ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 3, 2013

A q-ANALOGUE OF MODIFIED BETA OPERATORS

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ABSTRACT. In the present paper, we introduce a q-analogue of modified Beta operators. First we estimate moments of these operators and also obtain the relation for moments. We estimate some approximation properties of these operators.

1. Introduction. In 1997 Phillips [**11**] proposed an important generalization of the classical Bernstein polynomials based on q-integers. It is well known that many q-extensions of the classical objects arising in approximation theory (e.g., different types of Bernstein operators and Bernstein basis polynomials) have been recently introduced. See, e.g., [**12** and the references given there].

In 1967 Durrmeyer [**5**] introduced an integral-modification of Bernstein polynomials so as to approximate Lebesgue integrable functions on the interval $[0,1]$. These operators have been extensively studied by several researchers. In a similar manner, Deo [**4**] introduced the following modified beta operators so as to approximate integrable functions on the interval $[0,\infty)$,

$$M_n(f,x)=\frac{1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\infty}b_{n,k}(t)\,f(t)\,dt, \tag{1.1}$$

where

$$b_{n,k}(x)=\frac{1}{B(k+1,n)}\frac{x^k}{(1+x)^{n+k+1}}$$

and the beta function $B(\alpha,\beta)=[(\alpha-1)!(\beta-1)!]/(\alpha+\beta-1)!$.

2010 AMS *Mathematics subject classification.* Primary 41A25, 41A30.
Keywords and phrases. Beta operators, q-integers, q-beta function, modulus of continuity.
The third author is thankful to the "**Ministry of Human Resource and Development**" for financial support to carry out the above work.
Received by the editors on May 11, 2010 and in revised form on September 27, 2010.
DOI:10.1216/RMJ-2013-43-3-931

Following [**7**] and the references therein, for $q > 0$ and each nonnegative integer n, we have

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1, \end{cases}$$

and

$$[n]_q! = \begin{cases} [n]_q[n-1]_q[n-2]_q \cdots [1]_q, & n = 1, 2, \ldots \\ 1, & n = 0. \end{cases}$$

Further, we use the q-Pochhammer symbol, which is defined as

$$(b;q)_n = \prod_{j=0}^{n-1}(1+q^j b).$$

The q-Jackson integrals and q-improper integrals are given by (see [**8, 10**])

$$\int_0^a f(x)\, d_q x = (1-q)\, a \sum_{n=0}^{\infty} f(aq^n)\, q^n$$

and

$$\int_0^{\infty/A} f(x)\, d_q x = (1-q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, \quad A > 0,$$

provided the sums converge absolutely.

The q-gamma function [**2**] is defined by

$$\Gamma_q(t) = \int_0^{1/1-q} x^{t-1} E_q(-qx)\, d_q x, \quad t > 0,$$

where the q-exponential function E_q is defined as

$$E_q(z) = \prod_{j=0}^{\infty}(1+(1-q)q^j z) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{z^k}{[k]_q!}, \quad |q| < 1$$

(see e.g. [**9**]). Consequently,

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t), \quad \Gamma_q(1) = 1.$$

The q-beta function [**2**] is given by

$$B_q(t,s) = K(A,t) \int_0^{\infty/A} \frac{x^{t-1}}{(x;q)_{t+s}} \, d_q x,$$

where $K(x,t) = (1/1+x)x^t(x^{-1};q)_t(x;q)_{1-t}$. In particular, for any positive integer n,

$$K(x,n) = q^{n(n-1)/2}, \quad K(x,0) = 1$$

and

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}.$$

Motivated by the q-analogue of Bernstein-Durrmeyer polynomials [**7**], we introduce a similar q-analogue of the operators (1.1) as follows:

For $f \in C_B[0,\infty)$, the space of bounded and continuous functions on $[0,\infty)$, $q \in (0,1)$ and each $n \in N$, the positive linear operator $M_{n,q}$ is defined by

(1.2)
$$M_{n,q}(f,x) = \frac{1}{[n]_q} \sum_{k=0}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} q^k p_{n,k}^q(t) f(t) \, d_q t,$$

where

$$p_{n,k}^q(x) = \frac{q^{k(k-1)/2}}{B_q(k+1,n)} \frac{x^k}{(x,q)_{n+k+1}}, \quad x \in [0,\infty)$$

and $B_q(k+1,n) = [k]_q![n-1]_q!/[n+k]_q!$.

