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GLOBAL DYNAMICS FOR A HIGHER ORDER RATIONAL DIFFERENCE EQUATION

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ABSTRACT. In this paper, some properties of all positive solutions are considered for a higher order rational difference equation, mainly for the existence of eventual prime period two solutions, the existence and asymptotic behavior of non-oscillatory solutions and the global asymptotic stability of its equilibria. Our results show that a positive equilibrium point of this equation is a global attractor under appropriate conditions and that the origin of this equation is globally asymptotically stable. Our results also give an answer to the problem in [**9**].

1. Introduction and preliminaries. Consider the rational difference equation with higher order

$$x_{n+1} = \frac{px_n + x_{n-k}}{r + qx_n + x_{n-k}}, \quad n = 0, 1, 2, \dots, \tag{1.1}$$

where the parameters p, q and r are non-negative real numbers, k is a positive integer, and the initial conditions $x_{-k}, \dots, x_{-1}, x_0$ are nonnegative real numbers.

When $k = 1, 2$, equation (1.1) reduces respectively to

$$x_{n+1} = \frac{px_n + x_{n-1}}{r + qx_n + x_{n-1}}, \quad n = 0, 1, 2, \dots,$$

and

$$x_{n+1} = \frac{px_n + x_{n-2}}{r + qx_n + x_{n-2}}, \quad n = 0, 1, 2, \dots,$$

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whose properties have been investigated in [**8, 9**].

When $r = 0$, equation (1.1) reads

$$x_{n+1} = \frac{px_n + x_{n-k}}{qx_n + x_{n-k}}, \quad n = 0, 1, 2, \dots ,$$

whose properties have been thoroughly investigated by DeVault et al. in [**3**]. So, in the sequel, one will always assume that $r > 0$.

For(1.1), Kulenovic and Ladas presented the following question:

Open Problem 9.5.8 [**9**, page 166]. *Assume that*

$$p, q, r \in [0, \infty), \quad k \in \{2, 3, \dots\}.$$

Investigate the global behavior of all positive solutions of equation (1.1).

Motivated by the above open problem, our main aim in this paper is to investigate the global behavior of all positive solutions of (1.1) under the assumption that

$$p, q \in [0, \infty), \quad r \in (0, \infty), \; k \in \{1, 2, 3, \dots\}.$$

Firstly, we study in detail the properties of the origin of equation (1.1) in Section 2. Secondly, we formulate the necessary and sufficient conditions for the existence of eventual prime period two solutions of this equation in Section 3. Next, the global stability of positive equilibrium is stated in Section 4. Finally, Lerg's inclusion theorem is invoked and the existence and asymptotic behavior of non-oscillatory solutions of equation (1.1) are presented in Section 5.

The study of the rational difference equation (RDE, for short) is quite challenging and rewarding due to the fact that some results of rational difference equations offer prototypes for the development of the basic theory of global behavior of nonlinear difference equations; moreover, the investigations of rational difference equations are still in their infancy so far. To see this, refer to the monographs [**1, 8, 9**] and the papers [**3–7, 10–21**] and the references cited therein. For the work related to the express form of equation (1.1), see also [**3–7, 10–12, 14, 20, 21**].

Now, we present some basic concepts related to a higher-order difference equation and several key conclusions which will be used in the sequel.

Definition 1.1. Let I be some interval of real numbers and $f : I^2 \to I$ a continuously differentiable function.

Then, for every set of initial conditions $x_{-k}, \dots, x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-k}), \quad n = 0, 1, 2, \dots \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\overline{x}$ is called an equilibrium point of equation (1.2) if $\overline{x} = f(\overline{x}, \overline{x})$, that is, $x_n = \overline{x}$ for $n \geq 0$ is a solution of (1.2); or, equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.2. An interval $J \subset I$ is called an invariant interval of equation (1.2) if

$$x_{-k}, \cdots, x_{-1}, x_0 \in J \Longrightarrow x_n \in J \quad \text{for all } n > 0.$$

That is, every solution of (1.2) with initial conditions in J remains in J.

Let

$$P = \frac{\partial f}{\partial u}(\overline{x}, \overline{x}) \quad \text{and} \quad Q = \frac{\partial f}{\partial v}(\overline{x}, \overline{x})$$

denote the partial derivatives of $f(u, v)$ evaluated at an equilibrium $\overline{x}$ of equation (1.2). Then the equation

$$y_{n+1} = P y_n + Q y_{n-k}, \quad n = 0, 1, \dots \tag{1.3}$$

is called the linearized equation associated with (1.2) about $\overline{x}$.

Theorem A [**9**]. *Assume that $P, Q \in R$ and $k \in \{1, 2, \dots\}$. Then*

$$|P| + |Q| < 1 \tag{1.4}$$

is a sufficient condition for the asymptotic stability of equation (1.3). *Suppose, in addition, that one of the following two cases holds*:

(i) k *odd and* $Q > 0$;

(ii) k *even and* $PQ > 0$.

Then (1.4) *is also a necessary condition for the asymptotic stability of* (1.3).

