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ON LIMITING BEHAVIOR FOR ARRAYS OF ROWWISE LINEARLY NEGATIVE QUADRANT DEPENDENT RANDOM VARIABLES

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ABSTRACT. Let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise linearly negative quadrant dependent random variables, and let $\{a_{nk}, 1 \le k \le k_n, n \ge 1\}$ be a real number array. The limiting behavior for $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ is investigated, and some new results are obtained. The results extend and improve the corresponding result of Hu et al. [**3**].

1. Introduction and preliminaries. The concept of negative quadrant dependent (NQD, in short) random variables was introduced by Lehmann [**7**].

Definition 1.1. Two random variables X and Y are said to be NQD if, for any $x, y \in \mathbf{R}$,

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y). \tag{1.1}$$

A sequence of random variables $\{X_n, n \ge 1\}$ is said to be pairwise NQD if every pair of random variables in the sequence is NQD.

The concept of linearly negative quadrant dependent (LNQD, in short) random variables was introduced by Newman [**8**].

Definition 1.2. A sequence $\{X_n, n \ge 1\}$ of random variables is said to be LNQD if, for any disjoint subsets $A, B \subset \mathbf{Z}^+$ and positive $r_j's$, $\sum_{k\in A} r_k X_k$ and $\sum_{j \in B} r_j X_j$ are NQD.

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Remark 1.1. It is easily seen that, if $\{X_n, n \geq 1\}$ is a sequence of LNQD random variables, then $\{aX_n + b, n \geq 1\}$ is still a sequence of LNQD random variables, where a and b are real numbers.

Obviously, LNQD random variables are much weaker than independent random variables and negatively associated (NA, in short) (cf. [**4**]) random variables. Newman [**8**] established the central limit theorem for a strictly stationary LNQD process. Since the article of [**8**] appeared, Wang and Zhang [**11**] provided uniform rates of convergence in the central limit theorem for LNQD sequence, Ko, Cho, and Choi [**5**] obtained the Hoeffding-type inequality for LNQD sequence, Ko, Ryu and Kim [**6**] studied the strong convergence for weighted sums of LNQD arrays, Wang et al. [**12**] studied exponential inequalities, the complete convergence and almost sure convergence for LNQD sequence, and so forth.

Definition 1.3. An array of rowwise random variables $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ is said to be stochastically dominated by a nonnegative random variable X (write $\{X_{nk}\} \prec X$) if there exists a constant $C > 0$ such that

$$\sup_{n,k} P(|X_{nk}| > x) \leq CP(X > x), \quad \text{for all } x > 0. \tag{1.2}$$

Clearly, if $\{X_{nk}\} \prec X$, for $0 < p < \infty$ and any $1 \leq k \leq k_n, n \geq 1$, $E|X_{nk}|^p \leq CEX^p$.

A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant a if, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty.$$

In this case, we write $U_n \to a$ completely. This notion was given firstly by Hsu and Robbins [**2**].

Hu et al. [**3**] had obtained the following result in complete convergence.

Theorem A. *Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise independent random variables with $EX_{nk} = 0$. Suppose that $\{X_{nk}, 1 \leq$*

$k \le n, n \ge 1\}$ are uniformly bounded by some random variable X. If $E|X|^{2p} < \infty$ for some $1 \le p < 2$, then

$$n^{-1/p} \sum_{k=1}^{n} X_{nk} \longrightarrow 0 \quad \textit{completely},$$

if and only if $E|X_{11}|^{2p} < \infty$.

Let $\{Z_n, n \ge 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$, if

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \quad \text{for some or all } \varepsilon > 0; \tag{1.3}$$

then (1.3) was called the complete moment convergence by Chow [**1**]. Chow [**1**] investigated the complete moment convergence for independent random variables. Wan and Zhao [**10**] investigated the complete moment convergence for NA sequences. However, few articles have been written on this subject for LNQD random variable sequences.

In this work, we shall extend Theorem A by considering LNQD instead of independent variables. It is worth pointing out that our main methods differ from those used by Hu et al. [**3**]. In addition, we study the complete moment convergence, the convergence in probability and the mean convergence for the array of pairwise LNQD random variables under some appropriate conditions, which were not considered in [**3**].

In order to prove our results, we need the following lemmas.

Lemma 1.1 (cf. [**7**]). *Let $\{X_n, n \ge 1\}$ be a sequence of pairwise* NQD *random variables. Let $\{f_n, n \ge 1\}$ be a sequence of increasing functions. Then $\{f_n(X_n), n \ge 1\}$ is a sequence of pairwise* NQD *random variables.*

Lemma 1.2 (cf. [**6**]). *Suppose $X_1, \dots, X_n$ are* LNQD*. Then*

$$E \exp\left(\sum_{k=1}^{n} X_k\right) \le \prod_{k=1}^{n} E \exp(X_k).$$

Lemma 1.3. *Let* $\{X_n, n \geq 1\}$ *be an* LNQD *random variable sequence with zero mean and* $0 < B_n = \sum_{k=1}^n EX_k^2 < \infty$. *Then*

$$(1.4) \quad P(|S_n| \geq x) \leq \sum_{k=1}^n P(|X_k| \geq y) + 2\exp\left(\frac{x}{y} - \frac{x}{y}\ln\left(1 + \frac{xy}{B_n}\right)\right),$$

for all $x > 0, y > 0$.

By means of Lemma 1.2, this lemma is easily proved by following [**9**]. Here, we omit the details of the proof.

Below, C will denote generic positive constants, whose value may vary from one application to another, $I(A)$ will indicate the indicator function of A.

