= w^2 + 3, \\
f_5 &= w^4 + 6w^2 + 1, \\
f_6 &= w^6 + 6w^5 + 7w^4 + 20w^3 + 7w^2 + 6w + 1, \\
f_7 &= w^6 - 6w^5 + 7w^4 - 20w^3 + 7w^2 - 6w + 1, \\
f_8 &= w^6 + 2w^5 + 7w^4 - 4w^3 + 7w^2 + 2w + 1, \\
f_9 &= w^6 - 2w^5 + 7w^4 + 4w^3 + 7w^2 - 2w + 1, \\
f_{10} &= 5w^{12} + 30w^{10} + 55w^8 + 52w^6 + 99w^4 + 14w^2 + 1, \\
f_{11} &= w^{12} + 2w^{10} + 47w^8 + 156w^6 + 47w^4 + 2w^2 + 1, \\
f_{12} &= w^{12} + 26w^{10} + 79w^8 + 44w^6 + 79w^4 + 26w^2 + 1, \\
f_{13} &= w^{16} - 4w^{14} + 8w^{12} + 300w^{10} + 818w^8 + 564w^6 + 320w^4 + 36w^2 + 5.
\end{aligned}
$$

If we take $w_0 = 12$, than it is easy to check that the conditions of [**13,** Theorem 3.1], given above, are satisfied. We have

$$
\begin{aligned}
&E(12) : y^2 = x^3 - 51289727495763299303985770723092429x^2 \\
&+42118341771252682965694472808155483369289256234640658819712040104 9000x.
\end{aligned}
$$

Using `mwrank` [**5**], we compute that $\operatorname{rank}(E(12)(\mathbf{Q})) = 5$. Hence, we proved that

$$\operatorname{rank}\left(E(\mathbf{Q}(w))\right) = 5.$$

Moreover, `mwrank` is able to find free generators of $E(12)(\mathbf{Q})$ (we increased the default saturation bound from 100 to 350 to ensure that we get the full basis):

$$\begin{aligned}
P_1 = (&49375096502864171324367807393771554457526810/24389000,\\
&69361036898001232948930532710952449724275349458333827937855113/\\
&\qquad 24389000),\\
P_2 = (&126212544449321041708829273753136000000/1331,\\
&-6447422778736991144601152289405540432875988587062800000/1331),\\
P_3 = (&6360458707658937141339085290995689 0,\\
&875268515731315579215731210267478928388874598320191 0),\\
P_4 = (&674099810253858754263373534605278 90,\\
&10081869846217955542616878834848663934971209883931590),\\
P_5 = (&614701503627212706679544151188218 0,\\
&-939823161347572492401782456798530394750552461834180).
\end{aligned}$$

Let us denote by $Q_1, \dots, Q_5$ the points on E with X-coordinates X_1, X_2, X_3, X_4, X_5', respectively, and let $T_1 = (0,0)$, $T_2 = (e_2, 0)$ and $T_3 = (e_3, 0)$ be torsion points of E. The corresponding points on $E(12)$ obtained by the specialization $w = 12$ are denoted by $Q_1(12), \dots, Q_5(12), T_1(12), T_2(12), T_3(12)$. We check in SAGE that $E(12)(\mathbf{Q})_{\text{tors}} = \{\mathcal{O}, T_1(12), T_2(12), T_3(12)\}$. Now we express $Q_i(12)$, $i = 1, \dots, 5$, (modulo torsion) in the basis $P_1, \dots, P_5$. We get:

$$\begin{aligned}
Q_1(12) &= P_5 + T_3,\\
Q_2(12) &= -P_1 + P_2 + P_3 - P_4 - P_5,\\
Q_3(12) &= P_1 + P_4 + P_5,\\
Q_4(12) &= -P_1 + P_2 - P_5,\\
Q_5(12) &= P_3.
\end{aligned}$$

Since the transformation matrix has determinant equal to 1, we conclude that $Q_1(12), \dots, Q_5(12)$ also represent a full basis for $E(12)$. Finally, by **[13,** Corollary 3.2], we conclude that $Q_1, \dots, Q_5$ are free generators of $E(Q(w))$.

