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VALUE DISTRIBUTION OF CERTAIN DIFFERENCE POLYNOMIALS OF MEROMORPHIC FUNCTIONS

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ABSTRACT. In this paper, we establish a theorem concerning value distribution of certain difference polynomials of meromorphic functions, which extends [**14**, Theorem 2] and [**16**, Theorem 1.2]. Applying this result, we prove some uniqueness theorems of meromorphic functions whose certain difference polynomials share a non-zero polynomial, which extends [**18**, Theorems 1.1 and 1.2] and [**23**, Theorem 6], where the meromorphic functions are of finite order.

1. Introduction and main results. In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [**7, 13, 22**]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$, as $r \to \infty$ and $r \notin E$.

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a-points with the same multiplicities. We say that f and g share the value a IM, provided that f and g have the same a-points, and each common a-point of f and g is counted only once (see [**22**]). We say that a is a small function of f, if a is a meromorphic function satisfying $T(r,a) = S(r,f)$ (see [**22**]). Throughout this paper, we denote by $\rho(f)$ and $\rho_2(f)$ the order and the hyper-order of f, respectively (see [**7, 13, 22**]). We also need the following two definitions:

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Definition 1.1 [**12**, Definition 1]. Let f be a non-constant meromorphic function, let p be a positive integer, and let $a \in \mathbf{C} \cup \{\infty\}$. Then, by $N_{p)}(r, 1/(f-a))$, we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p, by $\overline{N}_{p)}(r, 1/(f-a))$, we denote the corresponding reduced counting function (ignoring multiplicities). By $N_{(p}(r, 1/(f-a))$, we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not less than p, by $\overline{N}_{(p}(r, 1/(f-a))$ we denote the corresponding reduced counting function (ignoring multiplicities), where $N_{p)}(r, 1/(f-a))$, $\overline{N}_{p)}(r, 1/(f-a))$, $N_{(p}(r, 1/(f-a))$ and $\overline{N}_{(p}(r, 1/(f-a))$ mean $N_{p)}(r, f)$, $\overline{N}_{p)}(r, f)$, $N_{(p}(r, f)$ and $\overline{N}_{(p}(r, f)$, respectively, if $a = \infty$.

Definition 1.2. Let f be a non-constant meromorphic function, let a be any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We define

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Much research on the uniqueness theory of meromorphic functions whose differential polynomials share one nonzero value has been done, for example, see [**3, 15, 19, 21**]. Recently the difference variant of the Nevanlinna theory has been established in [**1, 5**] and, in particular, in [**4**], by Halburd-Korhonen and by Chiang-Feng, independently. Using these theories, some mathematicians from Finland and China began to consider the uniqueness questions of meromorphic functions sharing values with their shifts, and produced many fine works (for example, see [**9, 10, 23**]). In this paper, we will consider the value distribution question and the uniqueness question of meromorphic functions whose difference polynomials share one nonzero value or an entire function of smaller order.

We recall the following result, which was proved by Clunie and Hayman, respectively:

Theorem A [**2, 8**]. *Let $f(z)$ be a transcendental entire function, and let $n \geq 1$ be a positive integer. Then $f^n(z)f'(z) - 1$ has infinitely many zeros.*

Regarding Theorem A, it is natural to ask the following question:

Question 1.1. What can be said about the conclusion of Theorem A if $f^n(z)f'(z)$ of Theorem A is replaced with $f^n(z)f(z+\eta)$ for a transcendental entire function $f(z)$, where η is a nonzero complex number?

Laine and Yang proved the following result dealing with Question 1.1:

Theorem B [**14**, Theorem 2]. *Let f be a transcendental entire function of a finite order, and let η be a nonzero complex number. Then $f(z)^n f(z+\eta)$ assumes every finite nonzero value a infinitely often for $n \geq 2$.*

We recall the following two examples:

Example A [**14**]. Let $f(z) = 1 + e^z$. Then $f(z)f(z+\pi i) - 1 = -e^{2z}$ has no zeros. This example shows that Theorem B does not remain valid if $n = 1$.

Example B [**16**, Remark 1]. Let $f(z) = e^{-e^z}$. Then $f(z)^2 f(z+\eta) - 2 = -1$ and $\rho(f) = \infty$, where η is a nonzero constant satisfying $e^\eta = -2$. Evidently, $f(z)^2 f(z+\eta) - 2$ have no zeros. This example shows that Theorem B does not remain valid if f is of infinite order.

Recently Liu and Yang proved the following result:

Theorem C [**16**, Theorem 1.2]. *Let f be a transcendental entire function of finite order, let η be a nonconstant complex number, and let $n \geq 2$ be an integer. Then $f^n(z)f(z+\eta) - P(z)$ has infinitely many zeros, where $P(z) \not\equiv 0$ is a polynomial.*

We recall the following example:

Example C [**16**, Remark 1]. Let $f(z) = e^{-e^z}$. Then $f(z)^n f(z+\eta) - P(z) = 1 - P(z)$ and $\rho(f) = \infty$, where η is a nonzero constant satisfying

$e^{\eta} = -n$, $P(z)$ is a nonconstant polynomial, n is a positive integer. Evidently, $f(z)^n f(z+\eta) - P(z)$ has finitely many zeros. This example shows that the condition "$\rho(f) < \infty$" in Theorem C is necessary.

One may ask, what can be said about the conclusion of Theorem C, if f is a transcendental meromorphic function? In this direction, we will prove:

Theorem 1.1. *Let f be a transcendental meromorphic function such that its order $\rho(f) = \rho < \infty$, let η be a nonzero complex number, and let $n \geq 1$ be an integer. Suppose that $P \not\equiv 0$ is a polynomial. Then*

(1.1)
$$\begin{aligned} nT(r, f(z)) + m(r, f(z)) \leq 2\overline{N}(r, f(z)) + 2\overline{N}\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(z)}\right) \\ + \overline{N}\left(r, \frac{1}{f^n(z)f(z+\eta) - P(z)}\right) \\ + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1), \end{aligned}$$

as $r \notin E$ and $r \to \infty$.

From Theorem 1.1, we can get the following results, which is an analogue of Theorem C for meromorphic functions of finite orders:

Corollary 1.1. *Let f be a transcendental meromorphic function such that its order $\rho(f) < \infty$, let η be a nonzero complex number, and let $n \geq 6$ be an integer. Suppose that $P \not\equiv 0$ is a polynomial. Then $f^n(z)f(z+\eta) - P(z)$ has infinitely many zeros.*

Proof. Noting that

$$\overline{N}\left(r, \frac{1}{f(z)}\right) \leq N\left(r, \frac{1}{f(z)}\right) \leq T(r, f(z)) + O(1), \tag{1.2}$$

$$\overline{N}(r, f(z)) \leq T(r, f(z)) + O(1) \tag{1.3}$$

and

$$m(r, f(z)) \geq 0,$$

we can get from (1.1) that

$$(n-5)T(r,f(z)) \le \overline{N}\left(r, \frac{1}{f^n(z)f(z+\eta)-P(z)}\right) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + O(1),$$

as $|z| = r \notin E$ and $r \to \infty$. This, together with the condition $n \ge 6$, reveals the conclusion of Corollary 1.1. □

Corollary 1.2. *Let f be a transcendental meromorphic function such that $\rho(f) < \infty$ and $\delta(\infty, f(z)) > 0$, let η be a nonzero complex number, and let $n \ge 5$ be an integer. Suppose that $P \not\equiv 0$ is a polynomial. Then $f^n(z)f(z+\eta) - P(z)$ has infinitely many zeros.*

Proof. Proceeding as in the proof of Corollary 1.1, we have (1.2) and (1.3). From the definition of deficiency $\delta(\infty, f(z))$, we have

$$m(r,f(z)) \ge (\delta(\infty,f(z))-\varepsilon)T(r,f(z)), \tag{1.4}$$

as $|z| = r \to \infty$. From (1.1)–(1.4), we have

$$(n+\delta(\infty,f(z))-5-\varepsilon)T(r,f(z)) \le \overline{N}\left(r, \frac{1}{f^n(z)f(z+\eta)-P(z)}\right) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + O(1),$$

as $|z| = r \notin E$ and $r \to \infty$. This, together with the conditions $\delta(\infty, f(z)) > 0$ and $n \ge 5$ implies the conclusion of Corollary 1.2.

