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EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS FOR FOURTH-ORDER m-POINT BOUNDARY VALUE PROBLEMS WITH TWO PARAMETERS

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ABSTRACT. This paper deals with the existence and uniqueness of positive solutions to fourth-order m-point boundary value problems with two parameters. The arguments are based upon a specially constructed cone and a fixed point theorem in a cone for a completely continuous operator, due to Krasnoselskii and Zabreiko. The results obtained herein generalize and complement the main results of [**7, 10**].

1. Introduction. In this paper, we will study the existence and uniqueness of positive solutions for the following fourth-order m-point boundary value problems with two parameters

$$(1.1)\qquad \begin{cases} u^{(4)} + \alpha u'' - \beta u = f(t,u), \quad 0 < t < 1, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i),\ u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u''(0) = \sum_{i=1}^{m-2} a_i u''(\xi_i),\ u''(1) = \sum_{i=1}^{m-2} b_i u''(\xi_i), \end{cases}$$

where $\alpha,\ \beta \in \mathbf{R}$, $\xi_i \in (0,1)$, $a_i, b_i \in \mathbf{R}^+$ for $i \in \{1, 2, \dots, m-2\}$ are given constants, $\mathbf{R}^+ = [0, +\infty)$, and $\alpha,\ \beta,\ f$ satisfy

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(H_1) $f : [0,1] \times \mathbf{R}^+ \to \mathbf{R}^+$ is continuous;

(H_2) $\alpha,\ \beta \in \mathbf{R}$ and $\alpha < 2\pi^2$, $\beta \geq -\alpha^2/4$, $(\alpha/\pi^2) + (\beta/\pi^4) < 1$.

Multi-point boundary value problems (BVPs) for ordinary differential equations arise in a variety of areas of applied mathematics and physics. For instance, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multi-point BVP in [**12**]; also, many problems in the theory of elastic stability can be handled by multi-point problems in [**13**]. Recently, the existence and multiplicity of positive solutions for second order or fourth order multi-point BVPs has been studied by many authors using the nonlinear alternative of Leray-Schauder, the method of upper and lower solutions, the Leggett-Williams fixed point theorem, the fixed point index theory and the fixed point theorem of expansion and compression type (in terms of norms) in a cone, see [**2–5, 9, 11, 15–19**] and references therein.

In 2003, Li [**7**] studied the existence of positive solutions for fourth order BVP with two parameters

$$\begin{cases} u^{(4)} + \alpha u'' - \beta u = f(t,u), \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{1.2}$$

Under the assumptions of (H_1), (H_2) and some other conditions, the existence results of positive solutions are obtained. Very recently, in [**10**] the fourth order m-point BVP (1.1) is considered under some conditions concerning the first eigenvalue of the relevant linear operator. The existence of positive solutions is studied by means of fixed point index theory for the case:

$$\liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda_*, \qquad \limsup_{u\to +\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda_*,$$

where λ_* is the first eigenvalue of the relevant linear operator. However, for another case:

$$\limsup_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda_*, \qquad \liminf_{u\to +\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda_*,$$

the author still does not know whether a similar result holds. Ma also established the uniqueness result for a kind of special case: $f(t,u) = h(t)u^r$, $r \in (0,1)$.

Motivated by the above works and [**1, 8, 14**], we will consider the fourth-order m-point BVPs (1.1). It is clear that, when $a_i = b_i = 0$, BVP (1.1) reduces to the BVP (1.2). In this paper, we establish the existence and uniqueness of positive solutions for BVP (1.1). The main tool used in the proofs of existence results is a fixed point theorem in a cone due to Krasnoselskii and Zabreiko. We remark that our methods are entirely different from those used in [**2–5, 7, 9–11, 15–19**]. Our results extend and complement corresponding ones in [**7, 10**].

By a positive solution of (1.1), we mean a function u which is positive on $(0,1)$ and satisfies the differential equation as well as the boundary conditions in (1.1).

The following fixed point theorem in a cone, due to Krasnoselskii and Zabreiko [**6**], is of crucial importance in our proofs.

Lemma 1.1. *Let E be a real Banach space and W a cone of E. Suppose that $A : (\overline{B}_R \backslash B_r) \cap W \to W$ is a completely continuous operator with $0 < r < R$, where $B_\rho = \{x \in E : \|x\| < \rho\}$ for $\rho > 0$. If either*:

(1) *$Au \not\leq u$ for each $u \in \partial B_r \cap W$ and $Au \not\geq u$ for each $u \in \partial B_R \cap W$, or*

(2) *$Au \not\geq u$ for each $u \in \partial B_r \cap W$ and $Au \not\leq u$ for each $u \in \partial B_R \cap W$,*

then A has at least one fixed point on $(B_R \backslash \overline{B}_r) \cap W$.

The remainder of this paper is organized as follows. In Section 2, we give some preliminary lemmas. Section 3 is devoted to the existence of positive solutions for (1.1). Section 4 is concerned with the uniqueness of positive solution for (1.1).