2. Main results. Now, we state our main results. The proofs will be given in Section 3.

Theorem 2.1. *Let* $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ *be an array of rowwise* LNQD *random variables such that* $EX_{nk} = 0$ *and* $\{X_{nk}\} \prec X$, *and* $\{a_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ *be a real number array such that* $\max_{1\leq k\leq k_n} |a_{nk}| = O(n^{-r})$ *for some* $r > 0$, *and let* $\{k_n\}$ *be a sequence such that* $k_n \leq bn^m$ *for some* $b, m > 0$. *If* $m < 2r$ *and* $EX^{(m+1)/r} < \infty$, *then*

$$(2.1) \qquad \sum_{k=1}^{k_n} a_{nk}X_{nk} \longrightarrow 0 \quad \textit{completely}.$$

Corollary 2.1. *Let* $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ *be an array of rowwise* LNQD *random variables such that* $EX_{nk} = 0$ *and* $\{X_{nk}\} \prec X$, $EX^{2p} < \infty$, $0 < p < 2$. *Then*

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \longrightarrow 0 \quad \textit{completely}.$$

Remark 2.1. Take $a_{nk} = n^{-r}$, $r = 1/p$ and $m = 1$ in Theorem 2.1; we can get the above corollary. Since LNQD random variables are much weaker than independent random variables, Theorem 2.1 extends Theorem A.

Theorem 2.2. *Let* $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ *be an array of rowwise* LNQD *random variables such that* $EX_{nk} = 0$ *and* $\{X_{nk}\} \prec X$, *and let* $\{k_n\}$ *be a sequence such that* $k_n \le bn^m$ *for some* $b, m > 0$. *If* $m < 2r$ *and* $EX^{(m+1)/r} < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{-rq} E\bigg\{\bigg|\sum_{k=1}^{k_n} X_{nk}\bigg| - \varepsilon n^r\bigg\}_+^q < \infty, \quad \text{for all } \varepsilon > 0, \tag{2.2}$$

where $0 < q < (m+1)/r$.

The following theorem shows that, under some appropriate conditions, we can obtain the convergence in probability for the array of rowwise LNQD random variables.

Theorem 2.3. *Let* $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ *be an array of rowwise* LNQD *random variables such that* $EX_{nk} = 0$ *and* $\{X_{nk}\} \prec X$, *and let* $\{k_n\}$ *be a sequence such that* $k_n \le bn^m$ *for some* $b, m > 0$. *If* $m < 2r$ *and*

$$\lim_{y\to\infty} n^{m-r} y P(X > y) = 0, \tag{2.3}$$

then

$$n^{-r} \sum_{k=1}^{k_n} X_{nk} \longrightarrow 0 \quad \text{in probability.} \tag{2.4}$$

.

Finally we state the mean convergence for the array of rowwise LNQD random variables, under some conditions which are stronger than those of Theorem 2.3.

Theorem 2.4. *Let* $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ *be an array of rowwise* LNQD *random variables such that* $EX_{nk} = 0$ *and* $\{X_{nk}\} \prec X$, *and let*

$\{k_n\}$ be a sequence such that $k_n \le bn^m$ for some $b, m > 0$. If $m < 2r$, $m+1 \ge r$ and

$$\lim_{y\to\infty} n^{m-r} EX^{(m+1)/r} I(X > y) = 0, \tag{2.5}$$

then

$$n^{-r} \sum_{k=1}^{k_n} X_{nk} \longrightarrow 0 \quad \text{in } L^{(m+1)/r}. \tag{2.6}$$

Remark 2.2. We point out that (2.5) is stronger than (2.3). Let $p = (m+1)/r$. Since

$$\begin{aligned} EX^p I(X > y) &= \left(\int_0^{y^p} + \int_{y^p}^{\infty} \right) P\big(X^p I(X > y) > t\big)\, \mathrm{d}t \\ &= \int_0^{y^p} P(X > y)\, \mathrm{d}t + \int_{y^p}^{\infty} P(X^p > t)\, \mathrm{d}t \\ &= y^p P(X > x) + \int_{y^p}^{\infty} P(X^p > t)\, \mathrm{d}t, \end{aligned}$$

we know (2.5) implies

$$\lim_{y\to\infty} n^{m-r} y^{(m+1)/r} P(X > y) = 0. \tag{2.7}$$

Noting that (2.7) implies (2.3) if $m+1 \ge r$. Therefore, (2.5) implies (2.3).

Remark 2.3. Since an independent or NA random variable sequence is a special LNQD sequence, Theorems 2.2–2.4 hold for arrays of rowwise independent or NA random variables.

3. Proofs.

Proof of Theorem 2.1. Let $T_{nk} = \sum_{k=1}^{k_n} a_{nk} X_{nk}$. Since $a_{nk} = a_{nk}^+ - a_{nk}^-$, $T_{nk} = \sum_{k=1}^{k_n} a_{nk}^+ X_{nk} - \sum_{k=1}^{k_n} a_{nk}^- X_{nk}$, and limit properties for $\sum_{k=1}^{k_n} a_{nk}^{+} X_{nk}$ are similar to those for $\sum_{k=1}^{k_n} a_{nk}^{-} X_{nk}$. Without loss of generality, we may assume that $0 < a_{nk} \le n^{-r}$. Let

$$\begin{aligned} X'_{nk} &= X_{nk} I(|X_{nk}| \le n^r) + n^r I(X_{nk} > n^r) \\ &\quad - n^r I(X_{nk} < -n^r), \\ X''_{nk} &= X_{nk} - X'_{nk} = (X_{nk} + n^r) I(X_{nk} < -n^r) \\ &\quad + (X_{nk} - n^r) I(X_{nk} > n^r). \end{aligned}$$