2. Preliminaries. In the sequel we shall require the following results.

Remark 1. Following the paper [**1**] and the q-derivative

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if } x \neq 0,$$

by a simple computation, we have

$$qx(1+x)D_q[p_{n,k}^q(x)] = \left(\frac{[k]_q}{q^{k-1}[n+1]_q} - qx\right)[n+1]_q p_{n,k}^q(qx)$$

and

$$\frac{t}{q}\left(1+\frac{t}{q}\right)D_q p_{n,k}^q\left(\frac{t}{q}\right) = \left(\frac{[k]_q}{q^{k-1}[n+1]_q} - t\right)\frac{[n+1]_q}{q^2}p_{n,k}^q(t),$$

where D_q denotes the q-derivative operator.

Lemma 1. *If we define the central moments as*

$$\begin{aligned} T_{n,m}(x) &:= M_{n,q}(t^m;x) \\ &= \frac{1}{[n]_q}\sum_{k=0}^{\infty} p_{n,k}^q(x)\int_0^{\infty/A} q^k p_{n,k}^q(t) t^m d_q t, \end{aligned}$$

then $T_{n,0}(x) = 1$ *and, for* $n > m+2$, *we have the following recurrence relation*:

$$\begin{aligned} &([n+1]_q - [m+2]_q)T_{n,m+1}(qx) \\ &\quad = qx(1+x)D_q[T_{n,m}(x)] + q([m+1]_q + [n+1]_q x)T_{n,m}(qx); \end{aligned}$$

consequently, we have

$$T_{n,1}(x) = \frac{[n+1]_q}{q^2[n-1]_q}x + \frac{1}{q[n-1]_q}$$

and

$$\begin{aligned} T_{n,2}(x) = &\frac{[n+1]_q[n+2]_q}{q^6[n-1]_q[n-2]_q}x^2 \\ &+ \frac{[n+1]_q(q+[3]_q)}{q^5[n-1]_q[n-2]_q}x \\ &+ \frac{[2]_q}{q^3[n-1]_q[n-2]_q}. \end{aligned}$$

Proof. Using Remark 1, we have

$$\begin{aligned}
E &:= qx(1+x)D_q[T_{n,m}(x)] \\
&= \frac{1}{[n]_q}\sum_{k=0}^{\infty} qx(1+x)D_q[p_{n,k}^q(x)]\int_0^{\infty/A} q^k p_{n,k}^q(t)t^m d_qt \\
&= \frac{[n+1]_q}{[n]_q}\sum_{k=0}^{\infty} p_{n,k}^q(qx)\int_0^{\infty/A} q^k\left(\frac{[k]_q}{q^{k-1}[n+1]_q}-qx\right)p_{n,k}^q(t)t^m d_qt \\
&= \frac{[n+1]_q}{[n]_q}\sum_{k=0}^{\infty} p_{n,k}^q(qx) \\
&\quad\times\int_0^{\infty/A} q^k\left(\frac{[k]_q}{q^{k-1}[n]_q}-t+t-qx\right)p_{n,k}^q(t)t^m d_qt.
\end{aligned}$$

Thus, by Remark 1, we have

$$\begin{aligned}
E &= \frac{1}{[n]_q}\sum_{k=0}^{\infty} p_{n,k}^q(qx)\int_0^{\infty/A} q^k q^2\left[\frac{t}{q}\left(1+\frac{t}{q}\right)\right]D_q[p_{n,k}^q(t)]t^m d_qt \\
&\quad + [n+1]_qT_{n,m+1}(qx) - q[n+1]_q xT_{n,m}(qx) \\
&= \frac{1}{[n]_q}\sum_{k=0}^{\infty} p_{n,k}^q(qx) \\
&\quad\times\int_0^{\infty/A} q^k\left[qt^{m+1}+t^{m+2}\right]D_q[p_{n,k}^q(t)]d_qt \\
&\quad + [n+1]_qT_{n,m+1}(qx) - q[n+1]_q xT_{n,m}(qx).
\end{aligned}$$