Theorem B [**9**]. *Let* $[a, b]$ *be an interval of real numbers, and assume that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b]$$

is a continuous function satisfying the following properties:

(i) $f(x, y)$ *is nondecreasing in each of its arguments*;

(ii) *The equation* $f(y, y) = y$ *has a unique positive solution.*

Then the difference equation

$$y_{n+1} = f(y_n, y_{n-k}), \quad n = 0, 1, \dots$$

where $k \in \{1, 2, \dots\}$ *has a unique equilibrium* $\overline{y} \in [a, b]$, *and every solution of this equation converges to* $\overline{y}$.

Theorem C [**8**]. *Consider the difference equation*

$$x_{n+1} = g(x_n, \dots, x_{n-k}), \quad n = 0, 1, \dots, \tag{1.5}$$

where $g \in C[(0, \infty)^{k+1}, (0, \infty)]$ *is increasing in each of its arguments and where the initial condition* $x_{-k}, \dots, x_0$ *is positive. Assume that equation* (1.5) *has a unique positive equilibrium* $\overline{x}$, *and suppose that the function* h *defined by*

$$h(x) = g(x, \dots, x), \quad x \in [0, \infty)$$

satisfies

$$(h(x) - x)(x - \overline{x}) < 0, \quad \textit{for } x \neq \overline{x}.$$

Then $\overline{x}$ *is a global attractor of all positive solutions of* (1.5).

Theorem D [**9**]. *Consider the difference equation*

$$x_{n+1} = \sum_{i=0}^{k} x_{n-i} F_i(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots \tag{1.6}$$

with the initial condition $x_{-k}, \dots, x_{-1}, x_0 \in [0, \infty)$, *where*

(i) $k \in \{1, 2, \dots\}$;

(ii) $F_0, F_1, \dots, F_k \in C[[0,\infty)^{k+1}, [0,1)]$;

(iii) $F_0, F_1, \dots, F_k$ *are non-increasing in each argument*;

(iv) $\sum_{i=0}^{k} F_i(y_0, y_1, \dots, y_k) < 1$ *for all* $(y_0, y_1, \dots, y_k) \in (0,\infty)^{k+1}$;

(v) $F_0(y, y, \dots, y) > 0$ *for all* $y \geq 0$.

Then $\overline{x} = 0$ *is a globally asymptotically stable equilibrium of* (1.6).

The equilibrium points of (1.1) are solutions of the equation

$$\overline{x} = \frac{p\overline{x} + \overline{x}}{r + q\overline{x} + \overline{x}},$$

from which one can see that (1.1) has only one equilibrium point, the origin, when $p + 1 \leq r$, and that (1.1) has another positive equilibrium point $\overline{x} = (p + 1 - r)/(q + 1)$ besides the origin, when $p + 1 > r$.

2. Property of the origin. Let $f(x, y) = (px + y)/(r + qx + y)$. Then

$$\frac{\partial f}{\partial x} = \frac{pr + (p - q)y}{(r + qx + y)^2}, \quad \frac{\partial f}{\partial y} = \frac{r + (q - p)x}{(r + qx + y)^2}.$$

One can see that the linearized equation associated with (1.1) about the origin is

$$y_{n+1} - \frac{p}{r} y_n - \frac{1}{r} y_{n-k} = 0, \quad n = 0, 1, \dots \tag{2.1}$$

with the characteristic equation

$$g(\lambda) = \lambda^{k+1} - \frac{p}{r}\lambda^k - \frac{1}{r} = 0. \tag{2.2}$$

It is easy to observe that the function $g(\cdot)$ possesses the following properties:

1) $g(-\infty) = (-1)^{k+1}(+\infty)$, $g(+\infty) = +\infty$;

2) $g(-1) = ((-1)^{k+1}(p + r) - 1)/r$, $g(0) = -1/r < 0$, $g(1) = (r - (p + 1))/r$;

3) $g(\lambda)$ is decreasing in $\lambda \in (0, \lambda_0)$ and increasing in $\lambda \in (\lambda_0, \infty)$, where $\lambda_0 = (k/k+1)(p/r)$. $g(\lambda_0) = -(k^k)/(k+1)^{k+1}(p/r)^{k+1} - (1/r) < 0$, which is the minimum of $g(\lambda)$ in $(-\infty, +\infty)$ when k is an odd integer whereas the local extreme small value of $g(\lambda)$ in $(-\infty, +\infty)$ when k is an even integer.

By Theorem A, one can see that the origin is locally asymptotically stable if $p + 1 < r$.

When $p + 1 = r$, $g(1) = (r - (p + 1))/r = 0$, namely, $\lambda = 1$ is a root of the characteristic equation $g(\lambda) = 0$; hence, the origin is not stable.

For $p + 1 > r$, because of $g(1) = (r - (p + 1))/r < 0$ and $g(+\infty) = (+\infty)$, The continuity of the function $g(\cdot)$ in $[1, \infty)$ implies that there exists a $\lambda \in (1, \infty)$ such that $g(\lambda) = 0$, i.e., $g(\cdot)$ has a root with modulus greater than 1; therefore, the origin is not stable, either.

Summarizing the above disputation, one gets the following conclusion.