2. Preliminaries and lemmas. To state and prove the main result of this paper, we need the following lemmas.

Lemma 2.1 [**7**]. *Let (H_2) hold. Then there exist unique φ_1, φ_2, ψ_1, and ψ_2 satisfying*:

$$\begin{cases} -\varphi_1'' + \lambda_1 \varphi_1 = 0, \\ \varphi_1(0) = 0, \quad \varphi_1(1) = 1; \end{cases} \qquad \begin{cases} -\varphi_2'' + \lambda_1 \varphi_2 = 0, \\ \varphi_2(0) = 1, \quad \varphi_2(1) = 0; \end{cases}$$
$$\begin{cases} -\psi_1'' + \lambda_2 \psi_1 = 0, \\ \psi_1(0) = 0, \quad \psi_1(1) = 1; \end{cases} \qquad \begin{cases} -\psi_2'' + \lambda_2 \psi_2 = 0, \\ \psi_2(0) = 1, \quad \psi_2(1) = 0, \end{cases}$$

respectively. And, on $[0,1]$, $\varphi_1,\ \varphi_2,\ \psi_1,\ \psi_2 \geq 0$, $\varphi_1'(0) > 0$, $\psi_1'(0) > 0$, *where* $\lambda_1,\ \lambda_2$ *are the roots for the polynomial equation* $\lambda^2 + \alpha\lambda - \beta = 0$. *That is,*

$$\lambda_1 = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta}}{2}, \qquad \lambda_2 = \frac{-\alpha - \sqrt{\alpha^2 + 4\beta}}{2}.$$

Notation. Set

$$\rho_1 = \varphi_1'(0), \qquad \rho_2 = \psi_1'(0),$$

$$\Delta_1 = \begin{vmatrix} \Sigma_{i=1}^{m-2} a_i \varphi_1(\xi_i) & \Sigma_{i=1}^{m-2} a_i \varphi_2(\xi_i) - 1 \\ \Sigma_{i=1}^{m-2} b_i \varphi_1(\xi_i) - 1 & \Sigma_{i=1}^{m-2} b_i \varphi_2(\xi_i) \end{vmatrix},$$
$$\Delta_2 = \begin{vmatrix} \Sigma_{i=1}^{m-2} a_i \psi_1(\xi_i) & \Sigma_{i=1}^{m-2} a_i \psi_2(\xi_i) - 1 \\ \Sigma_{i=1}^{m-2} b_i \psi_1(\xi_i) - 1 & \Sigma_{i=1}^{m-2} b_i \psi_2(\xi_i) \end{vmatrix}.$$

$$G_1(t,s) = \frac{1}{\rho_1} \begin{cases} \varphi_1(t)\varphi_2(s) & 0 \leq t \leq s \leq 1, \\ \varphi_1(s)\varphi_2(t) & 0 \leq s \leq t \leq 1, \end{cases} \tag{2.1}$$

$$G_2(t,s) = \frac{1}{\rho_2} \begin{cases} \psi_1(t)\psi_2(s) & 0 \leq t \leq s \leq 1, \\ \psi_1(s)\psi_2(t) & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.2}$$

Then $G_1(t,s)$ and $G_2(t,s)$ are the Green's function of the following linear BVP

$$\begin{cases} -u''(t) + \lambda_1 u(t) = 0, \\ u(0) = u(1) = 0; \end{cases} \quad \text{and} \quad \begin{cases} -u''(t) + \lambda_2 u(t) = 0, \\ u(0) = u(1) = 0, \end{cases}$$

respectively.

Lemma 2.2 [**7**]. *$G_i(t,s)$, $\varphi_i(t)$ and $\psi_i(t)$ $(i=1,2)$ have the following properties*:

(i) $G_i(t,s) > 0$, $t,s \in (0,1)$;

(ii) $G_i(t,s) \le C_i G_i(s,s)$ *for* $t,s \in [0,1]$; $\varphi_1(t) \le C_1$, $\varphi_2(t) \le C_1$, $\psi_1(t) \le C_2$, $\psi_2(t) \le C_2$ *for* $t \in [0,1]$;

(iii) $G_i(t,s) \ge \delta_i G_i(t,t) G_i(s,s)$ *for* $t,s \in [0,1]$; $\varphi_1(t) \ge \delta_1 G_1(t,t)$, $\varphi_2(t) \ge \delta_1 G_1(t,t)$, $\psi_1(t) \ge \delta_2 G_2(t,t)$, $\psi_2(t) \ge \delta_2 G_2(t,t)$ *for* $t \in [0,1]$,

where $C_i = 1$ *and* $\delta_i = w_i/(\sinh w_i)$ *if* $\lambda_i > 0$; $C_i = 1$ *and* $\delta_i = 1$ *if* $\lambda_i = 0$; $C_i = 1/(\sin w_1)$ *and* $\delta_i = w_i \sin w_i$ *if* $-\pi^2 < \lambda_i < 0$; $w_i = \sqrt{|\lambda_i|}$.

In the rest of the paper, we make the following assumptions:

(A_1) $\Sigma_{i=1}^{m-2} a_i \varphi_2(\xi_i) < 1$, $\Sigma_{i=1}^{m-2} b_i \varphi_1(\xi_i) < 1$;

(A_2) $\Sigma_{i=1}^{m-2} a_i \psi_2(\xi_i) < 1$, $\Sigma_{i=1}^{m-2} b_i \psi_1(\xi_i) < 1$;

(H_3) $\Delta_1 < 0$;

(H_4) $\Delta_2 < 0$.