2.6. Ranks 9 and 10. In [**14**] Izadi et al., using a result of Fine [**11**], found a subfamily of rank ≥ 3 and also several examples of elliptic curves with rank 7 over $\mathbf{Q}$ associated to Heron triangles. Here we will give some examples of such curves with rank 9 and 10.

Our starting point is the families of elliptic curves with rank ≥ 3 from subsections 2.2 and 2.3. We use the sieving method based on Mestre-Nagao sums

$$S(N,E) = \sum_{\substack{p \leq N \\ p \text{prime}}} \left(1 - \frac{p-1}{\#E(\mathbf{F}_p)}\right) \log(p)$$

(see [**6, 17, 18**]). For curves with large values of $S(N,E)$, with N up to 2000, we compute the Selmer rank, which is a well-known upper bound for the rank. We combine this information with the conjectural parity for the rank.

Finally, we try to compute the rank and find generators for the best candidates for large rank. We have implemented this procedure in PARI [**19**], using Cremona's program `mwrank` [**5**] for the computation of rank and Selmer rank.

In the following tables, we present examples of rank 9. We give the corresponding parameters n, w and also the sides of the corresponding Heron triangle (normalized such that they are coprime integers). Tables 1 and 2 correspond to the families from subsections 2.2 and 2.3, respectively.

TABLE 1. Heron triangles inducing curves with rank 9-first family.

n	w	a	b	c
221/48	4/17	18384649	31198450	15329769
41/194	11/41	64077917	35424617	30058860
79/20	4/33	65640625	110226402	57479537
87/52	3/41	77191201	86344565	54123476
71/107	13/17	122058701	109132469	26778900
178/117	9/20	309450037	335760200	160672963
179/81	5/31	6923377145	9144292673	5171712156
71/43	17/26	8074099721	8989245200	2850027831

TABLE 2. Heron triangles inducing curves with rank 9-second family.

n	w	a	b	c
167/33	4/5	626501	1318499	1594900
97/12	1/17	771626	2073001	1427235
109/73	3/4	1789609	686679	1934600
227/120	5/7	4746774	2834947	5724973
206/43	3/7	7055929	17271150	14397659
204/245	3/4	9383829	1931179	8839850
95/67	7/11	50547901	25054956	53109445
245/239	1/11	75291643	45686940	60989203
227/169	3/16	156821821	118236867	142364360
171/125	9/17	318902763	184309528	322125515
43/10	87/97	419985425	401156823	740852008
7/55	14/81	4128925645	1341216251	2816777076

For the parameters $(n, w) = (45/173, 1/95)$ in the family from subsection 2.2 we get by `mwrank` that rank is equal to 9 or 10, while the root number is 1, so according to the Parity conjecture the rank should be even and therefore equal to 10. Here the sides are

$$a = 49579585457, \quad b = 26029616561, \quad c = 25199344032,$$

and the equation in minimal Weierstrass form is

$$y^2 + xy = x^3 - 788122674655148921301606597951685 7217096x$$
$$- 26523602874420714675650466626040507305850107907496 77424137 60.$$

Finally, for the parameters $(n, w) = (21/328, 6/7)$ in the family from subsection 2.2, we get the curve with rank equal to 10, unconditionally. Here the sides are

$$a = 21151489, \quad b = 18364250, \quad c = 2807129.$$

The equation in minimal Weierstrass form is

$$y^2 = x^3 - x^2 - 369712768619708063464705575 20x$$

$$+\,27310847633588585011416495867764650699576\,32.$$

The ten independent points found by `mwrank` and further reduced by the LLL algorithm are:

$$\begin{aligned}
P_1 &= (63406532576504, 801113711642717115240),\\
P_2 &= (2237297792773394, 105445780346586956755050),\\
P_3 &= (-93593058891631, 231764885579979l495800),\\
P_4 &= (24036048058997, 1362471196229901177966),\\
P_5 &= (33228567635744, 1240673649293544991200),\\
P_6 &= (3850001393944, 1608975563080858775000),\\
P_7 &= (-79722558761326, 227416753449101789937\,0),\\
P_8 &= (93776244351274, 297865847675487142730),\\
P_9 &= (115077802151832, 2180294765772365205\,6),\\
P_{10} &= (1788254015658016/9, 48521092864923720268640/27)\,.
\end{aligned}$$

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