Theorem 2.1. *The origin of equation* (1.1) *is locally asymptotically stable for $p + 1 < r$, whereas it is not stable for $p + 1 \geq r$.*

For RDEs, there has been a folklore that local asymptotical stability of the equilibrium point implies its global asymptotical stability. Hence, many research projects in [**8**] and open problems and conjectures [**8, 9, 13**], we guess, are given based on this kind of idea. Indeed, most of the known results [**9, 13**] support this kind of folk parlance. How about equation (1.1)? We show that the folklore is true for the origin of (1.1); that is, the local asymptotical stability of the origin of (1.1) implies its global asymptotical stability. We will accurately prove the following results.

Theorem 2.2. *The origin of equation* (1.1) *is globally asymptotically stable for $p + 1 < r$.*

Proof. In view of Theorem 2.1, the origin of equation (1.1) is locally asymptotically stable for $p + 1 < r$. So, it suffices to show that the origin of (1.1) is globally attractive for $p + 1 < r$. Namely, it suffices to prove $\lim_{n\to\infty} x_n = 0$ for $p + 1 < r$. There are two cases to be considered:

1) $p = 0$;

2) $p > 0$.

Notice that $r > 1$ in either of the above two cases. Let us first consider case 1). In light of equation (1.1), one has

$$x_{n+1} = \frac{x_{n-k}}{r + qx_n + x_{n-k}} < 1, \quad n = 0, 1, \dots ,$$

namely, $x_n < 1$ for $n \geq 1$. Accordingly, for $n \geq k+1$,

$$x_{n+1} = \frac{x_{n-k}}{r + qx_n + x_{n-k}} < \frac{1}{1+r}, \quad n = k+1, k+2, \dots ,$$

namely, $x_n < 1/(1+r)$ for $n \geq k+1+1$.

Inductively, one has, for $N \in \{0, 1, \dots\}$,

$$x_n < \frac{1}{1 + r + r^2 + \cdots + r^N} = \frac{r-1}{r^{N+1} - 1} \quad \text{for } n \geq N(k+1) + 1.$$

From this, one can easily derive that $\lim_{n\to\infty} x_n = 0$.

Now one studies case 2). From (1.1), one has

$$x_{n+1} = \frac{p}{r + qx_n + x_{n-k}} x_n + \frac{1}{r + qx_n + x_{n-k}} x_{n-k}.$$

It is obvious that the functions $p/(r + qx + y)$ and $1/(r + qx + y)$ are nonincreasing in each of their arguments. So, for $p+1 < r$, one can see

$$\frac{p}{r + qx_n + x_{n-k}} + \frac{1}{r + qx_n + x_{n-k}} = \frac{p+1}{r + qx_n + x_{n-k}} \leq \frac{p+1}{r} < 1$$

and

$$\frac{p}{r + qy + y} > 0 \quad \text{for } y \geq 0.$$

By Theorem D, we can see that the origin of equation (1.1) is a global attractor. The proof is complete. □

3. Period two solution. In this section one will consider the existence of eventual non-negative prime period two solutions of equation (1.1). A solution $\{x_n\}_{n=-k}^{\infty}$ of (1.1) is said to be an eventually periodic two solution if there exists an $n_0 \in \{-k, -k+1, \dots\}$ such that $x_{n+2} = x_n$ for $n \geq n_0$.

Let

$$\dots, \phi, \psi, \phi, \psi, \dots$$

be an eventual prime period-two solution of (1.1), where ϕ and ψ are two distinct non-negative real numbers.

If k is odd, then $x_{n+1} = x_{n-k}$ is true eventually. So ϕ and ψ satisfy the system which consists of the following two equations:

$$\phi = \frac{p\psi + \phi}{r + q\psi + \phi}, \qquad \psi = \frac{p\phi + \psi}{r + q\phi + \psi}, \tag{3.1}$$

which indicates

$$(\phi - \psi)(\phi + \psi + r + p - 1) = 0. \tag{3.2}$$

Notice that $\phi, \psi \in [0, \infty)$ with $\phi \neq \psi$. So, it is impossible for equation (3.2) to have solutions for $p + r \geq 1$. When $p + r < 1$, (3.2) tells us $\phi + \psi + r + p - 1 = 0$. Or, equivalently,

$$\phi = 1 - p - r - \psi. \tag{3.3}$$

Substituting (3.3) into any one equation of (3.1), one has that ϕ or ψ satisfies

$$(q-1)x^2 + (q-1)(1-r-p)x - p(1-p-r) = 0. \tag{3.4}$$

From (3.4), one can see:

1) If $q = 1$, then $p = 0$. So $\phi \in [0, 1-r]$ with $\varphi \neq (1-r)/2$ and $\psi = 1 - r - \varphi$.