Lemma 2.3 [**10**]. *Let (A_1), (H_2) and (H_3) hold. Then, for any* $g \in C[0,1]$, BVP

$$\begin{cases} -u'' + \lambda_1 u = g(t), \quad 0 < t < 1, \\ u(0) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i), \ u(1) = \displaystyle\sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{2.3}$$

has a unique solution

$$u(t) = \int_0^1 G_1(t,s) g(s)\, ds + A(g)\varphi_1(t) + B(g)\varphi_2(t),$$

where

$$A(g) = -\frac{1}{\Delta_1} \begin{vmatrix} \Sigma_{i=1}^{m-2} a_i \int_0^1 G_1(\xi_i,s) g(s)\, ds & \Sigma_{i=1}^{m-2} a_i \varphi_2(\xi_i) - 1 \\ \Sigma_{i=1}^{m-2} b_i \int_0^1 G_1(\xi_i,s) g(s)\, ds & \Sigma_{i=1}^{m-2} b_i \varphi_2(\xi_i) \end{vmatrix}, \tag{2.4}$$

$$B(g) = -\frac{1}{\Delta_1} \begin{vmatrix} \Sigma_{i=1}^{m-2} a_i \varphi_1(\xi_i) & \Sigma_{i=1}^{m-2} a_i \int_0^1 G_1(\xi_i,s) g(s)\, ds \\ \Sigma_{i=1}^{m-2} b_i \varphi_1(\xi_i) - 1 & \Sigma_{i=1}^{m-2} b_i \int_0^1 G_1(\xi_i,s) g(s)\, ds \end{vmatrix}. \tag{2.5}$$

Furthermore, if $g \geq 0$, the unique solution u of problem (2.3) *satisfies $u(t) \geq 0$ for $t \in [0,1]$.*

Lemma 2.4 [**10**]. *Let* (A_2), (H_2) *and* (H_4) *hold. Then, for any $g \in C[0,1]$,* BVP

$$\begin{cases} -u'' + \lambda_2 u = g(t), \quad 0 < t < 1, \\ u(0) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i), \ u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{2.6}$$

has a unique solution

$$u(t) = \int_0^1 G_2(t,s) g(s)\, ds + C(g)\psi_1(t) + D(g)\psi_2(t),$$

where

$$C(g) = -\frac{1}{\Delta_2} \begin{vmatrix} \Sigma_{i=1}^{m-2} a_i \int_0^1 G_2(\xi_i, s) g(s)\, ds & \Sigma_{i=1}^{m-2} a_i \psi_2(\xi_i) - 1 \\ \Sigma_{i=1}^{m-2} b_i \int_0^1 G_2(\xi_i, s) g(s)\, ds & \Sigma_{i=1}^{m-2} b_i \psi_2(\xi_i) \end{vmatrix}, \tag{2.7}$$

$$D(g) = -\frac{1}{\Delta_2} \begin{vmatrix} \Sigma_{i=1}^{m-2} a_i \psi_1(\xi_i) & \Sigma_{i=1}^{m-2} a_i \int_0^1 G_2(\xi_i, s) g(s)\, ds \\ \Sigma_{i=1}^{m-2} b_i \psi_1(\xi_i) - 1 & \Sigma_{i=1}^{m-2} b_i \int_0^1 G_2(\xi_i, s) g(s)\, ds \end{vmatrix}. \tag{2.8}$$

Furthermore, if $g \geq 0$, the unique solution u of problem (2.6) *satisfies $u(t) \geq 0$ for $t \in [0,1]$.*

Now notice that

$$u^{(4)}(t) + \alpha u''(t) - \beta u(t) = \Big(-\frac{d^2}{dt^2} + \lambda_1 \Big)\Big(-\frac{d^2}{dt^2} + \lambda_2 \Big) u(t),$$

so we can easily get:

Lemma 2.5 [**10**]. *Let* (H_2), (H_3) *and* (H_4) *hold. Then, for any $g \in C[0,1]$,* BVP

$$\begin{cases} u^{(4)} + \alpha u'' - \beta u = g(t), \quad 0 < t < 1, \\ u(0) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i), \ u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u''(0) = \displaystyle\sum_{i=1}^{m-2} a_i u''(\xi_i), \ u''(1) = \sum_{i=1}^{m-2} b_i u''(\xi_i), \end{cases} \tag{2.9}$$

has a unique solution

$$u(t) = \int_0^1 G_2(t,s)h_2(s)\,ds + C(h_2)\psi_1(t) + D(h_2)\psi_2(t)$$
$$= \int_0^1 G_1(t,s)h_1(s)\,ds + A(h_1)\varphi_1(t) + B(h_1)\varphi_2(t),$$

where G_1*,* G_2*,* $A(g)$*,* $B(g)$*,* $C(g)$ *and* $D(g)$ *are defined as in* (2.1)*,* (2.2)*,* (2.4)*,* (2.5)*,* (2.7) *and* (2.8)*, and*

$$h_2(t) = \int_0^1 G_1(t,s)g(s)\,ds + A(g)\varphi_1(t) + B(g)\varphi_2(t),$$
$$h_1(t) = \int_0^1 G_2(t,s)g(s)\,ds + C(g)\psi_1(t) + D(g)\psi_2(t).$$

In addition, if (A_1) *and* (A_2) *hold and* $g \geq 0$*, then* $u(t) \geq 0$ *for* $t \in [0,1]$*.*

Remark 2.1. For any $g \in C[0,1]$ and $g \geq 0$, $A(g)$, $B(g)$, $C(g)$ and $D(g)$ are all linear functions and nondecreasing in g.