Corresponding to Theorem B, Qi, Yang and Liu [**18**] proved the following uniqueness results:

Theorem D [**18**, Theorem 1.1]. *Let f and g be two distinct transcendental entire functions of finite order. Suppose that η is a nonzero complex number and $n \ge 6$ is an integer. If $f(z)^n f(z+\eta) - z$ and $g(z)^n g(z+\eta) - z$ share* 0 CM*, then $f = tg$, where $t \ne 1$ is a constant satisfying $t^{n+1} = 1$.*

Theorem E [**18**, Theorem 1.2]. *Let f and g be two distinct transcendental entire functions of finite order. Suppose that η is a nonzero complex number and $n \geq 6$ is an integer. If $f(z)^n f(z+\eta)$ and $g(z)^n g(z+\eta)$ share* 1 CM, *then $f = tg$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$.*

One may ask, what can be said about the relationship between f and g, if f and g in Theorems D and E are meromorphic functions? In this direction, we will prove:

Theorem 1.2. *Let f and g be two distinct transcendental meromorphic functions of finite order, let η be a nonzero complex number, let $n \geq 14$ be an integer, and let $P \not\equiv 0$ be a polynomial such that* $2\deg(P) < n - 1$. *Suppose that $f(z)^n f(z+\eta) - P(z)$ and $g(z)^n g(z+\eta) - P(z)$ share* 0 CM. *Then*:

(I) *If $n \geq 10$ and if $f(z)^n f(z+\eta)/P(z)$ is a Möbius transformation of $g(z)^n g(z+\eta)/P(z)$, then one of the following two cases will hold*:

(i) *$f = tg$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$.*

(ii) *$fg = t$, where P reduces to a nonzero constant c, say, and t is a constant such that $t^{n+1} = c^2$.*

(II) *If $n \geq 14$, then one of the two cases* (I) (i) *and* (I) (ii) *will hold.*

Proceeding as in the proof of Theorem 1.2 in Section 3 of this paper, we can get the following result by Lemma 2.7 in Section 2 of this paper.

Theorem 1.3. *Let f and g be two distinct transcendental meromorphic functions of finite order, let η be a nonzero complex number, let $n \geq 12$ be an integer, and let $P \not\equiv 0$ be a polynomial such that* $2\deg(P) < n + 1$. *Suppose that f and g share ∞* IM, *$f(z)^n f(z+\eta) - P(z)$ and $g(z)^n g(z+\eta) - P(z)$ share* $0, \infty$ CM. *Then*:

(I) *If $n \geq 10$ and if $f(z)^n f(z+\eta)/P(z)$ is a Möbius transformation of $g(z)^n g(z+\eta)/P(z)$, then one of the following two cases will hold*:

(i) *$f = tg$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$.*

(ii) *$f = e^Q$ and $g = te^{-Q}$, where P reduces to a nonzero constant c, say, and t is a constant such that $t^{n+1} = c^2$, Q is a nonconstant polynomial.*

(II) *If $n \geq 12$, then one of the two cases* (I) (i) *and* (I) (ii) *will hold.*

From Theorems 1.1 and 1.2, we can get the following result:

Corollary 1.3. *Let f and g be two distinct nonconstant meromorphic functions of finite order. Suppose that η is a nonzero complex number and $n \geq 17$ is an integer. If $f(z)^n f(z+\eta) - z$ and $g(z)^n g(z+\eta) - z$ share* 0 CM, *then $f = tg$, where t is a constant satisfying $t^{n+1} = 1$ and $t \neq 1$.*

Proceeding as in the proof of Corollary 1.3 in Section 3 of this paper, we can deduce the following result by Theorem 1.3 and Lemma 2.8 in Section 2 of this paper:

Corollary 1.4. *Let f and g be two distinct nonconstant meromorphic functions of finite order, let η be a nonzero complex number, and let $n \geq 13$ be an integer. Suppose that f and g share ∞* IM, *$f(z)^n f(z+\eta) - z$ and $g(z)^n g(z+\eta) - z$ share* $0, \infty$ CM. *Then $f = tg$, where t is a constant satisfying $t^{n+1} = 1$ and $t \neq 1$.*

Recently Zhang proved the following result.

Theorem D [**23**, Theorem 6]. *Let f and g be two transcendental entire functions of finite order, and let α be a small function related to f and g. Suppose that η is a nonzero complex number and $n \geq 7$ is an integer. If $f(z)^n(f(z)-1)f(z+\eta) - \alpha(z)$ and $g(z)^n(g(z)-1)g(z+\eta) - \alpha(z)$ share* 0 CM, *then $f = g$.*

We will prove the following result, which is an analogue of Theorem D for meromorphic functions of finite order.

Theorem 1.4. *Let f and g be two transcendental meromorphic functions of finite order, let $\alpha \not\equiv 0$ be an entire function such that $\rho(\alpha) < \rho(f)$, let η be a nonzero complex number, and let n and m be two positive integers such that $n \geq m + 12$ and $m \geq 2$. Suppose that f and g share ∞* IM, *$f(z)^n(f(z)^m - 1)f(z+\eta) - \alpha(z)$ and $g(z)^n(g(z)^m - 1)g(z+\eta) - \alpha(z)$ share* $0, \infty$ CM. *Then $f = tg$, where t is a constant satisfying $t^m = 1$.*

2. Preliminaries. In this section, we introduce the following important lemmas for proving the main results in this paper.

Lemma 2.1 [**20**, Proof of Lemma 2]. *Let f be a nonconstant meromorphic function in the complex plane, and let*

$$P(f) = a_n f(z)^n + a_{n-1} f(z)^{n-1} + \cdots + a_1 f(z) + a_0, \tag{2.1}$$

where $a_0, a_1, \cdots, a_{n-1}$, a_n are constants and $a_n \neq 0$. Then

$$m(r, P(f)) = nm(r, f) + O(1).$$

Lemma 2.2 [**6**, Theorem 5.1]. *Let f be a nonconstant meromorphic function, and let η be a nonzero complex number. If f is of finite order, then*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = O\left(\frac{T(r, f(z)) \log r}{r}\right)$$

for all r outside of a set E satisfying

$$\limsup_{r \to \infty} \frac{\int_{E \cap [1,r)} dt/t}{\log r} = 0,$$

i.e., outside of a set E of zero logarithmic density. If $\rho_2(f) = \rho_2 < 1$ and $\varepsilon > 0$, then

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = o\left(\frac{T(r, f(z))}{r^{1-\rho_2-\varepsilon}}\right),$$

for all r outside of a finite logarithmic measure, where and in what follows, ε is an arbitrary positive number.

Lemma 2.3. *Let f be a nonconstant meromorphic function of order $\rho(f) < \infty$, let η be a nonzero complex number, and let $P(f)$ be defined as in* (2.1). *Suppose that $F(z) = P(f(z))f(z+\eta)$. Then*

$$m(r, F(z)) = (n+1)m(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1),$$

for all r outside of a finite logarithmic measure.

Proof. First of all, by the condition $\rho(f) < \infty$, we get $\rho_2(f) = 0$. This, together with Lemma 2.1, Lemma 2.2 and the assumptions of Lemma 2.3 gives

$$\begin{aligned}(n+1)m(r, f(z)) &= m(r, f(z)P(f(z))) + O(1)\\ &\le m\left(r, \frac{f(z)P(f(z))}{F(z)}\right) + m(r, F(z)) + O(1)\\ &= m\left(r, \frac{f(z)}{f(z+\eta)}\right) + m(r, F(z)) + O(1)\\ &\le m(r, F(z)) + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1),\end{aligned}$$

i.e.,

$$\begin{aligned}m(r, F(z)) \ge{}& (n+1)m(r, f(z))\\ &+ o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1).\end{aligned} \tag{2.2}$$

Next from Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}m(r, F(z)) &\le m(r, P(f(z))) + m\left(r, f(z)\cdot\frac{f(z+\eta)}{f(z)}\right)\\ &\le nm(r, f(z)) + m(r, f(z))\\ &\quad + m\left(r, \frac{f(z+\eta)}{f(z)}\right) + O(1)\\ &= (n+1)m(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1),\end{aligned}$$

i.e.,

$$\begin{aligned}m(r, F(z)) \le{}& (n+1)m(r, f(z))\\ &+ o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1).\end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we get the conclusion of Lemma 2.3.