Using the q integral by parts,

$$\int_a^b u(t)D_q(v(t))\,d_qt = [u(t)v(t)]_a^b - \int_a^b v(qt)D_q[u(t)]\,d_qt,$$

we have

$$\begin{aligned}
E &= -q[m+1]_qT_{n,m}(qx) - [m+2]_qT_{n,m+1}(qx) \\
&\quad + [n+1]_qT_{n,m+1}(qx) - q[n+1]_q xT_{n,m}(qx),
\end{aligned}$$

which completes the proof of recurrence relation. ∎

Corollary 1. *From Lemma* 1, *it follows that*

$$M_{n,q}((t-x),x) = \frac{[2]_q}{q^2[n-1]_q}x + \frac{1}{q[n-1]_q}$$
$$M_{n,q}((t-x)^2,x) = \left(\frac{[n+1]_q[n+2]_q}{q^6[n-1]_q[n-2]_q} - \frac{2[n+1]_q}{q^2[n-1]_q} + 1\right)x^2 + \left(\frac{(q+[3]_q)[n+1]_q}{q^5[n-1]_q[n-2]_q} - \frac{2}{q[n-1]_q}\right)x + \frac{[2]_q}{q^3[n-1]_q[n-2]_q}.$$

Further, $M_{n,q}((t-x)^m,x)$ *is a polynomial in* x *of degree* m *and, for every* $x \in [0,\infty)$,

$$M_{n,q}((t-x)^m,x) = O\left(\frac{1}{[n]_q^{[(m+1)/2]}}\right), \quad as\ , n \to \infty,$$

where $[\beta]$ *denotes the integer part of* β.

Lemma 2. *For* $n > 2$, *we have*

$$M_{n,q}((t-x)^2,x) \le \frac{15}{q^6[n-1]_q}\left(\varphi(x) + \frac{1}{[n-2]_q}\right),$$

where $\varphi^2(x) = x(1+x)$, $x \in [0,\infty)$.

Proof. By Corollary 1, we have

$$M_{n,q}((t-x)^2,x) = \left(\frac{[n+1]_q[n+2]_q}{q^6[n-1]_q[n-2]_q} - \frac{2[n+1]_q}{q^2[n-1]_q} + 1\right)x^2 + \left(\frac{(q+[3]_q)[n+1]_q}{q^5[n-1]_q[n-2]_q} - \frac{2}{q[n-1]_q}\right)x + \frac{[2]_q}{q^3[n-1]_q[n-2]_q}$$
$$= \left(\frac{[n+1]_q[n+2]_q - 2q^4[n+1]_q[n-2]_q}{q^6[n-1]_q[n-2]_q}\right.$$

$$
\begin{aligned}
&+ \frac{q^6[n-1]_q[n-2]_q}{q^6[n-1]_q[n-2]_q}\Bigg)x(1+x) \\
+ &\Bigg(\frac{q(q+[3]_q-[n+2]_q)[n+1]_q}{q^5[n-1]_q[n-2]_q} \\
&+ \frac{(2q^4[n+1]_q-2q^5-q^6[n-1]_q)[n-2]_q}{q^5[n-1]_q[n-2]_q}\Bigg)x \\
+ &\frac{[2]_q}{q^3[n-1]_q[n-2]_q}.
\end{aligned}
$$

Hence, for $n>2$, we have, using

$$
\begin{aligned}
[n-1]_q &= 1+q[n-2]_q, \\
[n+1]_q &= [3]_q+q^3[n-2]_q
\end{aligned}
$$

and

$$
[n+2]_q = [4]_q+q^4[n-2]_q
$$

$$
\begin{aligned}
&[n+1]_q[n+2]_q-2q^4[n+1]_q[n-2]_q+q^6[n-1]_q[n-2]_q \\
&\quad = ([3]_q+q^3[n-2]_q)([4]_q+q^4[n-2]_q)-2q^4([3]_q+q^3[n-2]_q)[n-2]_q \\
&\qquad + q^6(1+q[n-2]_q)[n-2]_q \\
&\quad = [3]_q[4]_q+[n-2]_q(q^3[4]_q-q^4[3]_q+q^6) \\
&\quad = [3]_q[4]_q+[n-2]_q(q^3+q^6-q^7)>0.
\end{aligned}
$$