2) If $q \neq 1$, then ϕ or ψ is the non-negative root of equation (3.4). More accurately, one has that, for $q \in [(2-p-2r)/(2(1-p-r)), \infty)$, $\psi = 1 - p - r - \varphi$ with $\varphi \neq (1-p-r)/2$, where

$$\varphi = \frac{\sqrt{(q-1)^2(1-p-r)^2 + 4p(q-1)(1-p-r)} - (q-1)(1-p-r)}{2(q-1)}.$$

If k is even, then $x_n = x_{n-k}$ holds eventually. Hence, ϕ and ψ satisfy the following system:

$$\phi = \frac{p\psi + \psi}{r + q\psi + \psi}, \qquad \psi = \frac{p\phi + \phi}{r + q\phi + \phi}.$$

From this, one can derive

$$(\phi - \psi)(p + r + 1) = 0.$$

This is impossible because $p+r+1 > 0$ and $\phi \neq \psi$. Hence, there do not exist eventual non-negative prime period two solutions of equation (1.1) for k even.

Combining the above discussions, one has the following result.

Theorem 3.1. *Equation* (1) *eventually has non-negative prime period-two solutions if and only if k is odd and $p + r < 1$ and $q \in \{1\} \cup [(2 - p - 2r)/(2(1 - p - r)), \infty)$. At this time the prime period-two solution of* (1) ,

$$\dots, \phi, \psi, \phi, \psi, \dots,$$

satisfies one of the following conditions:

(i) *For $q = 1$ (implying $p = 0$), $\phi \in [0, 1-r]$ with $\varphi \neq (1 - r)/2$ and $\psi = 1 - r - \varphi$.*

(ii) *For $q \in [(2 - p - 2r)/(2(1 - p - r)), \infty)$, $\psi = 1 - p - r - \varphi$ with $\varphi \neq (1 - p - r)/2$, where*

$$\varphi = \frac{\sqrt{(q-1)^2(1-p-r)^2 + 4p(q-1)(1-p-r)} - (q-1)(1-p-r)}{2(q-1)}.$$

4. Character of positive equilibrium point. In this section, we will study the global behavior of positive equilibrium of equation (1.1). Remember that it always assumes $p+1 > r$ and $\overline{x} = (p + 1 - r)/(q + 1)$. We first study the local asymptotical stability of $\overline{x}$. The result is expressed by the following proposition.

Theorem 4.1. *The positive equilibrium* $\overline{x} = (p+1-r)/(q+1)$ *is locally asymptotically stable, provided that one of the following conditions is satisfied*:

(i) $1 \leq q \leq p-r$ *or* $q \leq p-r < (3q+1)/(1-q)$;

(ii) $q > p-r$ *and* $q(1-r) \leq p$ *or* $q(1-r) > p > [(1-r)(q-1)]/(q+3)$.

Proof. The linearized equation associated with equation (1.1) about positive equilibrium point $\overline{x}$ is

$$y_{n+1} - \frac{pr+(p-q)\overline{x}}{(r+q\overline{x}+\overline{x})^2}y_n - \frac{r+(q-p)\overline{x}}{(r+q\overline{x}+\overline{x})^2}y_{n-k} = 0,$$

namely,

$$y_{n+1} - \frac{p+qr-q}{(p+1)(q+1)}y_n - \frac{r+q-p}{(p+1)(q+1)}y_{n-k} = 0. \tag{4.1}$$

If $1 \leq q \leq p-r$ or $q \leq p-r < (3q+1)/(1-q)$, then $(1-q)(p-r) < 3q+1$. Hence,

$$\begin{aligned}|P|+|Q| &= \left|\frac{p+qr-q}{(p+1)(q+1)}\right| + \left|\frac{r+q-p}{(p+1)(q+1)}\right| \\ &= \frac{p+qr-q+p-(q+r)}{(p+1)(q+1)} = \frac{2(p-q)+(q-1)r}{(p+1)(q+1)} < 1.\end{aligned}$$

If the case where $q > p-r$ and $q(1-r) \leq p$ occurs, then

$$\begin{aligned}|P|+|Q| &= \left|\frac{p+qr-q}{(p+1)(q+1)}\right| + \left|\frac{r+q-p}{(p+1)(q+1)}\right| \\ &= \frac{p+qr-q+(q+r)-p}{(p+1)(q+1)} = \frac{(q+1)r}{(p+1)(q+1)} = \frac{r}{p+1} < 1.\end{aligned}$$

If the case where $q > p-r$ and $q(1-r) > p > [(1-r)(q-1)]/(q+3)$ happens, then $(q-1)(1-r) < p(q+3)$. So,

$$\begin{aligned}|P|+|Q| &= \left|\frac{p+qr-q}{(p+1)(q+1)}\right| + \left|\frac{r+q-p}{(p+1)(q+1)}\right| \\ &= \frac{q-(p+qr)+(q+r)-p}{(p+1)(q+1)} = \frac{2(q-p)+r(1-q)}{(p+1)(q+1)} < 1.\end{aligned}$$

So, according to Theorem A, either of the above cases i) and ii) being satisfied leads to the local asymptotical stability of the positive equilibrium $\overline{x}$ of equation (1.1). $\square$

Next, we investigate the global attractivity of $\overline{x}$. We will consider three cases: $p = q$, $p < q$ and $p > q$, respectively.