Let

$$E = C[0,1], \qquad \|u\| = \max_{t\in[0,1]} |u(t)|,$$
$$P = \{u \in E : u(t) \geq 0, \quad \text{for all } t \in [0,1]\};$$

then $(E, \|\cdot\|)$ is a Banach space and P is a cone in E. Define an operator $T : E \to E$ by

$$(Tu)(t) = \int_0^1 G_2(t,s)e(f)(s)\,ds + C(e(f))\psi_1(t) + D(e(f))\psi_2(t), \tag{2.10}$$

where

$$(ef)(t) = \int_0^1 G_1(t,s)f(s,u(s))\,ds + A(f)\varphi_1(t) + B(f)\varphi_2(t).$$

From Lemma 2.4, we know that u is the nonzero fixed point of T in P equivalent to u is the positive solution of BVP (1.1). In addition, we have from (A_1), (A_2), (H_1), (H_2), (H_3) and (H_4) that $T : P \to P$ is completely continuous. Define the linear operators $L,\ L^* : E \to E$ by (2.11)

$$(Lu)(t) = \int_0^1 G_2(t,s)(l_2u)(s)\,ds + C(l_2(u))\psi_1(t) + D(l_2(u))\psi_2(t),$$
$$(L^*u)(t) = \int_0^1 G_1(t,s)(l_1u)(s)\,ds + A(l_1(u))\varphi_1(t) + B(l_1(u))\varphi_2(t),$$

where

(2.12)
$$(l_2u)(t) = \int_0^1 G_1(t,s)u(s)\,ds + A(u)\varphi_1(t) + B(u)\varphi_2(t),$$
$$(l_1u)(t) = \int_0^1 G_2(t,s)u(s)\,ds + C(u)\psi_1(t) + D(u)\psi_2(t).$$

It follows from (A_1), (A_2), (H_2), (H_3) and (H_4) that $L,\ L^* : P \to P$ are completely continuous.

Lemma 2.6 [**10**]. *Suppose that* (A_1), (A_2), (H_2), (H_3) *and* (H_4) *are satisfied. Then, for the operator* L *defined by* (2.11), *the spectral radius* $r(L) \neq 0$ *and* L *has a positive eigenfunction* ϕ^* *corresponding to its first eigenvalue* $\lambda^* = (r(L))^{-1}$, *that is,* $\phi^* = \lambda^* L\phi^*$.

Remark 2.2. It follows from Lemmas 2.5 and 2.6 that $L = L^*$, the spectral radius $r(L) = r(L^*)$ and $L,\ L^*$ have the same positive eigenfunction ϕ^* corresponding to their first eigenvalue $\lambda^* = (r(L))^{-1} = (r(L^*))^{-1}$, that is, $\phi^* = \lambda^* L\phi^*$, $\phi^* = \lambda^* L^*\phi^*$.

Let

$$C_0 = \int_0^1 G_2(t,t)G_1(t,t)\,dt + C(G_1(s,s)) + D(G_1(s,s)),$$
$$M = \int_0^1 G_2(t,t)\,dt + C(1) + D(1),$$
$$\kappa = \frac{\delta_1\delta_2 C_0}{C_1C_2M}\int_0^1 G_2(t,t)\phi^*(t)\,dt,$$

and

$$K = \left\{ u \in P : \int_0^1 \phi^*(t)u(t)\,dt \geq \kappa\|u\| \right\}.$$

Clearly, $M > 0$, $C_0 > 0$, $\kappa > 0$ and K is a cone in E.

Lemma 2.7. *Suppose that* (A_1), (A_2), (H_1), (H_2), (H_3) *and* (H_4) *are satisfied. Then* $L(P) \subset K$ *(in particular* $L(K) \subset K$*).*

Proof. For $u \in P$, by (2.11), (2.12) and Lemmas 2.2–2.4, $Lu(t) \geq 0$, and

(2.13)
$$\begin{aligned}(Lu)(t) &\leq C_1C_2\left[\int_0^1 G_2(s,s)\,ds + C(1) + D(1)\right] \\ &\qquad \times \left[\int_0^1 G_1(s,s)u(s)\,ds + A(u) + B(u)\right] \\ &\leq C_1C_2M\left[\int_0^1 G_1(s,s)u(s)\,ds + A(u) + B(u)\right], \quad t \in [0,1].\end{aligned}$$

Then

$$\|Lu\| \leq C_1C_2M\left[\int_0^1 G_1(s,s)u(s)\,ds + A(u) + B(u)\right].$$

On the other hand, Lemma 2.2 implies that

$$\begin{aligned}(l_2u)(t) &= \int_0^1 G_1(t,s)u(s)\,ds + A(u)\varphi_1(t) + B(u)\varphi_2(t) \\ &\geq \delta_1 G_1(t,t)\left[\int_0^1 G_1(s,s)u(s)ds + A(u) + B(u)\right],\end{aligned}$$

(2.14)

$$\begin{aligned}(Lu)(t) &\geq \delta_2 G_2(t,t)\left[\int_0^1 G_2(s,s)(l_2u)(s)\,ds + C(l_2u) + D(l_2u)\right]\\ &\geq \delta_1\delta_2 G_2(t,t)\left[\int_0^1 G_1(s,s)u(s)\,ds + A(u) + B(u)\right]\\ &\quad\times\left[\int_0^1 G_2(s,s)G_1(s,s)\,ds + C(G_1(s,s)) + D(G_1(s,s))\right]\\ &= \delta_1\delta_2 C_0 G_2(t,t)\left[\int_0^1 G_1(s,s)u(s)\,ds + A(u) + B(u)\right],\ t\in[0,1].\end{aligned}$$

So (2.13) and (2.14) imply that

$$(Lu)(t) \geq \frac{\delta_1\delta_2 C_0}{C_1C_2M} G_2(t,t)\|Lu\|, \quad t\in[0,1],$$

and thus,

$$\int_0^1 \phi^*(t)(Lu)(t)\,dt \geq \kappa\|Lu\|.$$

Hence, $L(P)\subset K$. $\square$

Remark 2.3. It follows, by Lemma 2.7, that $T(P)\subset K$ and, in particular, $T(K)\subset K$.