Next,

$$
\begin{aligned}
q(q+[3]_q-[n+2]_q)[n+1]_q &+ (2q^4[n+1]_q-2q^5-q^6[n-1]_q)[n-2]_q \\
&= q(q+[3]_q-[4]_q-q^4[n-2]_q)([3]_q+q^3[n-2]_q) \\
&\quad + [2q^4([3]_q+q^3[n-2]_q)-2q^5-q^6(1+q[n-2])][n-2]_q \\
&= q(q+[3]_q)[3]_q-[3]_q[4]_q+[n-2]_q \\
&\quad \times (q^4+q^5+q^5[2]_q-2q^5+q^4[3]_q-q^3[4]_q-q^6) \\
&= q(q+[3]_q)[3]_q-[3]_q[4]_q+[n-2]_q(q^4-q^3) \\
&= -1-q+q^3+q^4-[n-2]_q(q^3-q^4)<0, \quad \text{for } n>2.
\end{aligned}
$$

Thus, we have

$$\begin{aligned}
&M_{n,q}((t-x)^2,x)\\
&\le x(1+x)\left(\frac{[n+1]_q[n+2]_q-2q^4[n+1]_q[n-2]_q+q^6[n-1]_q[n-2]_q}{q^6[n-1]_q[n-2]_q}\right)\\
&\quad+\frac{[2]_q}{q^3[n-1]_q[n-2]_q}\\
&\le\frac{15[n-2]_q}{q^6[n-1]_q[n-2]_q}\varphi^2(x)+\frac{[2]_q}{q^3[n-1]_q[n-2]_q}\\
&\le\frac{15}{q^6[n-1]_q}\left(\varphi^2(x)+\frac{1}{[n-2]_q}\right)
\end{aligned}$$

for every $q\in(0,1)$ and $x\in[0,\infty)$. Hence, the result follows.

3. Main results. In this section we establish direct and local approximation theorems for the operators $M_{n,q}$.

Let the space $C_B[0,\infty)$ be endowed with the norm $\|f\|=\sup\{|f(x)|: x\in[0,\infty)\}$. Further, let us consider the following K-functional:

$$K_2(f,\delta)=\inf_{g\in W^2}\{\|f-g\|+\delta\|g''\|\},$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty): g',g''\in C_B[0,\infty)\}$. By [**3**, page 177, Theorem 2.4], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\le C\,\omega_2(f,\sqrt{\delta}),\tag{3.1}$$

where

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|$$

is the second order modulus of smoothness of $f\in C_B[0,\infty)$.

In what follows, we shall use the notations $\varphi(x)=\sqrt{x(1+x)}$, where $x\in[0,\infty)$ and $n\ge3$.

Theorem 1. *Let $f\in C_B[0,\infty)$ and $n\ge3$. Then, for every $x\in[0,\infty)$, we have*

$$|M_{n,q}(f,x)-f(x)|\le C\omega_2\left(f,\frac{\delta_n(x)}{\sqrt{q^6[n-1]_q}}\right)+\omega\left(f,\frac{q^{-2}[2]_qx+q^{-1}}{[n-1]_q}\right),$$

where

$$\delta_n^2(x) = \varphi^2(x) + \frac{1}{[n-2]_q}$$

and C is a positive constant.

Proof. We introduce the auxiliary operators $M_{n,q}^*$ defined by

$$M_{n,q}^*(f,x) = M_{n,q}(f,x) - f\left(x + \frac{q^{-2}[2]_q x + q^{-1}}{[n-1]_q}\right) + f(x), \tag{3.2}$$

$x \in [0,\infty)$. These operators are linear and preserve the linear functions in view of Lemma 2.

Let $g \in W^2$. From Taylor's expansion of g,

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du, \quad t \in [0,\infty)$$

we have

$$M_{n,q}^*(g,x) = g(x) + M_{n,q}^*\left(\int_x^t (t-u)g''(u)du, x\right).$$

Hence, by (3.2), we obtain

(3.3)