By Lemma 1.1, we know that $\{X'_{nk}, 1 \le k \le k_n, n \ge 1\}$ and $\{X''_{nk}, 1 \le k \le k_n, n \ge 1\}$ are arrays of rowwise LNQD. To prove (2.1), it suffices to show

$$I_1 = \sum_{n=1}^{\infty} P\left(\left| \sum_{k=1}^{k_n} a_{nk} (X'_{nk} - EX'_{nk}) \right| > \varepsilon/2 \right) < \infty, \tag{3.1}$$

$$I_2 = \sum_{n=1}^{\infty} P\left(\left| \sum_{k=1}^{k_n} a_{nk} (X''_{nk} - EX''_{nk}) \right| > \varepsilon/2 \right) < \infty. \tag{3.2}$$

First, we prove (3.1). Let $B'_n = \sum_{k=1}^{k_n} a_{nk}^2 E(X'_{nk} - EX'_{nk})^2$. Take $x = \varepsilon/2$, $y = \varepsilon/2\eta$ and $\eta > \max\{1/(2r-m), 1\}$. By Lemma 1.3, we have

$$\begin{aligned} I_1 &\le \sum_{n=1}^{\infty} \sum_{k=1}^{k_n} P(a_{nk} |X'_{nk} - EX'_{nk}| > \varepsilon/2\eta) \\ &\quad + C \sum_{n=1}^{\infty} \left(\frac{B'_n}{B'_n + \varepsilon^2/4\eta} \right)^{\eta} \\ &:= I_3 + I_4. \end{aligned}$$

Since $EX_{nk} = 0$, we have

$$\begin{aligned} a_{nk} |EX'_{nk}| = a_{nk} |EX''_{nk}| &\le n^{-r} E|X_{nk}| I(X_{nk} > n^r) \\ &\le n^{-m-1} E|X_{nk}|^{(m+1)/r} I(X_{nk} > n^r) \\ &\le n^{-m-1} EX^{(m+1)/r} \longrightarrow 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.3}$$

To prove $I_3 < \infty$, it suffices to show

$$I'_3 \widehat{=} \sum_{n=1}^{\infty} \sum_{k=1}^{k_n} P(a_{nk} |X'_{nk}| > \varepsilon/4\eta) < \infty. \tag{3.4}$$

Then

$$\begin{aligned}
I_3' &\le \sum_{n=1}^{\infty} \sum_{k=1}^{k_n} P(|X_{nk}'| > n^r \varepsilon / 4\eta) \\
&\le \sum_{n=1}^{\infty} \sum_{k=1}^{k_n} P(|X_{nk}| > n^r \varepsilon / 4\eta) \\
&\le C \sum_{n=1}^{\infty} n^m P(X > n^r \varepsilon / 4\eta) \\
&\le CEX^{(m+1)/r} < \infty.
\end{aligned}$$

By (3.3) and (3.4), we have $I_3 < \infty$.

Then we prove $I_4 < \infty$. Clearly, for $x \ge 0$, $y \ge 0$ and $\eta > 1$, $(x+y)^\eta \le 2^{\eta-1}(x^\eta + y^\eta)$. Hence, by the C_r-inequality, we have

$$\begin{aligned}
I_4 &\le C \sum_{n=1}^{\infty} \left(\sum_{k=1}^{k_n} a_{nk}^2 E(X_{nk}' - EX_{nk}')^2 \right)^\eta \\
&\le C \sum_{n=1}^{\infty} \left(\sum_{k=1}^{k_n} n^{-2r} E(X_{nk}')^2 \right)^\eta \\
&= C \sum_{n=1}^{\infty} \left(\sum_{k=1}^{k_n} n^{-2r} EX_{nk}^2 I(|X_{nk}| \le n^r) \right. \\
&\qquad \left. + \sum_{k=1}^{k_n} P(|X_{nk}| > n^r) \right)^\eta \\
&\le C \sum_{n=1}^{\infty} \left(\sum_{k=1}^{k_n} n^{-2r} EX_{nk}^2 I(|X_{nk}| \le n^r) \right)^\eta \\
&\qquad + C \sum_{n=1}^{\infty} \left(\sum_{k=1}^{k_n} P(|X_{nk}| > n^r) \right)^\eta \\
&:= I_{41} + I_{42}.
\end{aligned}$$

For I_{42}, we have

$$\begin{aligned}I_{42} &\le C\sum_{n=1}^{\infty}\left(n^m P(X>n^r)\right)^{\eta} \quad (\text{by } \eta>1)\\ &\le C\left(\sum_{n=1}^{\infty} n^m P(X>n^r)\right)^{\eta} \le C(EX^{(m+1)/r})^{\eta}<\infty.\end{aligned}$$

For I_{41}, we consider the following two cases. When $m+1<2r$, we have

$$\begin{aligned}I_{41} &\le C\sum_{n=1}^{\infty}\left(\sum_{k=1}^{k_n} n^{-2r}E(|X_{nk}|^{(m+1)/r}|X_{nk}|^{2-(m+1)/r})I(|X_{nk}|\le n^r)\right)^{\eta}\\ &\le C\sum_{n=1}^{\infty}\left(\sum_{k=1}^{k_n} n^{-m-1}E|X_{nk}|^{(m+1)/r}I(|X_{nk}|\le n^r)\right)^{\eta}\\ &\le C\sum_{n=1}^{\infty} n^{-\eta}(EX^{(m+1)/r})^{\eta}<\infty.\end{aligned}$$

When $m+1\ge 2r$, we have $EX^2<\infty$. By $\eta>1/(2r-m)$, we have

$$I_{41}\le C\sum_{n=1}^{\infty}\left(n^{m-2r}EX^2\right)^{\eta}\le C\sum_{n=1}^{\infty} n^{(m-2r)\eta}(EX^2)^{\eta}<\infty.$$

From the above proof, we know that $I_4<\infty$. Therefore, we prove that (3.1) holds.