Lemma 2.4 [**6**, Lemma 8.3]. *Let $T : [0, +\infty) \to [0, +\infty)$ be a nondecreasing continuous function, and let $s \in \mathbf{R}^+$. If the hyper-order of T is strictly less than one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \zeta < 1,$$

and $\delta \in (0, 1-\zeta)$, *then*

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right),$$

where r *runs to infinity outside of a set of finite logarithmic measure.*

Let F and G be two nonconstant meromorphic functions, let $a \in \mathbf{C} \cup \{\infty\}$, and let $\overline{N}_E(r,a)$ "count" those points in $\overline{N}(r, 1/(F-a))$, where a is taken by F and G with the same multiplicity, and each point is counted only once. We next denote by $\overline{N}_0(r,a)$ the reduced counting function of common a-points of F and G in $|z| < r$. We say that F and G share the value a CM*, if

$$\overline{N}\left(r, \frac{1}{F-a}\right) - \overline{N}_E(r,a) = S(r,F)$$

and

$$\overline{N}\left(r, \frac{1}{G-a}\right) - \overline{N}_E(r,a) = S(r,G),$$

where and in what follows, $\overline{N}(r, 1/(F-\infty))$ means $\overline{N}(r,F)$.

Lemma 2.5 [**22**, Lemma 7.1]. *Let* F *and* G *be two nonconstant meromorphic functions such that* G *is a Möbius transformation of* F*. Suppose that there exists a subset* $I \subset \mathbf{R}^+$ *with its linear measure* $\operatorname{mes} I = +\infty$ *such that*

$$\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,G) < (\lambda + o(1))T(r,F),$$

as $r \in I$ *and* $r \to \infty$, *where* $\lambda < 1$*. If there exists a point* $z_0 \in \mathbf{C}$ *such that* $F(z_0) = G(z_0) = 1$, *then* $F = G$ *or* $FG = 1$.

Lemma 2.6 [**22**, Theorem 7.10] or [**21**, Lemma 3]. *Let* F *and* G *be two nonconstant meromorphic functions such that* F *and* G *share* 1 CM*. Suppose that there exists some subset* $I \subset \mathbf{R}^+$ *with its linear measure* $\operatorname{mes} I = \infty$ *such that*

$$N_2(r,F)+N_2(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)<(\mu+o(1))T(r),$$

where $\mu<1$, $T(r)=\max\{T(r,F),T(r,G)\}$. *Then* $F=G$ *or* $FG=1$.

Lemma 2.7 [**22**, Proof of Theorem 7.10]. *Let* F *and* G *be two nonconstant meromorphic functions such that* F, G *share* $1,\infty$ CM*. *Suppose that there exists a subset* $I\subset\mathbf{R}^+$ *with linear measure* $\operatorname{mes} I=+\infty$ *such that*

$$N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2\overline{N}(r,F)<\lambda T(r)+S(r),$$

as $r\in I$ *and* $r\to\infty$, *where* $\lambda<1$, $T(r)=\max\{T(r,F),T(r,G)\}$ *and* $S(r)=o\{T(r)\}$, *as* $r\in I$ *and* $r\to\infty$. *Then* $F=G$ *or* $FG=1$.

Lemma 2.8 [**11**, Lemma 2.2]. *Let* $\varphi(r)$ *be a nondecreasing, continuous function on* $\mathbf{R}^+$. *Suppose that*

$$0<\rho<\limsup_{r\to\infty}\frac{\log\varphi(r)}{\log r},$$

and set

$$I:=\{r\in\mathbf{R}^+|\varphi(r)\geq r^\rho\}.$$

Then we have

$$\overline{\log\operatorname{dens} I}=\limsup_{r\to\infty}\frac{\int_{I\cap[1,r]}(dr)/r}{\log r}>0.$$

3. Proof of theorems.

Proof of Theorem 1.1. By Lemma 2.3, we have

$$\begin{aligned}m(r,f^n(z)f(z+\eta))&=(n+1)m(r,f(z))\\&\quad+o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right)+O(1),\end{aligned}\tag{3.1}$$

where $\rho = \rho(f)$ is the order of f. Noting that

$$\limsup_{r\to\infty} \frac{\log\log N(r,(1/f(z)))}{\log r} \le \rho_2(f) = 0,$$

we can get from Lemma 2.4 that

$$\begin{aligned} N\left(r, \frac{1}{f(z+\eta)}\right) &\le N\left(r+|\eta|, \frac{1}{f(z)}\right) \\ &= N\left(r, \frac{1}{f(z)}\right) \\ &\quad + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + O(1), \end{aligned} \tag{3.2}$$

as $r \notin E$ and $r \to \infty$, where and in what follows, $E \subset (1,\infty)$ denotes some subset with logarithmic measure $\log \operatorname{mes} E < \infty$. Similarly,

$$N\left(r, f(z+\eta)\right) \le N(r, f(z)) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + O(1), \tag{3.3}$$

$$\overline{N}\left(r, f(z+\eta)\right) \le \overline{N}(r, f(z)) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + O(1) \tag{3.4}$$

and

$$\overline{N}\left(r, \frac{1}{f(z+\eta)}\right) \le \overline{N}\left(r, \frac{1}{f(z)}\right) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + O(1), \tag{3.5}$$

as $r \notin E$ and $r \to \infty$.

Next we denote by $N(r, |f^n(z)f(z+\eta) = f(z) = \infty, f(z+\eta) \neq \infty)$ the counting function of those common poles of $f^n(z)f(z+\eta)$ and $f(z)$ in $|z| < r$, where each such point is not a pole of $f(z+\eta)$, and each such point is counted according to its multiplicity in $N(r, f^n(z)f(z+\eta))$, denote by $N(r, |f(z+\eta) = f(z) = \infty)$ the counting function of common poles of $f(z)$ and $f(z+\eta)$ in $|z| < r$, where each such point is counted according to its multiplicity in $N(r, f^n(z)f(z+\eta))$, and denote by

$N(r, |f^n(z)f(z+\eta)) = f(z+\eta) = \infty, f(z) \neq \infty)$ the counting function of those common poles of $f^n(z)f(z+\eta)$ and $f(z+\eta)$ in $|z| < r$, where each such point is not a pole of $f(z)$, and each such point is counted according to its multiplicity in $N(r, f^n(z)f(z+\eta))$. By observing, we have

$$
\begin{aligned}
(3.6)\quad & N(r, f^n(z)f(z+\eta)) \\
&= N(r, |f^n(z)f(z+\eta) = f(z) = \infty, f(z+\eta) \neq \infty) \\
&\quad + N(r, |f(z+\eta) = f(z) = \infty) \\
&\quad + N(r, |f^n(z)f(z+\eta)) = f(z+\eta) = \infty, f(z) \neq \infty).
\end{aligned}
$$

Then, from (3.2) and (3.6), we have

$$
\begin{aligned}
N(r, f^n(z)f(z+\eta)) &\geq nN(r, f(z)) - N\left(r, \frac{1}{f(z+\eta)}\right) \\
&\geq nN(r, f(z)) - N\left(r, \frac{1}{f(z)}\right) \\
&\quad + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1),
\end{aligned}
$$

as $r \notin E$ and $r \to \infty$, this together with (3.1) gives

$$
\begin{aligned}
T(r, f^n(z)f(z+\eta)) &= m(r, f^n(z)f(z+\eta)) \\
&\quad + N(r, f^n(z)f(z+\eta)) \\
&\geq nT(r, f(z)) + m(r, f(z)) - N\left(r, \frac{1}{f(z)}\right) \qquad (3.7) \\
&\quad + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1),
\end{aligned}
$$

as $r \notin E$ and $r \to \infty$. By (3.4), (3.5) and Nevanlinna's three small functions theorem (see [**22**, Theorem 1.36]), we have