$$\begin{aligned}
|M_{n,q}^*(g,x) - g(x)| &\le \left|M_{n,q}\left(\int_x^t (t-u)\,g''(u)\,du,\, x\right)\right| \\
&\quad + \left|\int_x^{x+(q^{-2}[2]_q x + q^{-1})/[n-1]_q} \left(x + \frac{q^{-2}[2]_q x + q^{-1}}{[n-1]_q} - u\right) g''(u)\,du\right| \\
&\le M_{n,q}\left(\left|\int_x^t |(t-u)|\,|g''(u)|\,du\right|, x\right) \\
&\quad + \int_x^{x+(q^{-2}[2]_q x + q^{-1})/[n-1]_q} \left(\left|x + \frac{q^{-2}[2]_q x + q^{-1}}{[n-1]_q}\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad - u\Big|\Big)|g''(u)|\,du\\
&\le M_{n,q}\left(\left|\int_x^t |(t-x)|\,|g''(u)|\,du\right|, x\right)\\
&\quad + \int_x^{x+(q^{-2}[2]_q x+q^{-1})/[n-1]_q} \left|\frac{q^{-2}[2]_q x+q^{-1}}{[n-1]_q}\right| |g''(u)|\,du\\
&\le \left[M_{n,q}((t-x)^2,x) + \left(\frac{q^{-2}[2]_q x+q^{-1}}{[n-1]_q}\right)^2\right]\|g''\|\\
&\le \frac{4}{[n-1]_q}.
\end{aligned}
$$

Applying Lemma 2, for $n \ge 3$, we get

$$
\begin{aligned}
M_{n,q}((t-x)^2,x) &+ \left(\frac{q^{-2}[2]_q x+q^{-1}}{[n-1]_q}\right)^2\\
&\le \frac{15}{q^6[n-1]_q}\left(\varphi^2(x) + \frac{1}{[n-2]_q}\right)\\
&\quad + \left(\frac{q^{-2}[2]_q x+q^{-1}}{[n-1]_q}\right)^2.
\end{aligned}
$$

Since

$$
\begin{aligned}
\left(\frac{q^{-2}[2]_q x+q^{-1}}{[n-1]_q}\right)^2 &\cdot [\delta_n^2(x)]^{-1}\\
&= \frac{(1+q)^2x^2+2q(1+q)x+q^2}{q^4[n-1]_q^2}\cdot\frac{[n-2]_q}{[n-2]_q x(1+x)+1}\\
&\le \frac{1}{q^4[n-1]_q}\cdot\frac{[n-2]_q}{[n-1]_q}\cdot\frac{4x^2+4x+1}{[n-2]_q x(1+x)+1},
\end{aligned}
$$

it follows that

$$
|M^*_{n,q}(g,x) - g(x)| \le \frac{19}{q^6[n-1]_q}\delta_n^2(x)\|g''\|. \tag{3.4}
$$

On the other hand, by (1.2), (3.2) and Lemma 1, we have

$$
|M^*_{n,q}(f,x)| \le |M_{n,q}(f,x)| + 2\|f\| \le \|f\| M_{n,q}(1,x) + 2\|f\| \le 3\|f\| \tag{3.5}
$$

Now (3.2), (3.4) and (3.5) imply

$$\begin{aligned}|M_{n,q}(f,x)-f(x)| &\le |M^*_{n,q}(f-g,x)-(f-g)(x)| \\ &\quad + |M^*_{n,q}(g,x)-g(x)| \\ &\quad + \left|f\left(x+\frac{q^{-2}[2]_q x+q^{-1}}{[n-1]_q}\right)-f(x)\right| \\ &\le 4\|f-g\| + \frac{19}{q^6[n-1]_q}\delta_n^2(x)\|g''\| \\ &\quad + \left|f\left(x+\frac{q^{-2}[2]_q x+q^{-1}}{[n-1]_q}\right)-f(x)\right|.\end{aligned}$$

Hence, taking infimum on the right hand side over all $g \in W^2$, we get

$$|M_{n,q}(f,x)-f(x)| \le 19K_2\left(f,\frac{1}{q^6[n-1]_q}\delta_n^2(x)\right)+\omega\left(f,\frac{q^{-2}[2]_q x+q^{-1}}{[n-1]_q}\right).$$

In view of (3.1), for every $q \in (0,1)$ we get

$$|M_{n,q}(f,x)-f(x)| \le C\omega_2\left(f,\frac{\delta_n(x)}{\sqrt{q^6[n-1]_q}}\right)+\omega\left(f,\frac{q^{-2}[2]_q x+q^{-1}}{[n-1]_q}\right).$$

This completes the proof of the theorem. □

Let $B_{x^2}[0,\infty) = \{f : \text{ for every } x \in [0,\infty), |f(x)| \le M_f(1+x^2),$ M_f being a constant depending upon $f\}$. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$. Also, $C^*_{x^2}[0,\infty)$ is the subspace of all functions $f \in C_{x^2}[0,\infty)$ for which $\lim_{x\to\infty} f(x)/1+x^2$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\infty)} |f(x)|/1+x^2$.