Then we prove (3.2). Note the definition of X''_{nk}. By a similar argument as in the proof of (3.4), we have

$$\begin{aligned}I_2 &\le \sum_{n=1}^{\infty} P\left(\exists k;\ 1\le k\le k_n,\ \text{such that } |X_{nk}|>n^r\right)\\ &\le \sum_{n=1}^{\infty}\sum_{k=1}^{k_n} P(|X_{nk}|>n^r)\le C\sum_{n=1}^{\infty} n^m P(X>n^r)\\ &\le CEX^{(m+1)/r}<\infty.\end{aligned}$$

The proof is complete. □

Proof of Theorem 2.2. Since

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-rq} E\Big(\Big|\sum_{k=1}^{k_n} X_{nk}\Big| - \varepsilon n^r\Big)_+^q \\
&\qquad = \sum_{n=1}^{\infty} n^{-rq} \int_0^{\infty} P\Big(\Big|\sum_{k=1}^{k_n} X_{nk}\Big| > \varepsilon n^r + t^{1/q}\Big)\,\mathrm{d}t \\
&\qquad = \sum_{n=1}^{\infty} n^{-rq} \Big(\int_0^{n^{rq}} P\Big(\Big|\sum_{k=1}^{k_n} X_{nk}\Big| > \varepsilon n^r + t^{1/q}\Big)\,\mathrm{d}t \\
&\qquad\quad + \int_{n^{rq}}^{\infty} P\Big(\Big|\sum_{k=1}^{k_n} X_{nk}\Big| > \varepsilon n^r + t^{1/q}\Big)\,\mathrm{d}t\Big) \\
&\qquad \le \sum_{n=1}^{\infty} P\Big(\Big|\sum_{k=1}^{k_n} X_{nk}\Big| > \varepsilon n^r\Big) \\
&\qquad\quad + \sum_{n=1}^{\infty} n^{-rq} \int_{n^{rq}}^{\infty} P\Big(\Big|\sum_{k=1}^{k_n} X_{nk}\Big| > t^{1/q}\Big)\,\mathrm{d}t \\
&\qquad := I_5 + I_6.
\end{aligned}
$$

Taking $a_{nk} = n^{-r}$ as in Theorem 2.1 of this paper, we have $I_5 < \infty$. To prove (2.2), it suffices to prove that $I_6 < \infty$. Let

$$
\begin{aligned}
X_{nk}^* = {} & X_{nk} I(|X_{nk}| \le t^{1/q}) + t^{1/q} I(X_{nk} > t^{1/q}) \\
& - t^{1/q} I(X_{nk} < -t^{1/q}).
\end{aligned}
$$

For

$$
\begin{aligned}
P\Big(\Big|\sum_{k=1}^{k_n} X_{nk}\Big| > t^{1/q}\Big) \le {} & \sum_{k=1}^{k_n} P(|X_{nk}| > t^{1/q}) \\
& + P\Big(\Big|\sum_{k=1}^{k_n} (X_{nk}^* - EX_{nk}^*)\Big| > t^{1/q}/2\Big).
\end{aligned}
$$

Then

$$
I_6 \le \sum_{n=1}^{\infty} n^{-rq} \int_{n^{rq}}^{\infty} \sum_{k=1}^{k_n} P(|X_{nk}| > t^{1/q})\,\mathrm{d}t
$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty} n^{-rq} \int_{n^{rq}}^{\infty} P\Big(\Big|\sum_{k=1}^{k_n}(X_{nk}^* - EX_{nk}^*)\Big| > t^{1/q}/2\Big)\,\mathrm{d}t \\
&:= I_{61} + I_{62}.
\end{aligned}
$$

By $\{X_{nk}\} \prec X$ and the mean value theorem of differentials, we have

$$
\begin{aligned}
I_{61} &\leq C \sum_{n=1}^{\infty} n^{m-rq} \int_{n^{rq}}^{\infty} P(X > t^{1/q})\,\mathrm{d}t \\
&= C \sum_{n=1}^{\infty} n^{m-rq} \sum_{s=n}^{\infty} \int_{s^{rq}}^{(s+1)^{rq}} P(X > t^{1/q})\,\mathrm{d}t \\
&\leq C \sum_{n=1}^{\infty} n^{m-rq} \sum_{s=n}^{\infty} s^{rq-1} P(X > s^r) \\
&\leq C \sum_{s=1}^{\infty} s^{rq-1} P(X > s^r) \sum_{n=1}^{s} n^{m-rq} \quad (\text{by } m - rq > -1) \\
&\leq C \sum_{s=1}^{\infty} s^m P(X > s^r) \leq CEX^{(m+1)/r} < \infty.
\end{aligned}
$$

Then we prove $I_{62} < \infty$. For $t \geq n^{rq}$, set

$$
\begin{aligned}
&\mathbf{N}_1 = \Big\{n : \sum_{k=1}^{k_n} P(|X_{nk}| > t^{1/q}) \geq P\Big(\Big|\sum_{k=1}^{k_n}(X_{nk}^* - EX_{nk}^*)\Big| > t^{1/q}/2\Big)\Big\}, \\
&\mathbf{N}_2 = \mathbf{N} - \mathbf{N}_1.
\end{aligned}
$$

Obviously if one of $\mathbf{N}_1$ and $\mathbf{N}_2$ is finite, we need only study the case that $n \in \mathbf{N}_1$ or $n \in \mathbf{N}_2$. Hence, without loss of generality, we may assume that $\mathbf{N}_1$ and $\mathbf{N}_2$ are all infinite.