Case 1. $p = q$**.** If $p = q = 0$, then $r < p + 1 = 1$ and equation (1.1) reads

$$x_{n+1} = \frac{x_{n-k}}{r + x_{n-k}}, n = 0, 1, \dots . \tag{4.1}$$

Obviously, $x_n < 1$ for $n = 1, 2, \dots$. Denote

$$m = \min\{x_{-k}, x_{-k+1}, \dots , x_{-1}, x_0\} > 0.$$

Then, $x_n \geq m/(m + r)$ for $n = 1, 2, \dots , k + 1$. One can inductively show that

$$x_n \geq \frac{m}{m + mr + mr^2 + \cdots + mr^{N-1} + r^N} = \frac{m(1 - r)}{m + r^N(1 - r - m)}$$

for $n = (N - 1)(k + 1) + 1$, $(N - 1)(k + 1) + 2, \dots , N(k + 1)$ and $N \in \{1, 2, \dots\}$. Therefore, any solution $\{x_n\}_{-k}^{\infty}$ of equation (4.1) is bounded and persists. Set

$$\lambda = \lim_{n\to\infty} \inf x_n \quad \text{and} \quad \mu = \lim_{n\to\infty} \sup x_n.$$

Then

$$0 < \lambda \leq \overline{x} \leq \mu < \infty. \tag{4.2}$$

In view of equation (4.1), one has

$$\lambda \geq \frac{\lambda}{r + \lambda} \quad \text{and} \quad \mu \leq \frac{\mu}{r + \mu},$$

that displays that $\lambda \geq 1 - r = \overline{x} \geq \mu$. This, together with (4.2), gives rise to $\lim_{n\to\infty} x_n = \overline{x}$.

If $p = q > 0$, equation (1.1) becomes

$$x_{n+1} = \frac{qx_n + x_{n-k}}{r + qx_n + x_{n-k}} =: f(x_n, x_{n-k}), \quad n = 0, 1, \dots .$$

It is evident that $x_n \in [0, 1]$ for all $n \geq 1$ and

$$\frac{\partial f}{\partial x_n} = \frac{pr}{(r + qx_n + x_{n-k})^2} > 0 \text{ and } \frac{\partial f}{\partial x_{n-k}} = \frac{r}{(r + qx_n) + x_{n-k}} > 0.$$

(Notice that equation (1.1) is an autonomous RDE. Without loss of generality, one can assume that the initial values $x_{-k}, \dots, x_{-1}, x_0 \in$ [0, 1]).

Put $h(x) = (qx + x)/(r + qx + x)$. We have

$$\begin{aligned} h(x) - x &= \frac{(q+1)x}{r + (q+1)x} - x = \frac{x(q + 1 - r - (q+1)x)}{r + (q+1)x} \\ &= \frac{x(q+1)((q+1-r)/(q+1) - x)}{r + (q+1)x} = \frac{x(q+1)(\overline{x} - x)}{r + (q+1)x}. \end{aligned}$$

Hence, $(h(x) - x)(x - \overline{x}) < 0$ for $x \neq \overline{x}$. By Theorem C, one can see that $\overline{x} = \frac{p+1-r}{p+1}$ is a global attractor. Accordingly, $\overline{x}$ is globally attractive for $p = q$.

Case 2. $p < q$**.** In this case, one has $x_n \in [0, 1]$ for $n \geq 1$ and

$$\frac{\partial f}{\partial x_{n-k}} = \frac{r + (q-p)x_n}{(r + qx_n + x_{n-k})^2} > 0.$$

If $pr + p \geq q$, then

$$\frac{\partial f}{\partial x_n} = \frac{pr + (p-q)x_{n-k}}{(r + qx_n + x_{n-k})^2} \geq \frac{pr + p - q}{(r + qx_n + x_{n-k})^2} \geq 0.$$

By Theorem B, the positive equilibrium is a global attractor.

Combining Cases 1 and 2, one gets the following results.

Theorem 4.2. *The positive equilibrium point* $\overline{x}$ *of equation* (1.1) *is a global attractor of all positive solutions of* (1.1) *for* $p \leq q \leq pr + p$.

Case 3. $p > q$**.** The following observations hold:

1) $x_{n+1} - (p/q) = [(q-p)x_{n-k} - pr]/[q(r + qx_n + x_{n-k})] < 0$, $n = 0, 1, \dots$;

2) $x_{n+1} - x_n = [qx_n((p - r/q) - x_n) + x_{n-k}(1 - x_n)]/[r + qx_n + x_{n-k}]$;

3) $x_{n+1} - 1 = [(p-q)(x_n - (r/p-q))]/[r + qx_n + x_{n-k}]$.

So, $x_n \in [0, (p/q)]$ for $n \geq 1$. Moreover, the following results can easily be derived.

Lemma 4.3. *For $p > q$ and for any positive solutions of* (1.1), *the following conclusions are true*:

(i) *If there exists an $N \in \{1, 2, \dots\}$ such that $x_N > \max\{1, (p-r)/q\}$, then $x_{N+1} < x_N$.*

(ii) *If there exists an $N \in \{1, 2, \dots\}$ such that $x_N < \min\{1, (p-r)/q\}$, then $x_{N+1} > x_N$.*

For the invariant interval of equation (1.1), one easily has the following results.