Remark 2.4. If u is the fixed point of T in P, then

$$u(t) \geq \frac{\delta_1\delta_2 C_0}{C_1C_2M} G_2(t,t)\|u\|, \quad t\in[0,1],$$

so $u(t)>0$, $t\in(0,1)$.

3. Existence results for the BVP (1.1).

Theorem 3.1. *Assume that* (A_1), (A_2), (H_1)–(H_4) *are satisfied, and*

$$\begin{aligned} f_\infty &= \liminf_{x\to+\infty}\min_{t\in[0,1]}\frac{f(t,x)}{x} > \lambda^*,\\ f^0 &= \limsup_{x\to 0^+}\max_{t\in[0,1]}\frac{f(t,x)}{x} < \lambda^*, \end{aligned} \tag{3.1}$$

where λ^* *is the first eigenvalue of* L *defined by* (2.11). *Then the* BVP (1.1) *has at least one positive solution.*

Proof. By $f_\infty > \lambda^*$, there exists an $\varepsilon_1 > 0$ such that

$$f(t,x) \geq (\lambda^* + \varepsilon_1)x, \quad 0 \leq t \leq 1,$$

when x is sufficiently large. By the continuity of f, there exists an $R_1 > 0$ such that

$$f(t,x) \geq (\lambda^* + \varepsilon_1)x - R_1, \quad t \in [0,1],\ x \in [0,+\infty).$$

Then, for any $u \in P$ and $t \in [0,1]$,

$$\begin{aligned}(ef)(t) &= \int_0^1 G_1(t,s)f(s,u(s))\,ds + A(f)\varphi_1(t) + B(f)\varphi_2(t)\\ &\geq (\lambda^* + \varepsilon_1)(l_2u)(t) - R_1(l_2\mathbf{1})(t),\end{aligned}$$

and

$$\begin{aligned}(Tu)(t) &= \int_0^1 G_2(t,s)e(f)(s)\,ds + C(e(f))\psi_1(t) + D(e(f))\psi_2(t)\\ &\geq (\lambda^* + \varepsilon_1)(Lu)(t) - R_1(L\mathbf{1})(t),\end{aligned} \tag{3.2}$$

where $\mathbf{1}$ stands for the constant function $\mathbf{1}(t) \equiv 1$. Let

$$\Omega_1 = \{u \in K : u \geq Tu\}.$$

We shall show that Ω_1 is a bounded subset of K. Indeed, if $u \in \Omega_1$, (3.2) implies that

$$u(t) \geq (\lambda^* + \varepsilon_1)(Lu)(t) - R_1(L\mathbf{1})(t).$$

Multiplying by $\phi^*(t)$ on both sides and integrating over $[0,1]$, then using integration by parts and exchanging the integral sequence, we deduce that

$$\begin{aligned}\int_0^1 u(t)\phi^*(t)\,dt \geq (\lambda^* + \varepsilon_1) &\int_0^1 (Lu)(t)\phi^*(t)\,dt \\ &- R_1 \int_0^1 (L\mathbf{1})(t)\phi^*(t)\,dt \\ = (\lambda^* + \varepsilon_1) &\int_0^1 u(t)(L^*\phi^*)(t)\,dt \\ &- R_1 \int_0^1 (L\phi^*)(t)\,dt \\ = (\lambda^* + \varepsilon_1) &\int_0^1 r(L)u(t)\phi^*(t)\,dt \\ &- R_1 \int_0^1 r(L)\phi^*(t)\,dt,\end{aligned}$$

and so

$$\int_0^1 u(t)\phi^*(t)\,dt \leq \frac{R_1 \int_0^1 \phi^*(t)\,dt}{\varepsilon_1}.$$

Hence,

$$\|u\| \leq \frac{R_1 \int_0^1 \phi^*(t)\,dt}{\kappa\varepsilon_1}, \quad \text{for all } u \in \Omega_1.$$

This proves the boundedness of Ω_1. Taking $r_1 > \sup_{u\in\Omega_1} \|u\|$, we see that

$$Tu \not\leq u, \quad u \in \partial B_{r_1} \cap K. \tag{3.3}$$

By $f^0 < \lambda^*$, there exist $0 < \varepsilon_2 < \lambda^*$ and $0 < r_2 < r_1$ such that

$$f(t,x) \leq (\lambda^* - \varepsilon_2)x, \quad t \in [0,1],\ x \in [0, r_2];$$