When $n \in \mathbf{N}_1$, by a similar argument as in the proof of $I_{61} < \infty$, we have

$$\begin{aligned}\sum_{n\in\mathbf{N}_1} n^{-rq}\int_{n^{rq}}^{\infty} P\bigg(\bigg|\sum_{k=1}^{k_n}(X^*_{nk}-EX^*_{nk})\bigg| > t^{1/q}/2\bigg)\,\mathrm{d}t \\ &\le C\sum_{n\in\mathbf{N}_1} n^{-rq}\int_{n^{rq}}^{\infty}\sum_{k=1}^{k_n} P(|X_{nk}|>t^{1/q})\,\mathrm{d}t \\ &\le C\sum_{n=1}^{\infty} n^{m-rq}\int_{n^{rq}}^{\infty} P(X>t^{1/q})\,\mathrm{d}t \\ &\le C\sum_{s=1}^{\infty} s^m P(X>s^r) \le CEX^{(m+1)/r}<\infty.\end{aligned}$$

Hence, to prove $I_{62}<\infty$, it suffices to prove that

$$I'_{62}=\sum_{n\in\mathbf{N}_2} n^{-rq}\int_{n^{rq}}^{\infty} P\bigg\{\bigg|\sum_{k=1}^{k_n}(X^*_{nk}-EX^*_{nk})\bigg|>t^{1/q}/2\bigg\}\,\mathrm{d}t<\infty.$$

Let $B''_n=\sum_{k=1}^{k_n} E(X^*_{nk}-EX^*_{nk})^2$. Take $x=t^{1/q}/2$, $y=t^{1/q}/2\delta$, and $\delta>\max\{q/2,\,1,\,1/(2r-m)\}$. By Lemma 1.3, we have

$$\begin{aligned}I'_{62} &\le C\sum_{n\in\mathbf{N}_2} n^{-rq}\int_{n^{rq}}^{\infty}\sum_{k=1}^{k_n} P\Big(\Big|X^*_{nk}-EX^*_{nk}\Big|>t^{1/q}/2\delta\Big)\,\mathrm{d}t \\ &\quad + C\sum_{n\in\mathbf{N}_2} n^{-rq}\int_{n^{rq}}^{\infty}\bigg(\frac{B''_n}{B''_n+t^{2/q}/4\delta}\bigg)^{\delta}\mathrm{d}t \\ &:= I_7+I_8.\end{aligned}$$

When $t\ge n^{rq}$, if $m+1\le r$, then

$$\begin{aligned}t^{-1/q}|EX^*_{nk}| &\le t^{-1/q}E|X_{nk}|I_{(|X_{nk}|\le t^{1/q})}+P(|X_{nk}|>t^{1/q}) \\ &\le t^{-(m+1)/(rq)}E|X_{nk}|^{(m+1)/r}\longrightarrow 0 \quad \text{as } n\to\infty.\end{aligned}$$

If $m+1>r$, by $EX_{nk}=0$, we have

$$\begin{aligned}t^{-1/q}|EX^*_{nk}| &\le t^{-1/q}E|X_{nk}|I_{(|X_{nk}|>t^{1/q})} \\ &\le t^{-(m+1)/(rq)}E|X_{nk}|^{(m+1)/r}\longrightarrow 0 \quad \text{as } n\to\infty.\end{aligned}$$

Therefore, when n is sufficiently large, we can get $|EX^*_{nk}| \le t^{1/q}/4\delta$. Hence, to prove $I_7 < \infty$, it suffices to prove that

$$I'_7 = C \sum_{n\in\mathbf{N}_2} n^{-rq} \int_{n^{rq}}^{\infty} \sum_{k=1}^{k_n} P\left(|X^*_{nk}| > t^{1/q}/4\delta\right) \mathrm{d}t < \infty.$$

The proof of $I'_7 < \infty$ is similar to that of $I_{61} < \infty$. Hence,

$$I'_7 \le C \sum_{n=1}^{\infty} n^{m-rq} \int_{n^{rq}}^{\infty} P(X \ge t^{1/q}/4\delta)\,\mathrm{d}t \le CEX^{(m+1)/r} < \infty.$$

Finally, we prove $I_8 < \infty$. By $\delta > 1$, we have

$$\begin{aligned}
I_8 &\le C \sum_{n\in\mathbf{N}_2} n^{-rq} \int_{n^{rq}}^{\infty} (B''_n)^{\delta} t^{-2\delta/q}\,\mathrm{d}t \\
&\le C \sum_{n\in\mathbf{N}_2} n^{-rq} \int_{n^{rq}}^{\infty} \left(\sum_{k=1}^{k_n} EX^2_{nk} I(|X_{nk}| \le t^{1/q}) \right.\\
&\quad \left. + \sum_{k=1}^{k_n} t^{2/q} P(|X_{nk}| > t^{1/q})\right)^{\delta} t^{-2\delta/q}\,\mathrm{d}t \\
&\le C \sum_{n\in\mathbf{N}_2} n^{-rq} \int_{n^{rq}}^{\infty} \left(\sum_{k=1}^{k_n} EX^2_{nk} I(|X_{nk}| \le t^{1/q})\right)^{\delta} t^{-2\delta/q}\,\mathrm{d}t \\
&\quad + C \sum_{n\in\mathbf{N}_2} n^{-rq} \int_{n^{rq}}^{\infty} \left(\sum_{k=1}^{k_n} P(|X_{nk}| > t^{1/q})\right)^{\delta} \mathrm{d}t \\
&:= I_{81} + I_{82}.
\end{aligned}$$