For any positive number a, by

$$\omega_a(f,\delta) = \sup_{|t-x|\le\delta}\ \sup_{x,t\in[0,a]} |f(t)-f(x)|,$$

we denote the usual modulus of continuity of f on the closed interval $[0,a]$. We know that, for a function $f \in C_{x^2}[0,\infty)$, modulus of continuity $\omega_a(f,\delta)$ tends to zero as $\delta \to 0$.

Theorem 2. *Let $f \in C_{x^2}[0,\infty)$, $q = q_n \in (0,1)$, be such that $q_n \to 1$ as $n \to \infty$, and let ω_{a+1} be its modulus of continuity on finite interval $[0, a+1] \subset [0,\infty)$ where $a > 0$. Then, for every $n > 2$,*

$$\|M_{n,q}(f) - f\|_{C[0,a]} \leq \frac{K}{q^6[n-2]_q} + 2\omega_{a+1}\left(f, \sqrt{\frac{K}{q^6[n-2]_q}}\right),$$

where $K = 114M_f(1+a^2)(1+a+a^2)$.

Proof. For $x \in [0,a]$ and $t > a+1$. Since $t - x > 1$, we have

$$\begin{aligned} |f(t) - f(x)| &\leq M_f(2 + x^2 + t^2) \\ &\leq M_f(2 + 3x^2 + 2(t-x)^2) \\ &\leq 3M_f(1 + x^2 + (t-x)^2) \\ &\leq 6M_f(1+x^2)(t-x)^2 \\ &\leq 6M_f(1+a^2)(t-x)^2. \end{aligned} \tag{3.6}$$

For $x \in [0,a]$ and $t \leq a+1$, we have

$$|f(t) - f(x)| \leq \omega_{a+1}(f, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta) \tag{3.7}$$

with $\delta > 0$. From (3.6) and (3.7), we can write

$$|f(t) - f(x)| \leq 6M_f(1+a^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta). \tag{3.8}$$

For $x \in [0,a]$ and $t \geq 0$,

$$\begin{aligned} |M_{n,q}(f,x) - f(x)| &\leq M_{n,q}(|f(t) - f(x)|, x) \\ &\leq 6M_f(1+a^2)M_{n,q}((t-x)^2, x) \\ &\quad + \omega_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}M_{n,q}((t-x)^2,x)^{1/2}\right). \end{aligned}$$

Hence, by Schwartz's inequality and Lemma 2, for every $q \in (0,1)$ and $x \in [0,a]$,

$$|M_{n,q}(f,x) - f(x)|$$

$$
\begin{aligned}
&\le \frac{114M_f(1+a^2)}{q^6[n-1]_q}\left(\varphi^2(x)+\frac{1}{[n-2]_q}\right)\\
&\quad + \omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{19}{q^6[n-1]_q}\left(\varphi^2(x)+\frac{1}{[n-2]_q}\right)}\right)\\
&\le \frac{K}{q^6[n-2]_q}+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{K}{q^6[n-2]_q}}\right),
\end{aligned}
$$

taking $\delta=\sqrt{K/q^6[n-2]_q}$

$$
\le \frac{K}{q^6[n-2]_q}+\omega_{a+1}\left(f,\sqrt{\frac{K}{q^6[n-2]_q}}\right).
$$

This completes the proof of Theorem 2. □

By $f\in \mathrm{Lip}_M\alpha$, we mean that the function f is said to satisfy Lipschitz condition of order α on $[a,b]$, if

$$
|f(x)-f(y)|\le M|x-y|^{\alpha},\quad \text{for all } x,y\in[a,b];\ \alpha>0
$$

Corollary 1. *If $f\in \mathrm{Lip}_M\alpha$ on $[0,a+1]$, then for $n>2$*

$$
\|M_{n,q_n}(f)-f\|_{C[0,a]}\le(1+2M)\sqrt{\frac{K}{q_n^6[n-2]_{q_n}}},
$$

where $\|f\|_{C[0,a]}=\sup_{x\in[0,a]}|f(x)|$.