(3.8)

$$
\begin{aligned}
T(r, f^n(z)f(z+\eta)) &\leq \overline{N}(r, f^n(z)f(z+\eta)) + \overline{N}\left(r, \frac{1}{f^n(z)f(z+\eta)}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{f^n(z)f(z+\eta) - P(z)}\right) + O(\log r)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \overline{N}(r, f(z)) + \overline{N}(r, f(z+\eta)) + \overline{N}\left(r, \frac{1}{f(z)}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{f(z+\eta)}\right) + \overline{N}\left(r, \frac{1}{f^n(z)f(z+\eta) - P(z)}\right) \\
&\quad + O(\log r) \\
&\leq 2\overline{N}(r, f(z)) + 2\overline{N}\left(r, \frac{1}{f(z)}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{f^n(z)f(z+\eta) - P(z)}\right) \\
&\quad + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1),
\end{aligned}
$$

as $r \notin E$ and $r \to \infty$. From (3.7) and (3.8), we get the conclusion of Theorem 1.1.

Theorem 1.1 is thus completely proved. □

Proof of Theorem 1.2. First of all, we set

$$
F_1(z) = \frac{f(z)^n f(z+\eta)}{P(z)}, \qquad G_1(z) = \frac{g(z)^n g(z+\eta)}{P(z)}. \tag{3.9}
$$

From the condition $n \geq 14$ and the condition that f, g are transcendental meromorphic functions, we can deduce from (3.9) and Lemma 2.4 that F_1, G_1 are transcendental meromorphic functions. Suppose that $z_0 \in \mathbf{C}$ is a zero of $F_1 - 1$ of multiplicity μ. Then, by the condition that $P \not\equiv 0$ is a polynomial, we can see that z_0 is a zero of $f(z)^n f(z+\eta) - P(z)$ of multiplicity $\mu + \nu$, where $\nu \geq 0$ is the multiplicity of z_0 as a zero of P. Hence, z_0 is a zero of $g(z)^n g(z+\eta) - P(z)$ of multiplicity $\mu + \nu$ by the value sharing assumption. Now one sees that z_0 is a zero of $G_1 - 1$ of multiplicity μ. This also works in the other direction. Therefore, F_1 and G_1 indeed share 1 CM. Since f, g are of finite order, it follows from (3.9) and Lemma 2.4 that the same is true for F_1 and G_1 as well. We discuss the following two cases.

Case 1. Suppose that F_1 is a Möbius transformation of G_1. Then it follows from the (3.9) and the standard Valiron-Mokhon'ko lemma (see

[**17**]) that

$$\begin{aligned} T(r,F_1(z)) &= T(r,f(z)^n f(z+\eta)) + O(\log r) \\ &= T(r,g(z)^n g(z+\eta)) + O(\log r) \\ &= T(r,G_1(z)) + O(1). \end{aligned} \tag{3.10}$$

From Theorem 1.1, we get

$$\begin{aligned} &(n-5)\,T(r,f(z)) + m(r,f(z)) \\ &\quad \le \overline{N}\left(r, \frac{1}{f^n(z)f(z+\eta)-P(z)}\right) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + O(1). \end{aligned} \tag{3.11}$$

This, together with Lemma 2.4 and the condition that $f(z)^n f(z+\eta) - P(z)$ and $g(z)^n g(z+\eta) - P(z)$ share 0 CM, gives

$$\begin{aligned} &(n-5)T(r,f(z)) + m(r,f(z)) \\ &\quad \le \overline{N}\left(r, \frac{1}{g^n(z)g(z+\eta)-P(z)}\right) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + O(1) \\ &\quad \le T(r,G_1(z)) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + O(\log r) \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\le (n+1)T(r,g(z)) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) \\ &\quad + o\left(\frac{T(r,g(z))}{r^{1-\varepsilon}}\right) + O(\log r), \end{aligned} \tag{3.13}$$

as $r \notin E$ and $r \to \infty$. Similarly,

$$\begin{aligned} &(n-5)T(r,g(z)) + m(r,g(z)) \\ &\quad \le \overline{N}\left(r, \frac{1}{f^n(z)f(z+\eta)-P(z)}\right) + o\left(\frac{T(r,g(z))}{r^{1-\varepsilon}}\right) + O(1) \\ &\quad \le T(r,F_1(z)) + o\left(\frac{T(r,g(z))}{r^{1-\varepsilon}}\right) + O(\log r) \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\le (n+1)T(r,f(z)) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) \\ &\quad + o\left(\frac{T(r,g(z))}{r^{1-\varepsilon}}\right) + O(\log r), \end{aligned} \tag{3.15}$$

as $r \notin E$ and $r \to \infty$. From (3.12), (3.13), the condition $n \geq 10$, Definition 1.1 and the standard reasoning of removing exceptional set (see [**13**, Lemma 1.1.2]) we get

$$\rho(f) \leq \rho(G_1) \leq \rho(g). \tag{3.16}$$

Similarly, from (3.14) and (3.15), we have

$$\rho(g) \leq \rho(F_1) \leq \rho(f). \tag{3.17}$$

From (3.16) and (3.17), we have

$$\rho(f) = \rho(g) = \rho(F_1) = \rho(G_1). \tag{3.18}$$

From (3.9), (3.18) and Lemma 2.4, we deduce

$$\begin{aligned}
\overline{N}(r, F_1(z)) + \overline{N}\left(r, \frac{1}{F_1(z)}\right) &\leq \overline{N}\,(r, f(z)) + \overline{N}\,(r, f(z+\eta)) \\
&\quad + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z+\eta)}\right) + O(\log r) \\
&\leq 2T(r, f(z)) + 2T(r, f(z+\eta)) + O(\log r) \\
&\leq 4T(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(\log r),
\end{aligned} \tag{3.19}$$

as $r \notin E$ and $r \to \infty$. Similarly,

$$\overline{N}(r, G_1(z)) + \overline{N}\left(r, \frac{1}{G_1(z)}\right) \leq 4T(r, g(z)) + o\left(\frac{T(r, g(z))}{r^{1-\varepsilon}}\right) + O(\log r), \tag{3.20}$$

as $r \notin E$ and $r \to \infty$. Proceeding as in the proof of Theorem 1.1, we can get (3.7). From (3.7) and (3.10), we have

$$T(r, f(z)) \leq \frac{1}{n-1} T(r, F_1(z)) + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(\log r) \tag{3.21}$$

and

$$T(r, g(z)) \leq \frac{1}{n-1} T(r, G_1(z)) + o\left(\frac{T(r, g(z))}{r^{1-\varepsilon}}\right) + O(\log r), \tag{3.22}$$

as $r \notin E$ and $r \to \infty$. From (3.19)–(3.22), we have

$$\begin{aligned}\overline{N}\left(r, \frac{1}{F_1}\right) + \overline{N}(r, F_1) + \overline{N}\left(r, \frac{1}{G_1}\right) + \overline{N}(r, G_1) &\\ \leq \frac{4}{n-1}\{T(r, F_1) + T(r, G_1)\}(1 + o(1)) &\\ = \frac{8}{n-1}T(r, F_1)(1 + o(1)), &\end{aligned} \tag{3.23}$$

as $r \notin E$ and $r \to \infty$. Again, from (3.9) and (3.11), we have

$$\begin{aligned}(n-5)\,T(r, f(z)) + m(r, f(z)) \leq \overline{N}\left(r, \frac{1}{F_1(z) - 1}\right) &\\ + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(\log r), &\end{aligned} \tag{3.24}$$

as $r \notin E$ and $r \to \infty$. Noting that F_1 and G_1 share 1 CM, we know from (3.24) and $n \geq 10$ that there exists some point $z_0 \in \mathbf{C}$ such that $F_1(z_0) = G_1(z_0) = 1$. This, together with (3.23) and Lemma 2.5, implies that either $F_1 = G_1$ or $F_1G_1 = 1$. We discuss the following two subcases.