When $n \in \mathbf{N}_2$, we know that $\sum_{k=1}^{k_n} P(|X_{nk}| > t^{1/q}) < 1$. By $\delta > 1$ and a similar argument as in the proof of $I_{61} < \infty$, we have

$$\begin{aligned}
I_{82} &\le C \sum_{n\in\mathbf{N}_2} n^{-rq} \int_{n^{rq}}^{\infty} \sum_{k=1}^{k_n} P(|X_{nk}| > t^{1/q})\,\mathrm{d}t \\
&\le C \sum_{n\in\mathbf{N}_2} n^{m-rq} \int_{n^{rq}}^{\infty} P(X > t^{1/q})\,\mathrm{d}t \le CEX^{(m+1)/r} < \infty.
\end{aligned}$$

Then we prove $I_{81} < \infty$. If $m+1 < 2r$, by $\delta > 1$ and $0 < q < (m+1)/r$, we have

$$
\begin{aligned}
I_{81} \le{} & C \sum_{n \in \mathbf{N}_2} n^{-rq} \int_{n^{rq}}^{\infty} \bigg(\sum_{k=1}^{k_n} E(|X_{nk}|^{(m+1)/r} |X_{nk}|^{2-(m+1)/r}) \\
& \times I(|X_{nk}| \le t^{1/q}) \bigg)^{\delta} t^{-2\delta/q} \mathrm{d}t \\
\le{} & C \sum_{n \in \mathbf{N}_2} n^{-rq} \int_{n^{rq}}^{\infty} \bigg(\sum_{k=1}^{k_n} E|X_{nk}|^{(m+1)/r} I(|X_{nk}| \le t^{1/q}) \bigg)^{\delta} t^{-(m+1)\delta/rq} \mathrm{d}t \\
\le{} & C \sum_{n \in \mathbf{N}_2} n^{m\delta - rq} (EX^{(m+1/r})^{\delta} \int_{n^{rq}}^{\infty} t^{-(m+1)\delta/rq} \mathrm{d}t \\
\le{} & C \sum_{n \in \mathbf{N}_2} n^{-\delta} (EX^{(m+1)/r})^{\delta} \le C \sum_{n=1}^{\infty} n^{-\delta} (EX^{(m+1)/r})^{\delta} < \infty.
\end{aligned}
$$

If $m < 2r \le m+1$, by $\delta > \max\{q/2, 1/(2r-m)\}$ and $EX_{nk}^2 I(|X_{nk}| \le t^{1/q}) \le EX^2 < \infty$, we have

$$
\begin{aligned}
I_{81} &\le C \sum_{n \in \mathbf{N}_2} n^{m\delta - rq} (EX^2)^{\delta} \int_{n^{rq}}^{\infty} t^{-2\delta/q} \mathrm{d}t \\
&\le C \sum_{n \in \mathbf{N}_2} n^{(m-2r)\delta} (EX^2)^{\delta} \\
&\le C \sum_{n=1}^{\infty} n^{(m-2r)\delta} (EX^2)^{\delta} < \infty.
\end{aligned}
$$

The proof is complete. □

Proof of Theorem 2.3. Following the notations of X'_{nk} and X''_{nk} in the proof of Theorem 2.1, we can get

$$
\begin{aligned}
P\bigg(\bigg| \sum_{k=1}^{k_n} X_{ni} \bigg| \ge n^r \varepsilon \bigg) \le{} & P\bigg(\bigg| \sum_{k=1}^{k_n} (X'_{ni} - EX'_{ni}) \bigg| \ge n^r \varepsilon/2 \bigg) \\
& + P\bigg(\bigg| \sum_{k=1}^{k_n} (X''_{ni} - EX''_{ni}) \bigg| \ge n^r \varepsilon/2 \bigg) \\
:={} & I_9 + I_{10}.
\end{aligned}
$$

To prove (2.4), it suffices to show that $I_9 \to 0$ and $I_{10} \to 0$ as $n \to \infty$. Let $C_n = \sum_{k=1}^{k_n} E(X'_{nk} - EX'_{nk})^2$, $x = y = n^r\varepsilon/2$. By Lemma 1.3 and the Markov inequality, we have

$$\begin{aligned}
I_9 &\le \sum_{k=1}^{k_n} P(|X'_{ni} - EX'_{ni}| \ge n^r\varepsilon/2) + C\frac{C_n}{C_n + n^{2r}\varepsilon^2/4} \\
&\le Cn^{-2r}\sum_{k=1}^{k_n} E(X'_{nk} - EX'_{nk})^2 \le Cn^{-2r}\sum_{k=1}^{k_n} E(X'_{nk})^2 \\
&\le Cn^{-2r}\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \le n^r) + C\sum_{k=1}^{k_n} P(|X_{nk}| > n^r) \\
&:= I_{91} + I_{92}.
\end{aligned}$$