Lemma 4.4. (i) *If $1 < r/(p-q)$ and $x_{-k}, \dots, x_{-1}, x_0 \in (0, r/(p-q))$, then $x_n < 1 < r/(p-q)$ for $n \geq 1$, i.e., $(0, r/(p-q))$ is an invariant interval of* (1.1).

(ii) *If $1 > r/(p-q)$ and $x_{-k}, \dots, x_{-1}, x_0 \in (r/(p-q), (p/q))$, then $x_n > 1 > r/(p-q)$ for $n \geq 1$. Namely, $(r/(p-q), (p/q))$ is an invariant interval of equation (1.1).*

We now give another result for the global attractivity of $\overline{x}$ of (1.1).

Theorem 4.5. *If $r/(p-q) > 1$, and $x_{-k}, \dots, x_{-1}, x_0 \in (0, r/(p-q))$, then the positive equilibrium $\overline{x}$ of equation* (1.1) *is a global attractor for all positive solutions of* (1.1).

Proof. $r/(p-q) > 1$ implies that $p > q$. By the above Lemma 4.4, $x_n < 1 < r/(p-q)$ for $n \geq 1$. So,

$$\frac{\partial f}{\partial x_n} = \frac{pr + (p-q)x_{n-k}}{(r + qx_n + x_{n-k})^2} > 0$$

and

$$\frac{\partial f}{\partial x_{n-k}} = (p-q)\frac{(r/(p-q)) - x_n}{(r+qx_n+x_{n-k})^2} > 0.$$

It is easy to see that $x = (px+x)/(r+qx+x)$ has a unique positive solution. Thus, by Theorem B, $\overline{x}$ is a global attractor.

5. Existence and asymptotic behavior of non-oscillatory solutions. In this section we consider the existence and asymptotic behavior of non-oscillatory solutions of (1.1). Relative to zero equilibrium, every positive solution of (1.1) is non-oscillatory. So, we only consider the positive equilibrium point.

Firstly, we have the following results.

Theorem 5.1. *Assume that* $p < q(1-r)$. *Then every non-oscillatory solution of* (1.1) *approaches* $\overline{x}$.

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be any one non-oscillatory solution of equation (1.1). Then, there exists an $n_0 \in \{-k, -k+1, \dots\}$, such that

$$x_n \geq \overline{x} \quad \text{for } n \geq n_0 \tag{5.1}$$

or

$$x_n < \overline{x} \quad \text{for } n \geq n_0. \tag{5.2}$$

We only prove the case where (5.1) holds. The proof for the case where (5.2) holds is similar and will be omitted. According to (5.1), for $n \geq n_0 + l$, one has

$$x_{n+1} = x_{n-k}\frac{px_n/x_{n-k}+1}{r+qx_n+x_{n-k}} \leq x_{n-k}\frac{px_n/\overline{x}+1}{r+qx_n+\overline{x}} \leq x_{n-k}.$$

So, $\{x_{j+(k+1)i}\}_{i=0}^{\infty}$ is decreasing for $j \in \{-k, -k+1, \dots, -1, 0\}$ with lower bound $\overline{x}$. So, $\lim_{i\to\infty} x_{j+(l+1)i}$ exists and is finite. Denote

$$\lim_{i\to\infty} x_{j+(k+1)i} = \alpha_j, \quad j \in \{-k, -k+1, \dots, -1, 0\}.$$

Then $\alpha_j \geq \overline{x}$ for all $j \in \{-k, -k+1, \dots, -1, 0\}$. Taking limits on both sides of (1.1), one has

$$\alpha_j = \frac{p\alpha_{j-1} + \alpha_j}{r + q\alpha_{j-1} + \alpha_j} \quad \text{for } j \in \{-k, -k+1, \dots, -1, 0\},$$

from which one may obtain $\alpha_{j-1} = [\alpha_j(r - 1 + \alpha_j)]/(p - q\alpha_j) =: F(\alpha_j)$. It is easy to see that $F'(x) = [-q(x - (p/q))^2 + (p/q)(p - q(1 - r))]/(p - qx)^2 < 0$ for $x \geq \overline{x} > (p/q)$. So, if, for some $j_0 \in \{-k, -k+1, \dots, -1, 0\}$, $\alpha_{j_0} > \overline{x}$, then

$$\alpha_{j_0-1} = F(\alpha_{j_0}) < F(\overline{x}) = \overline{x}.$$

This contradicts the fact that $\alpha_j \geq \overline{x}$ for all $j \in \{-k, -k+1, \dots, -1, 0\}$. Hence, one has $\alpha_j = \overline{x}, j \in \{-k, -k+1, \dots, -1, 0\}$. This demonstrates $\lim_{n\to\infty} x_n = \overline{x}$ and completes this proof. ◻

A problem naturally rises: Are there non-oscillatory solutions of equation (1.1)? Next, we will positively answer this question. Our result is as follows.