Subcase 1.1. Suppose that $F_1 = G_1$. Then it follows from (3.9) that

$$f(z)^n f(z + \eta) = g(z)^n g(z + \eta) \tag{3.25}$$

for all $z \in \mathbf{C}$. Let

$$h = \frac{f}{g}. \tag{3.26}$$

From (3.25) and (3.26), we get

$$h(z)^n h(z + \eta) = 1 \tag{3.27}$$

for all $z \in \mathbf{C}$.

First suppose that h is rational. If h has a zero at some point z_0, then h has a pole at $z_0 + \eta$ by (3.27). Continuing, $h(z_0 + 2\eta) = 0$,

$h(z_0 + 3\eta) = \infty$, and so on. Therefore, h would have infinitely many zeros and poles, which is impossible. Hence, h has neither zeros nor poles, meaning that h has to be a constant, say $h = t$. By (3.27), $t^{n+1} = 1$. This together with (3.26) gives the conclusion (I) (i) of Theorem 1.2.

Next suppose that h is a transcendental meromorphic function. Since f, g are of finite order, the same is true for h as well. Thus it follows from (3.27), Lemma 2.4 and the standard Valiron-Mokhon'ko lemma (see [**17**]) that

$$\begin{aligned} nT(r, h(z)) &= T\left(r, \frac{1}{h(z+\eta)}\right) \\ &= T(r, h(z+\eta)) + O(1) \\ &\leq T(r + |\eta|, h(z)) + O(1) \\ &= T(r, h(z)) + o\left(\frac{T(r, h(z))}{r^{1-\varepsilon}}\right) + O(1), \end{aligned}$$

and so

$$(n-1)T(r, h(z)) = o\left(\frac{T(r, h(z))}{r^{1-\varepsilon}}\right) + O(1), \tag{3.28}$$

as $r \notin E$ and $r \to \infty$. From (3.28) and the condition $n \geq 10$, we deduce that h is a constant, which is impossible.

Subcase 1.2. Suppose that $F_1 G_1 = 1$, while $F_1 \not\equiv G_1$. By substituting (3.9) into $F_1 G_1 = 1$, we get

$$f(z)^n f(z+\eta) g(z)^n g(z+\eta) = P(z)^2 \tag{3.29}$$

for all $z \in \mathbf{C}$. Proceeding as in Subcase 1.1, we can deduce from (3.29), Lemma 2.4 and the condition $n \geq 10$ that fg is a nonzero rational function. Let

$$fg = R, \tag{3.30}$$

where R is a nonzero rational function. Then, by (3.30), we know that (3.29) can be rewritten as

$$R(z)^n R(z+\eta) = P(z)^2 \tag{3.31}$$

for all $z \in \mathbf{C}$.

Suppose that R is not a constant. Then

$$R = \frac{P_1}{P_2}, \tag{3.32}$$

P_1 and P_2 are two nonzero relatively prime polynomials. From (3.32), we have

$$T(r, R) = \max\{\deg(P_1), \deg(P_2)\} \log r + O(1). \tag{3.33}$$

From (3.31)–(3.33), we get

$$\begin{aligned} n \max\{\deg(P_1), \deg(P_2)\} \log r \\ &= T(r, R(z)^n) + O(1) \\ &\leq T(r, R(z+\eta)) + 2T(r, P(z)) + O(1) \\ &= \max\{\deg(P_1), \deg(P_2)\} \log r + 2\deg(P) \log r + O(1). \end{aligned} \tag{3.34}$$

Noting that $\max\{\deg(P_1), \deg(P_2)\} \geq 1$, we deduce from (3.34) that $n - 1 \leq 2\deg(P)$, which contradicts the condition $2\deg(P) < n - 1$. Therefore, $R =: t$ is a nonzero constant. This, together with (3.31), reveals the conclusion (I) (ii) of Theorem 1.2.

Case 2. Suppose that $n \geq 14$. First of all, in the same manner as in the proof of Case 1, we can get (3.21) and (3.22). From (3.9) and Lemma 2.4, we have

(3.35)
$$\begin{aligned} N_2(r, F_1(z)) + N_2\left(r, \frac{1}{F_1(z)}\right) \\ &\leq 2\overline{N}(r, f(z)) + N(r, f(z+\eta)) \\ &\quad + 2\overline{N}\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(z+\eta)}\right) + O(\log r) \\ &\leq 4T(r, f(z)) + 2T(r, f(z+\eta)) + O(\log r) \\ &\leq 6T(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(\log r) \end{aligned}$$

and

$$N_2(r, G_1(z)) + N_2\left(r, \frac{1}{G_1(z)}\right) \leq 6T(r, g(z)) + o\left(\frac{T(r, g(z))}{r^{1-\varepsilon}}\right) + O(\log r), \tag{3.36}$$

as $r \notin E$ and $r \to \infty$. From (3.21), (3.22), (3.35) and (3.36), we have

$$N_2(r, F_1(z)) + N_2\left(r, \frac{1}{F_1(z)}\right) \leq \frac{6}{n-1}T(r, F_1(z)) + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(\log r) \tag{3.37}$$

and

$$N_2(r, G_1(z)) + N_2\left(r, \frac{1}{G_1(z)}\right) \leq \frac{6}{n-1}T(r, G_1(z)) + o\left(\frac{T(r, g(z))}{r^{1-\varepsilon}}\right) + O(\log r), \tag{3.38}$$

as $r \notin E$ and $r \to \infty$. From (3.37) and (3.38), we have

$$N_2(r, F_1(z)) + N_2\left(r, \frac{1}{F_1(z)}\right) + N_2(r, G_1(z)) + N_2\left(r, \frac{1}{G_1(z)}\right) \leq \frac{12}{n-1}T_1(r)(1 + o(1)), \tag{3.39}$$

as $r \notin E$ and $r \to \infty$, where $T_1(r) = \max\{T(r, F_1), T(r, G_1)\}$. From (3.39), Lemma 2.6 and the condition $n \geq 14$, we have $F_1 = G_1$ or $F_1 G_1 = 1$. Next, in the same manner as in Subcases 1.1 and 1.2, we can get the conclusion (II) of Theorem 1.2.

Theorem 1.2 is thus completely proved. □

Proof of Corollary 1.3. We discuss the following cases.

Case 1. Suppose that one of f and g is a rational function, the other one of f and g is a transcendental meromorphic function. Without loss of generality, we suppose that f is a transcendental meromorphic function and g is a rational function. Then, on the one hand, we get from Theorem 1.1 that $f(z)^n f(z+\eta) - z$ has infinitely many zeros in $\mathbf{C}$. On the other hand, by the supposition that g is a rational function, we know that $g(z)^n g(z+\eta) - z$ is also a rational function, and so $g(z)^n g(z+\eta) - z$ has at most finitely many zeros in $\mathbf{C}$. This contradicts the condition that $f(z)^n f(z+\eta) - z$ and $g(z)^n g(z+\eta) - z$ share 0 CM.

Case 2. Suppose that f and g are transcendental meromorphic functions. Then, from Theorem 1.2 and the assumptions of Corollary 1.3, we get the conclusion of Corollary 1.3.