For I_{92}, by taking $y = n^r$ as in (2.3), we have

$$I_{92} \le Cn^m P(X > n^r) \longrightarrow 0 \quad \text{as } n \to \infty.$$

For I_{91}, we have

$$\begin{aligned}
I_{91} &= Cn^{-2r}\sum_{k=1}^{k_n}\int_0^{n^{2r}} P(X_{nk}^2 I(|X_{nk}| \le n^r) > t)\,\mathrm{d}t \\
&\le Cn^{-2r}\sum_{k=1}^{k_n}\int_0^{n^{2r}} P(X_{nk}^2 > t)\,\mathrm{d}t \quad (\text{let } t = y^2) \\
&= Cn^{-2r}\sum_{k=1}^{k_n}\int_0^{n^r} yP(|X_{nk}| > y)\,\mathrm{d}y \\
&\le Cn^{m-2r}\int_0^{n^r} yP(X > y)\,\mathrm{d}y.
\end{aligned}$$

From (2.3), we know that there exists an $M > 0$ such that $n^{m-r}yP(X > y) \le \varepsilon$ if $n^r > M$. Then

$$\begin{aligned}
I_{91} &\le Cn^{m-2r}\int_0^M yP(X > y)\,\mathrm{d}y + Cn^{-r}\int_M^{n^r} \varepsilon\,\mathrm{d}y \\
&\le Cn^{m-2r} + C\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and $m < 2r$, $I_{91} \to 0$ as $n \to \infty$.

Then we prove $I_{10} \to 0$ as $n \to \infty$. By the definition of X''_{nk} and the similar argument as in the proof $I_{92} \to 0$, we have

$$\begin{aligned} I_{10} &\le P(\exists k;\ 1 \le k \le k_n,\ \text{such that } |X_{nk}| > n^r) \\ &\le \sum_{k=1}^{k_n} P(|X_{nk}| > n^r) \\ &\le Cn^m P(X > n^r) \longrightarrow 0 \quad \text{as } n \to \infty. \end{aligned}$$

The proof is complete. □

Proof of Theorem 2.4. Let $p = (m+1)/r$ and

$$\begin{aligned} X'_{nk} &= X_{nk} I(|X_{nk}| \le t^{1/p}) + t^{1/p} I(X_{nk} > t^{1/p}) \\ &\quad - t^{1/p} I(X_{nk} < -t^{1/p}), \\ X''_{nk} &= X_{nk} - X'_{nk} = (X_{nk} - t^{1/p}) I(X_{nk} > t^{1/p}) \\ &\quad + (X_{nk} + t^{1/p}) I(X_{nk} < -t^{1/p}). \end{aligned}$$

For all $\varepsilon > 0$,

$$\begin{aligned} E\left|n^{-r}\sum_{k=1}^{n} X_{nk}\right|^p &= n^{-m-1}\int_0^\infty P\left(\left|\sum_{k=1}^{k_n} X_{nk}\right| > t^{1/p}\right) \mathrm{d}t \\ &\le \varepsilon + n^{-m-1}\int_{n^{m+1}\varepsilon}^\infty P\left(\left|\sum_{k=1}^{k_n} X_{nk}\right| > t^{1/p}\right) \mathrm{d}t \\ &\le \varepsilon + n^{-m-1}\int_{n^{m+1}\varepsilon}^\infty \sum_{k=1}^{k_n} P(|X_{nk}| > t^{1/p})\, \mathrm{d}t \\ &\quad + n^{-m-1}\int_{n^{m+1}\varepsilon}^\infty P\left(\left|\sum_{k=1}^{k_n}(X'_{nk} - EX'_{nk})\right| > t^{1/p}/2\right) \mathrm{d}t \\ &:= \varepsilon + I_{11} + I_{12}. \end{aligned}$$

To prove (2.6), it suffices to show that $I_{11} \to 0$ and $I_{12} \to 0$ as $n \to \infty$.

For I_{11}, by (2.5) we have

$$\begin{aligned} I_{11} &\le n^{-1}\int_{n^{m+1}\varepsilon}^{\infty} P(X > t^{1/p})\,\mathrm{d}t \\ &\le n^{-1}EX^{(m+1)/r}I(X > n^r\varepsilon^{r/(m+1)}) \quad (\text{by } m-r \ge -1) \\ &\le n^{m-r}EX^{(m+1)/r}I(X > n^r\varepsilon^{r/(m+1)}) \longrightarrow 0 \quad \text{as } n \to \infty. \end{aligned}$$

Then we prove $I_{12} \to 0$. Let $D_n = \sum_{k=1}^{k_n} E(X'_{nk} - EX'_{nk})^2$, $x = t^{1/p}/2$, $y = t^{1/p}/2\gamma$ and $\gamma > \max\{1, p/2\}$. By Lemma 1.3, we have

$$\begin{aligned} I_{12} &\le n^{-m-1}\int_{n^{m+1}\varepsilon}^{\infty}\sum_{k=1}^{k_n} P\left(|X'_{nk} - EX'_{nk}| > t^{1/p}/2\gamma\right)\mathrm{d}t \\ &\quad + Cn^{-m-1}\int_{n^{m+1}\varepsilon}^{\infty}\left(\frac{D_n}{D_n + t^{2/p}/4\gamma}\right)^{\gamma}\mathrm{d}t \\ &:= I_{13} + I_{14}. \end{aligned}$$

By $EX_{nk} = 0$ and (2.5), we know

$$\begin{aligned} t^{-1/p}|EX'_{nk}| &= t^{-1/p}|EX''_{nk}| \\ &\le t^{-1/p}E|X_{nk}|I(|X_{nk}| > t^{1/p}) \\ &\le t^{-1}E|X_{nk}|^pI(|X_{nk}| > t^{1/p}) \quad (\text{by } t \ge n^{m+1}\varepsilon) \\ &\le n^{-m-1}\varepsilon^{-1}EX^{(m+1)/r}I(X > n^r\varepsilon^{r/(m+1)}) \longrightarrow 0 \\ &\qquad \text{as } n \to \infty. \end{aligned}$$