Theorem 5.2. *Assume that $p < q(1-r)$. Then* (1.1) *possesses non-oscillatory solutions asymptotically approaching its equilibrium point.*

The main tool to prove this theorem is to make use of Berg's inclusion theorem [**2**]. Now, for the sake of convenience of the statement, we first state some preliminaries. Consider a general real nonlinear difference equation of order $l \geq 1$ with the form

$$F(x_n, x_{n+1}, \dots, x_{n+l}) = 0, \tag{5.3}$$

where $F : \mathbf{R}^{l+1} \mapsto \mathbf{R}$, $n \in \mathbf{N}_0$. Let φ_n and ψ_n be two sequences satisfying $\psi_n > 0$ and $\psi_n = o(\varphi_n)$ as $n \to \infty$. Then, (maybe under certain additional conditions), for any given $\varepsilon > 0$, there exist a solution $\{x_n\}_{n=-l}^{\infty}$ of equation (5.3) and an $n_0(\varepsilon) \in \mathbf{N}$ such that

$$\varphi_n - \varepsilon\psi_n \leq x_n \leq \varphi_n + \varepsilon\psi_n, \quad n \geq n_0(\varepsilon). \tag{5.4}$$

Denote

$$X(\varepsilon) = \{x_n : \varphi_n - \varepsilon\psi_n \leq x_n \leq \varphi_n + \varepsilon\psi_n, \quad n \geq n_0(\varepsilon)\},$$

which is called an asymptotic stripe. So, if $x_n \in X(\varepsilon)$, then it is implied that there exists a real sequence C_n such that $x_n = \varphi_n + C_n\psi_n$ and $|C_n| \le \varepsilon$ for $n \ge n_0(\varepsilon)$.

We now state the inclusion theorem [**2**].

Lemma 5.3. *Let $F(\omega_0, \omega_1, \dots, \omega_l)$ be continuously differentiable when $\omega_i = y_{n+i}$, for $i = 0, 1, \dots, l$, and $y_n \in X(1)$. Let the partial derivatives of F satisfy*

$$F_{\omega_i}(y_n, y_{n+1}, \dots, y_{n+l}) \sim F_{\omega_i}(\varphi_n, \varphi_{n+1}, \dots, \varphi_{n+l})$$

as $n \to \infty$ uniformly in C_j for $|C_j| \le 1$, $n \le j \le n+l$, as far as $F_{\omega_i} \not\equiv 0$. Assume that there exist a sequence $f_n > 0$ and constants $A_0, A_1, \dots, A_l$ such that both

$$F(\varphi_n, \dots, \varphi_{n+l}) = o(f_n)$$

and

$$\psi_{n+i}F_{w_i}(\varphi_n, \dots, \varphi_{n+l}) \sim A_i f_n$$

for $i = 0, 1, \dots, l$ as $n \to \infty$, and suppose there exists an integer s, with $0 \le s \le l$, such that

$$|A_0| + \cdots + |A_{s-1}| + |A_{s+1}| + \cdots + |A_l| < |A_s|.$$

Then, for sufficiently large n, there exists a solution $\{x_n\}_{n=-l}^{\infty}$ of (5.3) *satisfying* (5.4).

Proof of Theorem 5.2. Put $y_n = x_n - \overline{x}$. Then equation (1.1) is transformed into

$$(y_{n+1}+\overline{x})(qy_n+y_{n-k}+p+1)-(py_n+y_{n-k}+(p+1)\overline{x}) = 0, \quad n = 0, 1, \dots .$$

Namely, for $n = -k, -k+1, \dots,$

$$(5.5)\quad (y_{n+k+1} + \overline{x})(qy_{n+k} + y_n + p + 1) - (py_{n+k} + y_n + (p+1)\overline{x}) = 0.$$

An approximate equation of (5.5) is the equation

$$(p+1)z_{n+k+1} + (q\overline{x} - p)z_{n+k} + (\overline{x} - 1)z_n = 0, \quad n = -k, -k+1, \dots, \tag{5.6}$$

provided that $z_n \to 0$ as $n \to \infty$. The general solution to (5.6) is $z_n = \sum_{i=0}^{k} c_i t_i^n$, where $c_i, i = 0, 1, \dots, k$, are complex numbers and $t_i, i = 0, 1, \dots, k$, are the $(k+1)$ roots of the polynomial

$$P(t) = (p+1)t^{k+1} + (q\overline{x} - p)t^k + \overline{x} - 1.$$

Obviously, $P(0)P(1) = \overline{x}(p-q-r) < 0$. Hence, $P(t) = 0$ has a solution $t_0 \in (0, 1)$. Now, choose the solution $z_n = t_0^n$ for this $t_0 \in (0, 1)$. For some $\lambda \in (1, 2)$, define the sequences $\{\varphi_n\}$ and $\{\psi_n\}$, respectively, as follows

$$\varphi_n = t_0^n \quad \text{and} \quad \psi_n = t_0^{\lambda n}. \tag{5.7}$$

Obviously, $\psi_n > 0$ and $\psi_n = o(\varphi_n)$ as $n \to \infty$.