Case 3. Suppose that f and g are nonconstant rational functions. Set

$$F_2(z) = \frac{f(z)^n f(z+\eta)}{z}, \qquad G_2(z) = \frac{g(z)^n g(z+\eta)}{z}. \tag{3.40}$$

Let

$$f = \frac{P_3}{P_4}, \tag{3.41}$$

where P_3, P_4 are two nonzero relatively prime polynomials. Proceeding as in Subcase 1.2 of the proof of Theorem 1.2, we can deduce from (3.41) and $n \geq 17$ that F_2 and G_2 are not constants. Proceeding as in the beginning of the proof of Theorem 1.2, we can deduce that F_2 and G_2 share 1 CM. From (3.40), (3.41), the condition $n \geq 17$ and the standard Valiron-Mokhon'ko lemma, we have

$$\begin{aligned}
&N_2(r, F_2(z)) + N_2\left(r, \frac{1}{F_2(z)}\right)\\
&\quad\leq 2\overline{N}(r, f(z)) + N(r, f(z+\eta))\\
&\qquad + 2\overline{N}\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(z+\eta)}\right) + \log r + O(1)\\
&\quad\leq 3\deg(P_4)\log r + 3\deg(P_3)\log r + \log r + O(1)\\
&\quad\leq 6\max\{\deg(P_3), \deg(P_4)\}\log r + \log r + O(1)
\end{aligned} \tag{3.42}$$

$$= 6T(r, f(z)) + \log r + O(1).$$

Similarly,

$$N_2(r, G_2(z)) + N_2\left(r, \frac{1}{G_2(z)}\right) \leq 6T(r, g(z)) + \log r + O(1). \tag{3.43}$$

Noting that f is a nonconstant rational function, we deduce

$$m(r, f^n(z)f(z+\eta)) = (n+1)m(r, f(z)) + O(1). \tag{3.44}$$

From (3.41) and the standard Valiron-Mokhon'ko lemma, we have

(3.45)
$$\begin{aligned} N(r, f^n(z)f(z+\eta)) &\geq n\deg(P_4)\log r - \deg(P_3)\log r + O(1) \\ &\geq nN(r, f(z)) - \max\{\deg(P_3), \deg(P_4)\}\log r + O(1) \\ &= nN(r, f(z)) - T(r, f(z)) + O(1). \end{aligned}$$

From (3.40), (3.44) and (3.45), we have

$$\begin{aligned} (n-1)T(r, f(z)) + m(r, f(z)) &\leq T(r, f^n(z)f(z+\eta)) + O(1) \\ &\leq T(r, F_2(z)) + \log r + O(1). \end{aligned} \tag{3.46}$$

From (3.42) and (3.46), we get

$$\begin{aligned} N_2(r, F_2(z)) + N_2\left(r, \frac{1}{F_2(z)}\right) &\leq \frac{6}{n-1}T(r, F_2(z)) \\ &\quad + \frac{n+5}{n-1}\log r + O(1). \end{aligned} \tag{3.47}$$

Similarly,

$$\begin{aligned} N_2(r, G_2(z)) + N_2\left(r, \frac{1}{G_2(z)}\right) &\leq \frac{6}{n-1}T(r, G_2(z)) \\ &\quad + \frac{n+5}{n-1}\log r + O(1). \end{aligned} \tag{3.48}$$

Noting that $T(r, f(z)) \geq \log r + O(1)$, we get from (3.46) that

$$\log r \leq \frac{1}{n-2}T(r, F_2(z)) + O(1). \tag{3.49}$$

Similarly,

$$\log r \leq \frac{1}{n-2} T(r, G_2(z)) + O(1). \tag{3.50}$$

From (3.47)–(3.50), we get

$$\begin{aligned} N_2(r, F_2(z)) + N_2\left(r, \frac{1}{F_2(z)}\right) \\ + N_2(r, G_2(z)) + N_2\left(r, \frac{1}{G_2(z)}\right) \\ \leq \frac{14n-14}{(n-1)(n-2)} T_2(r)(1+o(1)), \end{aligned} \tag{3.51}$$

where $T_2(r) = \max\{T(r, F_2), T(r, G_2)\}$. Noting that F_2 and G_2 share 1 CM, we have from (3.51), Lemma 2.6 and $n \geq 17$ that $F_2 = G_2$ or $F_2 G_2 = 1$. Next, in the same manner as in Subcases 1.1 and 1.2 of the proof of Theorem 1.2, we can get the conclusion of Corollary 1.3 from (3.40).

This proves Corollary 1.3. □

Proof of Theorem 1.4. First of all, we set

$$\begin{aligned} F_3(z) &= \frac{f(z)^n (f(z)^m - 1) f(z+\eta)}{\alpha(z)}, \\ G_3(z) &= \frac{g(z)^n (g(z)^m - 1) g(z+\eta)}{\alpha(z)} \end{aligned} \tag{3.52}$$

for all $z \in \mathbf{C}$. Proceeding as in the beginning of the proof of Theorem 1.2, we can deduce that F_3 and G_3 share 1 CM. From Lemma 2.3, we have

$$\begin{aligned} m(r, f(z)^n (f(z)^m - 1) f(z+\eta)) = (m+n+1) m(r, f(z)) \\ + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1), \end{aligned} \tag{3.53}$$

as $r \notin E$ and $r \to \infty$. In the same manner as in the proof of Theorem 1.1, we can get from Lemma 2.4 that

(3.54)

$$N(r, f(z)^n (f(z)^m - 1) f(z+\eta))$$

$$
\begin{aligned}
&\geq (m+n)N(r, f(z)) - N\left(r, \frac{1}{f(z+\eta)}\right)\\
&\geq (m+n)N(r, f(z)) - N\left(r+|\eta|, \frac{1}{f(z)}\right)\\
&= (m+n)N(r, f(z)) - N\left(r, \frac{1}{f(z)}\right) + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right)\\
&\geq (m+n)N(r, f(z)) - T(r, f(z))\\
&\quad + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1),
\end{aligned}
$$

as $r \notin E$ and $r \to \infty$. From (3.53) and (3.54), we have

$$
\begin{aligned}
T(r, f(z)^n(f(z)^m - 1)f(z+\eta))
&\geq (m+n-1)T(r, f(z)) + m(r, f(z))\\
&\quad + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(1),
\end{aligned} \tag{3.55}
$$

as $r \notin E$ and $r \to \infty$. By (3.52), Lemma 2.4, the condition $\rho(\alpha) < \rho(f) < \infty$, the standard Valiron-Mokhon'ko lemma and the condition that f is a transcendental meromorphic function, we deduce that F_3 is a transcendental meromorphic function. This, together with Lemma 2.4, the second fundamental theorem and the fact that F_3, G_3 share 1 CM, gives

(3.56)

$$
\begin{aligned}
T(r, F_3(z)) &\leq \overline{N}(r, F_3(z)) + \overline{N}\left(r, \frac{1}{F_3(z)}\right)\\
&\quad + \overline{N}\left(r, \frac{1}{F_3(z)-1}\right) + O(\log r)\\
&\leq \overline{N}(r, f(z)^n(f(z)^m - 1)f(z+\eta))\\
&\quad + \overline{N}\left(r, \frac{1}{f(z)^n(f(z)^m - 1)f(z+\eta)}\right)\\
&\quad + \overline{N}\left(r, \frac{1}{G_3(z)-1}\right) + \overline{N}\left(r, \frac{1}{\alpha(z)}\right) + O(\log r)\\
&\leq \overline{N}(r, f(z)) + \overline{N}(r, f(z+\eta))\\
&\quad + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z+\eta)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \overline{N}\left(r, \frac{1}{f(z)^m - 1}\right) + T(r, G_3(z)) + O(r^{\rho(\alpha)+\varepsilon}) \\
&+ O(\log r) \\
\le\ & (m+4)T(r, f(z)) + T(r, G_3(z)) \\
&+ o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(r^{\rho(\alpha)+\varepsilon}) + O(\log r) \\
\le\ & (m+4)T(r, f(z)) + (m+n+1)T(r, g(z)) \\
&+ o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) \\
&+ o\left(\frac{T(r, g(z))}{r^{1-\varepsilon}}\right) + O(r^{\rho(\alpha)+\varepsilon}) + O(\log r),
\end{aligned}
$$

as $r \notin E$ and $r \to \infty$. Also, from (3.52), we have

$$
\begin{aligned}
T(r, f(z)^n (f(z)^m - 1) f(z+\eta)) &\le T(r, F_3(z)) + T(r, \alpha(z)) \\
&\le T(r, F_3(z)) + O(r^{\rho(\alpha)+\varepsilon}).
\end{aligned} \tag{3.57}
$$