then, for any $u \in \overline{B}_{r_2} \cap P$ and $t \in [0,1]$,

$$(ef)(t) \leq (\lambda^* - \varepsilon_2)(l_2 u)(t),$$

and

$$(Tu)(t) \leq (\lambda^* - \varepsilon_2)(Lu)(t). \tag{3.4}$$

Let

$$\Omega_2 = \{u \in \overline{B}_{r_2} \cap K : u \le Tu\}.$$

We claim that $\Omega_2 = \{0\}$. Indeed, if $u \in \Omega_2$, by (3.4),

$$u(t) \le (\lambda^* - \varepsilon_2)(Lu)(t), \quad t \in [0,1].$$

Multiplying by $\phi^*(t)$ on both sides and integrating over $[0,1]$, then using integration by parts and exchanging the integral sequence, we get

$$\begin{aligned}\int_0^1 u(t)\phi^*(t)\,dt &\le (\lambda^* - \varepsilon_2)\int_0^1 (Lu)(t)\phi^*(t)\,dt \\ &= (\lambda^* - \varepsilon_2)\int_0^1 u(t)(L^*\phi^*)(t)\,dt \\ &= (\lambda^* - \varepsilon_2)\int_0^1 r(L)u(t)\phi^*(t)\,dt,\end{aligned}$$

and so $\int_0^1 u(t)\phi^*(t)\,dt = 0$. $u \in K$ implies $u = 0$. This proves that $\Omega_2 = \{0\}$. Therefore,

$$Tu \not\ge u, \quad u \in \partial B_{r_2} \cap K. \tag{3.5}$$

Now, (3.3) and (3.5) together with Lemma 1.1 imply that T has at least one fixed point $u^* \in (B_{r_1} \setminus \overline{B}_{r_2}) \cap K$. From Remark 2.4, we know that u^* is a positive solution of BVP (1.1). □

Theorem 3.2. *Assume that* (A_1), (A_2), (H_1)–(H_4) *are satisfied, and*

$$\begin{aligned} f_0 &= \liminf_{x\to 0^+} \min_{t\in[0,1]} \frac{f(t,x)}{x} > \lambda^*, \\ f^\infty &= \limsup_{x\to +\infty} \max_{t\in[0,1]} \frac{f(t,x)}{x} < \lambda^*, \end{aligned} \tag{3.6}$$

where λ^* *is the first eigenvalue of* L *defined by* (2.11). *Then the* BVP (1.1) *has at least one positive solution.*

Proof. Since $f_0 > \lambda^*$, there exist $\varepsilon_3 > 0$ and $r_3 > 0$ such that

$$f(t,x) \ge (\lambda^* + \varepsilon_3)x, \quad t \in [0,1],\ x \in [0, r_3].$$

Then, for any $u \in \overline{B}_{r_3} \cap P$ and $t \in [0,1]$, we have

$$(ef)(t) \geq (\lambda^* + \varepsilon_3)(l_2 u)(t),$$

and

$$(Tu)(t) \geq (\lambda^* + \varepsilon_3)(Lu)(t). \tag{3.7}$$

Let

$$\Omega_3 = \{u \in \overline{B}_{r_3} \cap K : u \geq Tu\}.$$

We shall show that $\Omega_3 = \{0\}$. Indeed, if $u \in \Omega_3$, it follows from (3.7) that

$$u(t) \geq (\lambda^* + \varepsilon_3)(Lu)(t), \quad t \in [0,1].$$

Multiplying by $\phi^*(t)$ on both sides and integrating over $[0,1]$, then using integration by parts on the right side and exchanging the integral sequence, we have

$$\begin{aligned}\int_0^1 u(t)\phi^*(t)\,dt &\geq (\lambda^* + \varepsilon_3)\int_0^1 (Lu)(t)\phi^*(t)\,dt \\ &= (\lambda^* + \varepsilon_3)\int_0^1 u(t)(L^*\phi^*)(t)\,dt \\ &= (\lambda^* + \varepsilon_3)\int_0^1 r(L)u(t)\phi^*(t)\,dt,\end{aligned}$$

and so $\int_0^1 u(t)\phi^*(t)\,dt = 0$. $u \in K$ implies that $u = 0$. This proves that $\Omega_3 = \{0\}$. Hence,

$$Tu \not\leq u, \quad u \in \partial B_{r_3} \cap K. \tag{3.8}$$

On the other hand, by $f^\infty < \lambda^*$, there exists a $0 < \varepsilon_4 < \lambda^*$ such that

$$f(t,x) \leq (\lambda^* - \varepsilon_4)x, \quad 0 \leq t \leq 1,$$

when x is sufficiently large. By the continuity of f, there exists an $R_2 > 0$ such that

$$f(t,x) \leq (\lambda^* - \varepsilon_4)x + R_2, \quad t \in [0,1],\ x \in [0,+\infty).$$

Then, for any $u \in P$ and $t \in [0,1]$,

$$(ef)(t) \leq (\lambda^* - \varepsilon_4)(l_2 u)(t) + R_2(l_2\mathbf{1})(t),$$

and

$$(Tu)(t) \leq (\lambda^* - \varepsilon_4)(Lu)(t) + R_2(L\mathbf{1})(t). \tag{3.9}$$