Therefore, while n is sufficiently large, by a similar argument as in the proof $I_{11} \to 0$, we have

$$\begin{aligned} I_{13} &\le n^{-m-1}\int_{n^{m+1}\varepsilon}^{\infty}\sum_{k=1}^{k_n} P\left(|X'_{nk}| > t^{1/p}/4\gamma\right)\mathrm{d}t \\ &\le n^{-m-1}\int_{n^{m+1}\varepsilon}^{\infty}\sum_{k=1}^{k_n} P\left(|X_{nk}| > t^{1/p}/4\gamma\right)\mathrm{d}t \\ &\le n^{-1}\int_{n^{m+1}\varepsilon}^{\infty} P\left(X > t^{1/p}/4\gamma\right)\mathrm{d}t \longrightarrow 0 \quad \text{as } n \to \infty. \end{aligned}$$

Then we prove $I_{14} \to 0$. By C_r-inequality and $\gamma > 1$, we have

$$
\begin{aligned}
I_{14} &\le Cn^{-m-1}\int_{n^{m+1}\varepsilon}^{\infty}\Big(t^{-2/p}\sum_{k=1}^{k_n}E(X'_{nk})^2\Big)^{\gamma}\mathrm{d}t \\
&= Cn^{-m-1}\int_{n^{m+1}\varepsilon}^{\infty}\Big(t^{-2/p}\sum_{k=1}^{k_n}EX_{nk}^2I(|X_{nk}|\le n^r) \\
&\quad + t^{-2/p}\sum_{k=1}^{k_n}EX_{nk}^2I(n^r<|X_{nk}|\le t^{1/p}) \\
&\quad + \sum_{k=1}^{k_n}P(|X_{nk}|>t^{1/p})\Big)^{\gamma}\mathrm{d}t \\
&\le Cn^{-m-1}\int_{n^{m+1}\varepsilon}^{\infty}\Big(t^{-2/p}\sum_{k=1}^{k_n}EX_{nk}^2I(|X_{nk}|\le n^r)\Big)^{\gamma}\mathrm{d}t \\
&\quad + Cn^{-m-1}\int_{n^{m+1}\varepsilon}^{\infty}\Big(t^{-1}\sum_{k=1}^{k_n}E|X_{nk}|^pI(n^r<|X_{nk}|\le t^{1/p})\Big)^{\gamma}\mathrm{d}t \\
&\quad + Cn^{-m-1}\int_{n^{m+1}\varepsilon}^{\infty}\Big(\sum_{k=1}^{k_n}P(|X_{nk}|>t^{1/p})\Big)^{\gamma}\mathrm{d}t \\
&:= I'_{14}+I''_{14}+I'''_{14}.
\end{aligned}
$$

For I'''_{14}, by the similar argument as in the proof $I_{11} \to 0$, we have

$$
I'''_{14} \le Cn^{-m-1-m\gamma}\int_{n^{m+1}\varepsilon}^{\infty}P(X>t^{1/p})\,\mathrm{d}t \longrightarrow 0 \quad \text{as } n\to\infty.
$$

By Remark 2.2 we know that (2.5) implies (2.3). Therefore, by a similar argument as in the proof $I_{91} \to 0$, we have

$$
\begin{aligned}
I'_{14} &\le sCn^{-m-1}\sum_{k=1}^{k_n}EX_{nk}^2I(|X_{nk}|\le n^r) \\
&\quad \times\int_{n^{m+1}\varepsilon}^{\infty}t^{-2\gamma/p}\mathrm{d}t \quad (\text{by } \gamma>p/2) \\
&\le C\varepsilon^{1-2\gamma/p}n^{-2\gamma r}\sum_{k=1}^{k_n}EX_{nk}^2I(|X_{nk}|\le n^r) \quad (\text{by } \gamma>1)
\end{aligned}
$$

$$\leq C\varepsilon^{1-2\gamma/p}n^{-2r}\sum_{k=1}^{k_n}EX_{nk}^2I(|X_{nk}|\leq n^r)\longrightarrow 0\quad\text{as } n\to\infty.$$

For I''_{14}, by (2.5), we have

$$\begin{aligned}
I''_{14} &\leq Cn^{-m-1}\int_{n^{m+1}\varepsilon}^{\infty}\left(t^{-1}\sum_{k=1}^{k_n}E|X_{nk}|^pI(|X_{nk}|>n^r)\right)^{\gamma}\mathrm{d}t\\
&\leq Cn^{-m-1}\sum_{k=1}^{k_n}E|X_{nk}|^pI(|X_{nk}|>n^r)\int_{n^{m+1}\varepsilon}^{\infty}t^{-\gamma}\mathrm{d}t\\
&\leq C\varepsilon^{1-\gamma}n^{-(m+1)\gamma}\sum_{k=1}^{k_n}E|X_{nk}|^pI(|X_{nk}|>n^r)\\
&\leq C\varepsilon^{1-\gamma}n^{m-(m+1)\gamma}EX^pI(X>n^r)\quad(\text{by } \gamma>1 \text{ and } m+1\geq r)\\
&\leq C\varepsilon^{1-\gamma}n^{m-r}EX^pI(X>n^r)\longrightarrow 0\quad\text{as } n\to\infty.
\end{aligned}$$

The proof is complete. $\square$

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