From (3.55)–(3.57), have

$$
\begin{aligned}
(m+n-1)&T(r, f(z)) \\
&\le (m+4)T(r, f(z)) + T(r, G_3(z)) \\
&\quad + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + O(r^{\rho(\alpha)+\varepsilon}) + O(\log r) \\
&\le (m+4)T(r, f(z)) + (m+n+1)T(r, g(z) \\
&\quad + o\left(\frac{T(r, f(z))}{r^{1-\varepsilon}}\right) + o\left(\frac{T(r, g(z))}{r^{1-\varepsilon}}\right) \\
&\quad + O(r^{\rho(\alpha)+\varepsilon}) + O(\log r),
\end{aligned} \tag{3.58}
$$

as $r \notin E$ and $r \to \infty$. From (3.58), the condition $\rho(\alpha) < \rho(f)$ and the standard reasoning of removing exceptional set (see [**13**, Lemma 1.1.2]) we deduce

$$
\rho(f) \le \rho(G_3) \le \rho(g). \tag{3.59}
$$

Similarly

$$
\begin{aligned}
(3.60)\quad (m+n-1)T(r,g(z)) &\le (m+4)T(r,g(z)) + T(r,F_3(z)) \\
&\quad + o\left(\frac{T(r,g(z))}{r^{1-\varepsilon}}\right) + O(r^{\rho(\alpha)+\varepsilon}) + O(\log r) \\
&\le (m+4)T(r,g(z)) + (m+n+1)T(r,f(z)) \\
&\quad + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + o\left(\frac{T(r,g(z))}{r^{1-\varepsilon}}\right) \\
&\quad + O(r^{\rho(\alpha)+\varepsilon}) + O(\log r),
\end{aligned}
$$

as $r \notin E$ and $r \to \infty$, and so

$$
(3.61)\qquad \rho(g) \le \rho(F_3) \le \rho(f).
$$

From (3.59) and (3.61) we have

$$
(3.62)\qquad \rho(f) = \rho(g) = \rho(F_3) = \rho(G_3).
$$

Noting that $\lambda(f) \le \rho(f)$, we can get from (3.52), (3.55) and $\rho(\alpha) < \rho(f)$ that

$$
\begin{aligned}
(3.63)\quad (m+n-1)T(r,f(z)) &\le T(r, f(z)^n(f(z)^m-1)f(z+\eta)) \\
&\quad + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) \\
&\le T(r,F_3(z)) + T(r,\alpha(z)) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) \\
&\le T(r,F_3(z)) + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + r^{\rho(\alpha)+\varepsilon},
\end{aligned}
$$

as $r \notin E$ and $r \to \infty$. From (3.58), (3.60), (3.62), (3.63), Lemma 2.4, Lemma 2.8, the conditions $\rho(\alpha) < \rho(f) < \infty$ and $n \ge m+6$, we know that there exists some subset $I \subseteq \mathbf{R}^+$ with linear measure $\operatorname{mes} I = \infty$

such that

$$r^{\rho(\alpha)+\varepsilon} = o\{T(r, F_3)\}, \tag{3.64}$$

$$r^{\rho(\alpha)+\varepsilon} = o\{T(r, G_3)\}, \tag{3.65}$$

$$\overline{N}\left(r, \frac{1}{F_3 - 1}\right) - \overline{N}_E(r, 1) = 0, \tag{3.66}$$

$$\overline{N}\left(r, \frac{1}{G_3 - 1}\right) - \overline{N}_E(r, 1) = 0, \tag{3.67}$$

$$\overline{N}(r, F_3) - \overline{N}_E(r, \infty) = o\{T(r, F_3)\} \tag{3.68}$$

and

$$\overline{N}(r, G_3) - \overline{N}_E(r, \infty) = o\{T(r, F_3)\}, \tag{3.69}$$

as $r \in I$ and $r \to \infty$, such that

(3.70)
$$\begin{aligned} N_2\left(r, \frac{1}{F_3}\right) &+ 2\overline{N}(r, F_3) \\ &\leq 2\overline{N}\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(z)^m - 1}\right) \\ &\quad + N\left(r, \frac{1}{f(z+\eta)}\right) + 2\overline{N}(r, f(z)) \\ &\quad + 2\overline{N}(r, f(z+\eta)) + 2\overline{N}\left(r, \frac{1}{\alpha(z)}\right) \\ &\leq (m+7)T(r, f(z)) + o\{T(r, f)\} \\ &= \frac{m+7}{m+n-1} T(r, F_3(z)) + o\{T(r, F_3(z))\} \end{aligned}$$

and

$$N_2\left(r, \frac{1}{G_3}\right) \leq \frac{m+3}{m+n-1} T(r, G_3) + o\{T(r, G_3)\}, \tag{3.71}$$

as $r \to \infty$ and $r \in I$. From (3.70) and (3.71), we get

$$\begin{aligned} N_2\left(r, \frac{1}{F_3}\right) + N_2\left(r, \frac{1}{G_3}\right) &+ 2\overline{N}(r, F_3) \\ &\leq \frac{2m+10}{m+n-1} T_3(r) + o\{T_3(r)\}, \end{aligned} \tag{3.72}$$

as $r \to \infty$ and $r \in I$, where $T_3(r) = \max\{T(r, F_3), T(r, G_3)\}$. From (3.72), Lemma 2.7 and the condition $n \geq m + 12$, we have $F_3 = G_3$ or $F_3 G_3 = 1$. We discuss the following two cases:

Case 1. Suppose that $F_3 = G_3$. Then, it follows from (3.52) that

$$f(z)^n(f(z)^m - 1)f(z + \eta) = g(z)^n(g(z)^m - 1)g(z + \eta) \tag{3.73}$$

for all $z \in \mathbf{C}$. Let h be defined as (3.26). From (3.26) and (3.73) we get

$$\{h(z)^{m+n}h(z + \eta) - 1\}g(z)^m = h(z)^n h(z + \eta) - 1 \tag{3.74}$$

for all $z \in \mathbf{C}$.

Suppose that $h(z)^{m+n}h(z + \eta) - 1 = 0$ for all $z \in \mathbf{C}$. Then, from (3.74), we have $h(z)^n h(z + \eta) - 1 = 0$ for all $z \in \mathbf{C}$, and so $h^m = 1$. This together with (3.26) gives the conclusion of Theorem 1.2. Next we suppose that $h(z)^{m+n}h(z + \eta) - 1 \not\equiv 0$. Then, (3.74) can be rewritten as

$$g(z)^m = \frac{h(z)^n h(z + \eta) - 1}{h(z)^{m+n}h(z + \eta) - 1} \tag{3.75}$$

for all $z \in \mathbf{C}$.

First suppose that h is rational. Then, from (3.75) we know that g is a rational function, which is impossible.