Let

$$\Omega_4 = \{u \in K : u \leq Tu\}.$$

We are going to prove that Ω_4 is a bounded subset of K. Indeed, if $u \in \Omega_4$, (3.9) implies that

$$u(t) \leq (\lambda^* - \varepsilon_4)(Lu)(t) + R_2(L\mathbf{1})(t).$$

Multiplying by $\phi^*(t)$ on both sides and integrating over $[0,1]$, then using integration by parts on the right side and exchanging the integral sequence, we get

$$\begin{aligned}\int_0^1 u(t)\phi^*(t)\,dt &\leq (\lambda^* - \varepsilon_4)\int_0^1 (Lu)(t)\phi^*(t)\,dt \\ &\quad + R_2\int_0^1 (L\mathbf{1})(t)\phi^*(t)\,dt \\ &= (\lambda^* - \varepsilon_4)\int_0^1 u(t)(L^*\phi^*)(t)\,dt \\ &\quad + R_2\int_0^1 (L\phi^*)(t)\,dt \\ &= (\lambda^* - \varepsilon_4)\int_0^1 r(L)u(t)\phi^*(t)\,dt \\ &\quad + R_2\int_0^1 r(L)\phi^*(t)\,dt,\end{aligned}$$

and so

$$\int_0^1 u(t)\phi^*(t)\,dt \leq \frac{R_2\int_0^1 \phi^*(t)\,dt}{\varepsilon_4}.$$

$u \in K$ implies that

$$\|u\| \leq \frac{R_2 \int_0^1 \phi^*(t)\, dt}{\kappa \varepsilon_4}, \quad \text{for all } u \in \Omega_4.$$

This proves the boundedness of Ω_4. Taking $r_4 > \max\{\sup_{u\in\Omega_4} \|u\|, r_3\}$, we then conclude that

$$Tu \not\geq u, \quad u \in \partial B_{r_4} \cap K. \tag{3.10}$$

Hence, (3.8), (3.10) and Lemma 1.1 imply that T has at least one fixed point on $(B_{r_4} \setminus \overline{B}_{r_3}) \cap K$, which is a positive solution of BVP (1.1). □

Remark 3.1. In [**10**], the author only established the existence of positive solution for $f_0 > \lambda^*$, $f^\infty < \lambda^*$, but for another case $f^0 < \lambda^*$, $f_\infty > \lambda^*$, it is still not known whether a similar result holds. Hence, Theorem 3.1 is new for [**10**].

Remark 3.2. $\lambda^* = \pi^4 - \alpha\pi^2 - \beta$ when $a_i = b_i = 0$ $(i = 1, 2, \dots, m-2)$. In such a case, Theorems 3.1 and 3.2 are reduced to Theorem 1.1 in [**7**], so our results extend the main results of [**7**]. Moreover, our method is different from [**7, 10**].

Remark 3.3. Since λ^* is the first eigenvalue of the linear problem corresponding to BVP (1.1), the positive result cannot be guaranteed when the strict inequality is weakened to nonstrict inequality. So our result is optimal.

4. Uniqueness of positive solutions for BVP (1.1).

Theorem 4.1. *Assume that (A_1), (A_2), (H_1)–(H_4) are satisfied, and*

(F_1) *for every $t \in [0, 1]$, $f(t, \cdot)$ is increasing in $\mathbf{R}^+$.*

(F_2) *There is a function $\gamma : (0, 1) \to \mathbf{R}^+$ such that, for every $t \in [0, 1]$, $x > 0$, $\lambda \in (0, 1)$, one has $\gamma(\lambda) > \lambda$ and $f(t, \lambda x) \geq \gamma(\lambda) f(t, x)$.*

Then the BVP (1.1) *has at most one positive solution.*

Proof. Suppose u_1 and u_2 are two positive solutions of BVP (1.1). Then

$$u_i(t) = \int_0^1 G_2(t,s)e(f_i)(s)\,ds + C(e(f_i))\psi_1(t) + D(e(f_i))\psi_2(t),$$

where

$$(ef_i)(t) = \int_0^1 G_1(t,s)f_i(s,u(s))\,ds + A(f_i)\varphi_1(t) + B(f_i)\varphi_2(t),$$

$$f_i(t,u) = f(t,u_i), \quad i = 1,2.$$

Set

$$\mu_0 = \sup\{\mu > 0 : u_1(t) \geq \mu u_2(t),\ t \in [0,1]\}.$$

Then $\mu_0 > 0$ and $u_1(t) \geq \mu_0 u_2(t)$, $t \in [0,1]$. We claim that $\mu_0 \geq 1$. Suppose the contrary, $\mu_0 < 1$. Using (F_1) and (F_2), for any $t \in [0,1]$, we deduce that

$$\begin{aligned}(ef_1)(t) &= \int_0^1 G_1(t,s)f(s,u_1(s))\,ds + A(f_1)\varphi_1(t) + B(f_1)\varphi_2(t)\\ &\geq \int_0^1 G_1(t,s)f(s,\mu_0 u_2(s))\,ds + A(f_*)\varphi_1(t) + B(f_*)\varphi_2(t)\\ &\geq \gamma(\mu_0)\left[\int_0^1 G_1(t,s)f(s,u_2(s))\,ds + A(f_2)\varphi_1(t) + B(f_2)\varphi_2(t)\right]\\ &= \gamma(\mu_0)(ef_2)(t),\end{aligned}$$

where

$$f_*(t,u) = f(t,\mu_0 u_2).$$

Then

$$\begin{aligned}u_1(t) &= \int_0^1 G_2(t,s)e(f_1)(s)\,ds + C(e(f_1))\psi_1(t) + D(e(f_1))\psi_2(t)\\ &\geq \gamma(\mu_0)\bigg[\int_0^1 G_2(t,s)e(f_2)(s)\,ds\\ &\qquad\qquad + C(e(f_2))\psi_1(t) + D(e(f_2))\psi_2(t)\bigg] = \gamma(\mu_0)u_2(t),\end{aligned}$$

contradicting the definition of μ_0. So, $\mu_0 \geq 1$ and $u_1(t) \geq \mu_0 u_2(t) \geq u_2(t)$ for $t \in [0,1]$. Similarly, $u_2(t) \geq u_1(t)$, $t \in [0,1]$. Consequently, $u_1 = u_2$. □