Now, again define the function

$$F(\omega_0, \omega_1, \dots, \omega_k, \omega_{k+1}) = (\omega_{k+1} + \overline{x})(q\omega_k + \omega_0 + p + 1) - (p\omega_k + \omega_0 + (p+1)\overline{x}). \tag{5.8}$$

Then the partial derivatives of F with respect to $\omega_0, \omega_1, \dots, \omega_{k+1}$, respectively, are

$$\begin{aligned} F_{\omega_0} &= \omega_{k+1} + \overline{x} - 1, \\ F_{\omega_k} &= q\omega_{k+1} + q\overline{x} - p, \\ F_{\omega_{k+1}} &= q\omega_k + \omega_0 + p + 1, \\ F_{\omega_i} &= 0, \quad i = 1, \dots, k-1. \end{aligned} \tag{5.9}$$

When $y_n \in X(1)$, $y_n \sim \varphi_n$. So,

$$F_{\omega_i}(y_n, y_{n+1}, \dots, y_{n+k+1}) \sim F_{\omega_i}(\varphi_n, \varphi_{n+1}, \dots, \varphi_{n+k+1}), \\ i = 0, 1, \dots, l+1,$$

as $n \to \infty$ uniformly in C_j for $|C_j| \le 1$, $n \le j \le n+k+1$.

Moreover, from definition (5.8) of the function F and equation (5.7), after some calculation, we find

$$\begin{aligned} F(\varphi_n, \varphi_{n+1}, \dots, \varphi_{n+k+1}) &= q t_0^{2(n+k)+1} + t_0^{2n+k+1} + (p+1) t_0^{n+k+1} \\ &\quad + (q\overline{x} - p) t_0^{n+k} + (\overline{x} - 1) t_0^n \\ &= q t_0^{2(n+k)+1} + t_0^{2n+k+1}. \end{aligned}$$

Now, choose $f_n = t_0^{\lambda n}$. Then one can easily see that

$$F(\varphi_n, \dots, \varphi_{n+l+1}) = o(f_n) \quad \text{as } n \to \infty.$$

Again, from (5.7) and (5.9), one has that

$$\begin{aligned} &\psi_n F_{\omega_0}(\varphi_n, \varphi_{n+1}, \dots, \varphi_{n+l+1}) = t_0^{\lambda n}(t_0^{n+k+1} + \overline{x} - 1), \\ &\psi_{n+k} F_{\omega_k}(\varphi_n, \varphi_{n+1}, \dots, \varphi_{n+l+1}) = t_0^{\lambda(n+k)}(q t_0^{n+k+1} + q\overline{x} - p), \\ &\psi_{n+k+1} F_{\omega_{k+1}}(\varphi_n, \varphi_{n+1}, \dots, \varphi_{n+l+1}) = t_0^{\lambda(n+k+1)}(q t_0^{n+k} + t_0^n + p + 1). \end{aligned}$$

Hence,

$$\psi_n F_{\omega_0} \sim A_0 f_n, \qquad \psi_{n+k} F_{\omega_k} \sim A_k f_n, \qquad \psi_{n+k+1} F_{\omega_{k+1}} \sim A_{k+1} f_n,$$

where

$$A_0 = \overline{x} - 1, \qquad A_k = (q\overline{x} - p) t_0^{\lambda k}, \qquad A_{k+1} = (p+1) t_0^{\lambda(k+1)}.$$

Therefore, one has

$$\begin{aligned} |A_1| + \dots + |A_{k+1}| &= (q\overline{x} - p) t_0^{\lambda k} + (p+1) t_0^{\lambda(k+1)} \\ &< (q\overline{x} - p) t_0^{k} + (p+1) t_0^{k+1} \\ &= 1 - \overline{x} = |A_0|. \end{aligned}$$

Up to here, all conditions of Lemma 5.3 with $l = k+1$ and $s = 0$ are satisfied. Accordingly, we see that, for arbitrary $\varepsilon \in (0,1)$ and for sufficiently large n, say $n \geq N_0 \in \mathbf{N}$, (5.5) has a solution $\{y_n\}_{n=-k}^{\infty}$ in the stripe $\varphi_n - \varepsilon\psi_n \leq y_n \leq \varphi_n + \varepsilon\psi_n$, $n \geq N_0$, where φ_n and ψ_n are as defined in (5.7). Because $\varphi_n - \varepsilon\psi_n > \varphi_n - \psi_n = t_0^n - t_0^{\lambda n} > 0$, $y_n > 0$ for $n \geq N_0$. Thus, equation (1.1) has a solution $\{x_n\}_{n=-k}^{\infty}$

satisfying $x_n = y_n + \overline{x} > \overline{x}$ for $n \geq N_0$. Since (1.1) is an autonomous equation, $\{x_{n+N_0+k}\}_{n=-k}^{\infty}$ still is its solution, which evidently satisfies $x_{n+N_0+k} > \overline{x}$ for $n \geq -k$. Therefore, the proof is complete. □

Remark 5.4. If we take $\varphi_n = -t_0^n$ in (5.7), then $\varphi_n + \varepsilon\psi_n < -t_0^n + t_0^{\lambda n} < 0$. At this time, (1.1) possesses solutions $\{x_n\}_{n=-k}^{\infty}$ which remain below its equilibrium for all $n \geq -k$, i.e., (1.1) has solutions with a single negative semi-cycle.

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