Proof. For a sufficiently large n

$$
\frac{K}{q_n^6[n-2]_{q_n}}\le\sqrt{\frac{K}{q_n^6[n-2]_{q_n}}}
$$

because $\lim_{n\to\infty}[n-2]_{q_n}=\infty$. Hence, by $f\in\mathrm{Lip}_M\alpha$, we obtain the assertion of the corollary. □

Now we shall discuss the weighted approximation theorem, where the approximation formula holds on the interval $[0,\infty)$.

Theorem 3. *Let $q = q_n$ satisfy $0 < q_n < 1$, and let $q_n \to 1$ as $n \to \infty$. For each $f \in C^*_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty} \|M_{n,q_n}(f) - f\|_{x^2} = 0.$$

Proof. Using [**6**], we see that it is sufficient to verify the following conditions

$$\lim_{n\to\infty} \|M_{n,q_n}(t^\nu, x) - x^\nu\|_{x^2} = 0, \quad \nu = 0,1,2. \tag{3.9}$$

Since $M_{n,q_n}(1,x) = 1$, therefore for $\nu = 0$, (3.9) holds. By Lemma 2, for $n > 1$, we have

$$\begin{aligned}\|M_{n,q_n}(t,x) - x\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{|M_{n,q_n}(t,x) - x|}{1+x^2}\\ &\le \left(\frac{[n+1]_{q_n}}{q_n^2[n-1]_{q_n}} - 1\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{1}{q_n[n-1]_{q_n}}\\ &\le \left(\frac{[n+1]_{q_n}}{q_n^2[n-1]_{q_n}} - 1\right) + \frac{1}{q_n[n-1]_{q_n}}.\end{aligned}$$

Condition (3.9) holds for $\nu = 1$ as $n \to \infty$. Again, by Lemma 2 for $n > 2$, we have

$$\begin{aligned}&\|M_{n,q_n}(t^2,x) - x^2\|_{x^2}\\ &= \sup_{x\in[0,\infty)} \frac{|M_{n,q_n}(t^2,x) - x^2|}{1+x^2} \le \left(\frac{[n+1]_{q_n}[n+2]_{q_n}}{q_n^6[n-1]_{q_n}[n-2]_{q_n}} - 1\right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2}\\ &\quad + \left(\frac{(q_n + [3]_{q_n})[n+1]_{q_n}}{q_n^5[n-1]_{q_n}[n-2]_{q_n}}\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{[2]_{q_n}}{q_n^3[n-1]_{q_n}[n-2]_{q_n}}\\ &\le \left(\frac{[n+1]_{q_n}[n+2]_{q_n}}{q_n^6[n-1]_{q_n}[n-2]_{q_n}} - 1\right) + \left(\frac{(q_n + [3]_{q_n})[n+1]_{q_n}}{q_n^5[n-1]_{q_n}[n-2]_{q_n}}\right)\\ &\quad + \frac{[2]_{q_n}}{q_n^3[n-1]_{q_n}[n-2]_{q_n}}.\end{aligned}$$

Condition (3.9) holds for $\nu = 2$ as $n \to \infty$. Hence, the theorem. ◻

Corollary 2. *Let $q = q_n$ satisfy $0 < q_n < 1$, and let $q_n \to 1$ as $n \to \infty$. For each $f \in C_{x^2}[0,\infty)$ and $\alpha > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|M_{n,q_n}(f,x) - f(x)|}{(1+x^2)^\alpha} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{|M_{n,q_n}(f,x)-f(x)|}{(1+x^2)^\alpha} &\le \sup_{x\le x_0} \frac{|M_{n,q_n}(f,x)-f(x)|}{(1+x^2)^\alpha} + \sup_{x\ge x_0} \frac{|M_{n,q_n}(f,x)-f(x)|}{(1+x^2)^\alpha} \\ &\le \|M_{n,q_n}(f)-f\|_{C[0,x_0]} + \|f\|_{x^2} \sup_{x\ge x_0} \frac{|M_{n,q_n}(1+t^2,x)|}{(1+x^2)^\alpha} \\ &\quad + \sup_{x\ge x_0} \frac{|f(x)|}{(1+x^2)^\alpha}.\end{aligned}$$

The first term of the above inequality tends to zero by Theorem 2. By Lemma 1, for any fixed x_0, it is easily seen that

$$\sup_{x\ge x_0} \frac{|M_{n,q_n}(1+t^2,x)|}{(1+x^2)^\alpha}$$

tends to zero as $n \to \infty$. We can choose x_0 so large that the last part of the above inequality can be made small enough.