Corollary 4.1. *Assume that* (A_1), (A_2), (H_1)–(H_4), (F_1), (F_2) *and* (3.6) *are satisfied. Then* BVP (1.1) *has exactly one positive solution.*

Remark 4.1. If we let $f(t,u) = h(t)u^r$, $r \in (0,1)$, it is easy to check that $f(t,u)$ satisfies the conditions (F_1) and (F_2) for $\gamma(\lambda) = \lambda^r$. Then [**10**, Theorem 4.1] becomes a special case of Theorem 4.1 presented in this paper. Obviously, our result includes and improves the uniqueness result of [**10**].

5. Examples. We give three explicit examples to illustrate our main results.

Example 5.1. We consider the BVP (1.1) with $m = 3$, $a_1 = 0$, $b_1 = 1/2$, $\xi_1 = 1/4$, $\alpha = -1$, $\beta = 0$ and $f(t,x) = \mu t \ln(1+x) + x^2$, fixing $\mu > 0$ sufficiently small. BVP (1.1) becomes the fourth order three-point BVP

$$(5.1) \qquad \begin{cases} u^{(4)}(t) - u''(t) = \mu t \ln(1+u(t)) + u^2(t), & t \in (0,1), \\ u(0) = 0,\ u(1) = \frac{1}{2}u(\frac{1}{4}),\ u''(0) = 0,\ u''(1) = \frac{1}{2}u''(\frac{1}{4}). \end{cases}$$

For this case, $\varphi_1(t) = e/(e^2-1)e^t - e/(e^2-1)e^{-t}$, $\varphi_2(t) = -1/(e^2-1)e^t + e^2/(e^2-1)e^{-t}$, $\psi_1(t) = t$, $\psi_2(t) = 1-t$, $b_1\varphi_1(\xi_1) = (1/2)e/(e^2-1)(e^{1/4} - e^{-1/4}) < 1$, $b_1\psi_1(\xi_1) = (1/2)\times(1/4) < 1$, $\Delta_1 = (1/2)e/(e^2-1)(e^{1/4} - e^{-1/4}) - 1 < 0$, $\Delta_2 = -7/8$.

On the other hand, by direct calculations, we can obtain that $f_\infty = \infty$, $f^0 = \mu$. Therefore, the assumptions of Theorem 3.1 are satisfied. Thus, BVP (5.1) has at least one positive solution.

Example 5.2. Consider the BVP (1.1) with $m = 3$, $a_1 = 0$, $b_1 = 1/2$, $\xi_1 = 1/4$, $\alpha = -1$, $\beta = 0$ and $f(t,x) = x^2 e^{-x} + \mu \sin x$, fixing $\mu > 0$ sufficiently large. BVP (1.1) becomes the fourth order three-point BVP

$$(5.2) \qquad \begin{cases} u^{(4)}(t) - u''(t) = u^2(t)e^{-u(t)} + \mu \sin u(t), & t \in (0,1), \\ u(0) = 0,\ u(1) = \frac{1}{2}u(\frac{1}{4}),\ u''(0) = 0,\ u''(1) = \frac{1}{2}u''(\frac{1}{4}). \end{cases}$$

For this case, $b_1\varphi_1(\xi_1) < 1$, $b_1\psi_1(\xi_1) < 1$, $\Delta_1 = (1/2)e/(e^2-1)(e^{1/4} - e^{-1/4}) - 1 < 0$, $\Delta_2 = -7/8$. By direct calculations, we have $f_0 = \mu$, $f^\infty = 0$. Therefore, the assumptions of Theorem 3.2 are satisfied. Thus, BVP (5.2) has at least one positive solution.

Example 5.3. Consider the BVP (1.1) with $m = 3$, $a_1 = 0$, $b_1 = 1/2$, $\xi_1 = 1/4$, $\alpha = -1$, $\beta = 0$ and $f(t,x) = (2+t)x^{1/2}$. BVP (1.1) becomes the fourth order three-point BVP

$$\begin{cases} u^{(4)}(t) - u''(t) = (2+t)u^{1/2}(t), \quad t \in (0,1), \\ u(0) = 0,\ u(1) = \frac{1}{2}u(\frac{1}{4}),\ u''(0) = 0,\ u''(1) = \frac{1}{2}u''(\frac{1}{4}). \end{cases} \tag{5.3}$$

For this case, $b_1\varphi_1(\xi_1) < 1$, $b_1\psi_1(\xi_1) < 1$, $\Delta_1 = (1/2)e/(e^2-1)(e^{1/4} - e^{-1/4}) - 1 < 0$, $\Delta_2 = -7/8$. By direct calculations, we have $f_0 = \infty$, $f^\infty = 0$. It is easy to check that $f(t,x)$ satisfies the conditions (F_1) and (F_2) for $\gamma(\lambda) = \lambda^{1/2}$, $\lambda \in (0,1)$. Therefore, the assumptions of Corollary 4.1 are satisfied. Thus, BVP (5.3) has a unique positive solution.

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