Next suppose that h is a transcendental meromorphic function. Since f, g are of finite order, the same is true for h as well. Set

$$H_1(z) = h(z)^n h(z + \eta), \qquad H_2(z) = h(z)^{m+n}h(z + \eta) \tag{3.76}$$

for all $z \in \mathbf{C}$. Let $z_0 \in \mathbf{C}$ be such a point that $H_2(z_0) - 1 = 0$ and $H_1(z_0) - 1 \neq 0$. Then, from (3.75) and (3.76), we deduce that z_0 is a zero of $H_2(z) - 1$ with multiplicity $\geq m$. Let $z_1 \in \mathbf{C}$ be a common zero of $H_1(z) - 1$ and $H_2(z) - 1$. Then, from (3.76), we deduce that $h(z_1)^m = 1$. Therefore, from Lemma 2.4 and the standard Valiron-Mokhon'ko lemma, we have

(3.77)
$$\overline{N}\left(r, \frac{1}{H_2(z) - 1}\right) \leq \overline{N}(r, |H_2(z) - 1 = 0, H_1(z) - 1 \neq 0)$$

$$
\begin{aligned}
&+\overline{N}\left(r, \frac{1}{h(z)^m - 1}\right)\\
\le\ & \frac{1}{m} N\left(r, \frac{1}{h(z)^{m+n}h(z+\eta)}\right) + mT(r, h(z)) + O(1)\\
\le\ & \frac{m+n+1}{m} T(r, h(z)) + mT(r, h(z))\\
&+ o\left(\frac{T(r, h(z))}{r^{1-\varepsilon}}\right) + O(1)\\
=\ & \frac{m^2+m+n+1}{m} T(r, h(z))\\
&+ o\left(\frac{T(r, h(z))}{r^{1-\varepsilon}}\right) + O(1),
\end{aligned}
$$

where $\overline{N}(r, |H_2(z)-1=0, H_1(z)-1 \neq 0)$ denotes the reduced counting function of those points in $N(r, 1/(H_2(z)-1))$, where each such point is not a zero of $H_1(z)-1$. Since h is of finite order, it follows from (3.76) and Lemma 2.4 that the same is true for H_2 as well. Hence, from (3.76), (3.77), Lemma 2.4 and the second fundamental theorem, we get

$$
\begin{aligned}
T(r, H_2) \le\ & \overline{N}(r, H_2) + \overline{N}\left(r, \frac{1}{H_2}\right)\\
&+ \overline{N}\left(r, \frac{1}{H_2 - 1}\right) + O(\log r)\\
\le\ & \overline{N}(r, h(z)) + \overline{N}(r, h(z+\eta))\\
&+ \overline{N}\left(r, \frac{1}{h(z)}\right) + \overline{N}\left(r, \frac{1}{h(z+\eta)}\right)\\
&+ \frac{m^2+m+n+1}{m} T(r, h(z))\\
&+ o\left(\frac{T(r, h(z))}{r^{1-\varepsilon}}\right) + O(\log r)\\
\le\ & \left(\frac{m^2+m+n+1}{m} + 4\right) T(r, h(z))\\
&+ o\left(\frac{T(r, h(z))}{r^{1-\varepsilon}}\right) + O(\log r),
\end{aligned}
\tag{3.78}
$$

as $r \notin E$ and $r \to \infty$. From (3.76), (3.78), Lemma 2.4 and the standard Valiron-Mokhon'ko lemma, we get

$$\begin{aligned}(3.79)\quad (m+n+1)T(r,h(z)) &= T(r,h(z)^{m+n+1})+O(1)\\ &\leq T(r,H_2(z))+T\left(r,\frac{h(z)^{m+n+1}}{H_2(z)}\right)+O(1)\\ &= T(r,H_2(z))+T\left(r,\frac{h(z)}{h(z+\eta)}\right)+O(1)\\ &\leq T(r,H_2(z))+2T(r,h(z))\\ &\quad+o\left(\frac{T(r,h(z))}{r^{1-\varepsilon}}\right)+O(\log r)\\ &\leq \left(\frac{m^2+m+n+1}{m}+6\right)T(r,h(z))\\ &\quad+o\left(\frac{T(r,h(z))}{r^{1-\varepsilon}}\right)+O(\log r),\end{aligned}$$

as $r \notin E$ and $r \to \infty$. By the conditions $n \geq m+12$ and $m \geq 2$ we deduce

$$m+n+1 > \frac{m^2+m+n+1}{m}+6,$$

which together with (3.79) gives

$$T(r,h) \leq o\left(\frac{T(r,h(z))}{r^{1-\varepsilon}}\right)+O(\log r), \tag{3.80}$$

as $r \notin E$ and $r \to \infty$, which means that h is a rational function. This contradicts the above supposition.

Case 2. Suppose that $F_3G_3 = 1$ and $F_3 \not\equiv G_3$. Then it follows from (3.52) that

$$f(z)^n(f(z)^m-1)f(z+\eta)g(z)^n(g(z)^m-1)g(z+\eta)=\alpha(z)^2 \tag{3.81}$$

for all $z \in \mathbf{C}$. From (3.58), (3.60), (3.62), the condition $\rho(\alpha) < \rho(f)$ and Lemma 2.8, we know that there exists some subset $I \subseteq \mathbf{R}^+$ with linear measure $\operatorname{mes} I = \infty$ such that

$$\begin{aligned}T(r,g(z)) &= O(T(r,f(z))),\\ T(r,f(z)) &= O(T(r,g(z)))\end{aligned} \tag{3.82}$$

and

$$T(r,\alpha(z)) = o\{T(r,f(z))\}, \tag{3.83}$$

as $r \in I$ and $r \to \infty$. By rewriting (3.81), we have

$$f(z)^n(f(z)^m - 1)g(z)^n(g(z)^m - 1) = \frac{\alpha(z)^2}{f(z+\eta)g(z+\eta)}. \tag{3.84}$$

From (3.84), Lemma 2.4, the fact $\lambda(f) \leq \rho(f) < \infty$ and the condition that f and g share ∞ IM we have

$$\begin{aligned}(m+n)[N(r,f(z)) + N(r,g(z))] &\leq N\left(r, \frac{1}{f(z+\eta)}\right) + N\left(r, \frac{1}{g(z+\eta)}\right) \\ &\leq N\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{g(z)}\right) \\ &\quad + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + o\left(\frac{T(r,g(z))}{r^{1-\varepsilon}}\right),\end{aligned} \tag{3.85}$$

as $r \notin E$ and $r \to \infty$. By rewriting (3.81), we have

$$\frac{1}{f(z)^n(f(z)^m - 1)g(z)^n(g(z)^m - 1)} = \frac{f(z+\eta)g(z+\eta)}{\alpha(z)^2}. \tag{3.86}$$

From (3.82), (3.83), (3.86), Lemma 2.4, the fact $\lambda(f) \leq \rho(f) < \infty$ and the condition that f, g share ∞ IM we get

$$\begin{aligned}n\left\{N\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{g(z)}\right)\right\}& \\ + N\left(r, \frac{1}{f(z)^m - 1}\right) + N\left(r, \frac{1}{g(z)^m - 1}\right)& \\ \leq N(r, f(z+\eta)) + N(r, g(z+\eta)) + 2N\left(r, \frac{1}{\alpha(z)}\right)& \\ \leq N(r,f(z)) + N(r,g(z)) + 2T(r,\alpha(z))& \\ + o\left(\frac{T(r,f(z))}{r^{1-\varepsilon}}\right) + o\left(\frac{T(r,g(z))}{r^{1-\varepsilon}}\right) + O(1)& \\ \leq N(r,f(z)) + N(r,g(z)) + o\{T(r,f(z))\},&\end{aligned} \tag{3.87}$$

as $r \in I$ and $r \to \infty$. From (3.82), (3.83), (3.85) and (3.87), we have

$$\begin{aligned}(3.88)\quad N(r, f(z)) + N(r, g(z)) + N\left(r, \frac{1}{f(z)}\right) \\ + N\left(r, \frac{1}{g(z)}\right) + N\left(r, \frac{1}{f(z)^m - 1}\right) \\ + N\left(r, \frac{1}{g(z)^m - 1}\right) \\ = o\{T(r, f(z))\},\end{aligned}$$

as $r \in I$ and $r \to \infty$. By (3.88) and the second fundamental theorem, we have

$$\begin{aligned}mT(r, f) &\leq \overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^{m} \overline{N}\left(r, \frac{1}{f - \omega_j}\right) + O(\log r) \\ &= o\{T(r, f)\},\end{aligned}$$

as $r \in I$ and $r \to \infty$, where $\omega_1, \omega_2, \dots, \omega_m$ stand for the roots of $\omega^m = 1$. This is impossible.

Theorem 1.4 is thus completely proved.

4. Concluding remarks.

From Example A and the condition "$n \geq 6$" of Theorem 1.1, we give the following question:

Question 4.1. What can be said about the conclusion of Theorem 1.1, if we replace the condition "$n \geq 6$" with "$2 \leq n \leq 5$"?

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