Thus the proof is completed. □

Voronovskaja type theorem. In this section we establish a Voronovskaja type asymptotic formula for the operators $M_{n,q}$.

Lemma 3. *Assume that $q_n \in (0,1)$, $q_n \to 1$ as $n \to \infty$. Then, for every $x \in [0,\infty)$, the following hold*:

$$\lim_{n\to\infty} [n-1]_{q_n} M_{n,q_n}(t-x,x) = 2x+1$$

and

$$\lim_{n\to\infty} [n-1]_{q_n} M_{n,q_n}((t-x)^2,x) = 2x(1+x).$$

Theorem 4. *For $q_n \in (0,1)$, the sequence $M_{n,q_n}(f)$ converges to f uniformly on $[0,A]$ for each $f \in C^*_{x^2}[0,\infty)$ if and only if* $\lim_{n\to\infty} q_n = 1$.

Proof. The proof is similar to that of Theorem 2 [**7**]. □

Theorem 5. *Assume that* $q_n \in (0,1)$, $q_n \to 1$ *as* $n \to \infty$. *Then, for any* $f \in C^*_{x^2}[0,\infty)$ *such that* $f', f'' \in C_{x^2}[0,\infty)$ *and* $x \in [0,\infty)$, *we have*

$$\lim_{n\to\infty}[n-1]_{q_n}(M_{n,q_n}(f,x) - f(x)) = (2x+1)f'(x) + x(1+x)f''(x)$$

for every $x \geq 0$.

Proof. Let $f, f', f'' \in C^*_{x^2}[0,\infty)$ and $x \in [0,\infty)$ be fixed. By the Taylor expansion, we can write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2, \tag{4.1}$$

where $r(t,x)$ is the Peano form of the remainder, $r(\cdot,x) \in C^*_{x^2}[0,\infty)$ and $\lim_{t\to x} r(t,x) = 0$. Applying M_{n,q_n} to the above, we obtain

$$\begin{aligned}[n-1]_{q_n}(M_{n,q_n}(f,x) - f(x)) &= f'(x)[n-1]_{q_n}M_{n,q_n}(t-x,x) \\ &\quad + \frac{1}{2}f''(x)[n-1]_{q_n}M_{n,q_n}((t-x)^2,x) \\ &\quad + [n-1]_{q_n}M_{n,q_n}(r(t,x)(t-x)^2,x).\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$M_{n,q_n}(r(t,x)(t-x)^2,x) \leq \sqrt{M_{n,q_n}(r(t,x)^2,x)}\sqrt{M_{n,q_n}((t-x)^4,x)}. \tag{4.2}$$

Observe that $r^2(x,x) = 0$ and $r^2(\cdot,x) \in C^*_{x^2}[0,\infty)$. Then it follows from Theorem 5 that

$$\lim_{n\to\infty}[n-1]_{q_n}M_{n,q_n}(r(t,x)^2,x) = r^2(x,x) = 0 \tag{4.3}$$

uniformly with respect to $x \in [0,A]$. Now, from (4.2), (4.3) and Lemma 3, we obtain

$$\lim_{n\to\infty}[n-1]_{q_n}M_{n,q_n}(r(t,x)(t-x)^2,x) = 0,$$

$$\begin{aligned}&\lim_{n\to\infty}[n-1]_{q_n}(M_{n,q_n}(f,x) - f(x)) \\ &= \lim_{n\to\infty}\bigg(f'(x)[n-1]_{q_n}M_{n,q_n}(t-x,x) + \frac{1}{2}f''(x)[n-1]_{q_n}M_{n,q_n} \\ &\qquad \times ((t-x)^2,x) + [n-1]_{q_n}M_{n,q_n}(r(t,x)(t-x)^2,x)\bigg) \\ &= (2x+1)f'(x) + x(1+x)f''(x).\end{aligned}$$

Acknowledgments. The authors are thankful to the reviewer for making valuable suggestions leading to overall improvements in the paper.

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