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ASYMPTOTIC ANALYSIS OF A FAMILY OF POLYNOMIALS ASSOCIATED WITH THE INVERSE ERROR FUNCTION

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ABSTRACT. We analyze the sequence of polynomials defined by the differential-difference equation $P_{n+1}(x) = P_n'(x)+x(n+1)P_n(x)$ asymptotically as $n \to \infty$. The polynomials $P_n(x)$ arise in the computation of higher derivatives of the inverse error function $\operatorname{inverf}(x)$. We use singularity analysis and discrete versions of the WKB and ray methods and give numerical results showing the accuracy of our formulas.

1. Introduction. The error function $\operatorname{erf}(x)$ is defined by [**1**]

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right)\, dt \tag{1}$$

and its inverse $\operatorname{inverf}(x)$, which we will denote by $\mathfrak{I}(x)$, satisfies $\mathfrak{I}[\operatorname{erf}(x)] = \operatorname{erf}[\mathfrak{I}(x)] = x$. The function $\mathfrak{I}(x)$ appears in several problems of applied mathematics and mathematical physics [**8**].

In [**4**] we considered the function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2}\, dt \tag{2}$$

and its inverse $S(x)$, satisfying

$$S\left[N(x)\right] = N\left[S(x)\right] = x.$$

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It is clear from (1) and (2) that

$$N(x) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) + 1\right],$$

and therefore,

$$S(x) = \sqrt{2}\mathfrak{I}\left(2x - 1\right). \tag{3}$$

In [**4**], we showed that

$$S'(x) = \sqrt{2\pi}\exp\left[\frac{1}{2}S^2(x)\right] \tag{4}$$

and

$$S^{(n)} = P_{n-1}(S)(S')^n, \quad n \geq 1, \tag{5}$$

where $P_n(x)$ is a polynomial of degree n satisfying the recurrence

$$P_0(x) = 1, \qquad P_{n+1}(x) = P_n'(x) + x(n+1)P_n(x), \quad n \geq 1. \tag{6}$$

The same approach was employed by Carlitz in [**2**]. From (6), it follows easily that for a fixed value of n

$$P_n(x) \sim n!\, x^n, \quad x \to \infty. \tag{7}$$

From (3) and (5), we conclude that

$$\mathfrak{I}^{(n)} = 2^{(n-1)/2} P_{n-1}\left(\sqrt{2}\mathfrak{I}\right)\left(\mathfrak{I}'\right)^n \quad n \geq 1.$$

Since

$$\mathfrak{I}(0) = 0, \qquad \mathfrak{I}'(0) = \frac{\sqrt{\pi}}{2},$$

we have

$$\mathfrak{I}^{(n)}(0) = \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}\right)^{n/2} P_{n-1}(0). \tag{8}$$

It follows from (8) that estimating $\mathfrak{I}^{(n)}(0)$ for large values of n is equivalent to finding an asymptotic approximation of the polynomials $P_n(x)$ when $x = 0$.

The objective of this work is to study $P_n(x)$ asymptotically as $n \to \infty$ for various ranges of x. We shall obtain different asymptotic expansions for $n \to \infty$ and (i) $0 < x < \infty$, (ii) $x = O(n^{-1})$ and (iii) $x = O(\sqrt{\ln(n)})$. The paper is organized as follows: in Section 2 we approach the problem using a singularity analysis of the generating function [**14**] of the polynomials $P_n(x)$. In Section 3 we apply the WKB method to the differential-difference equation (6). In [**15**], we used this approach in the asymptotic analysis of computer science problems and in [**7**] to study the Krawtchouk polynomials. Finally, in Section 4 we analyze (6) again using the ray method [**13**] and obtain an asymptotic approximation valid in various regions of the (x, n) domain. In [**5, 6, 9**], we employed the same technique to asymptotically analyze other families of polynomials and in [**10, 11**] to study some queueing problems.

2. Singularity analysis. In [**4**] we obtained the exponential generating function

$$\sum_{n=0}^{\infty} P_n(x)\frac{z^n}{n!} = \exp\left\{\frac{1}{2}S^2[N(x) + zN'(x)] - \frac{x^2}{2}\right\},$$

which implies that

$$P_n(x) = e^{-x^2/2}\frac{n!}{2\pi \mathrm{i}}\oint_{|z|<r} \exp\left\{\frac{1}{2}S^2[N(x) + zN'(x)]\right\}\frac{dz}{z^{n+1}},$$

where the integration contour is a small loop around the origin in the complex plane. Using (4), we have

$$\begin{aligned} P_n(x) &= e^{-x^2/2}\frac{n!}{2\pi \mathrm{i}}\oint_{|z|<r} \frac{1}{\sqrt{2\pi}}S'\left[N(x) + zN'(x)\right]\frac{dz}{z^{n+1}} \\ &= e^{-x^2/2}\frac{1}{\sqrt{2\pi}N'(x)}\frac{n!}{2\pi \mathrm{i}}\oint_{|z|<r}\frac{1}{z^{n+1}}\,dS[N(x) + zN'(x)] \\ &= \frac{n!}{2\pi \mathrm{i}}\oint_{|z|<r}\frac{n+1}{z^{n+2}}S\left[N(x) + zN'(x)\right]\,dz, \end{aligned}$$

and therefore,

$$P_n(x) = \frac{(n+1)!}{2\pi \mathrm{i}} \oint_{|z|<r} S\left[N(x) + zN'(x)\right] \frac{dz}{z^{n+2}}. \tag{9}$$

Since $S(x)$ has singularities at $x = 0$ and $x = 1$, we consider the functions

$$Z_1(x) = \frac{1 - N(x)}{N'(x)}, \qquad Z_0(x) = -\frac{N(x)}{N'(x)}.$$

We have

$$Z_1(-x) = -\sqrt{2\pi} \exp\left(\frac{x^2}{2}\right) - Z_1(x) \tag{10}$$

and

$$Z_1(x) = x^{-1} + O\left(x^{-3}\right), \quad x \to \infty,$$
$$Z_0(x) = -\sqrt{2\pi} \exp\left(\frac{x^2}{2}\right) + x^{-1} + O\left(x^{-3}\right), \quad x \to \infty.$$

Changing variables to

$$\omega = \left[z - Z_1(x)\right] N'(x)$$

in (9), we obtain

$$P_n(x) = \frac{1}{N'(x)} \frac{(n+1)!}{2\pi \mathrm{i}} \oint_C S(\omega + 1) \left[\frac{\omega}{N'(x)} + Z_1(x)\right]^{-(n+2)} d\omega,$$

or, setting $\omega = w/2$,

$$P_n(x) = \sqrt{\pi} e^{x^2/2} \frac{(n+1)!}{2\pi \mathrm{i}} \oint_C \frac{\mathfrak{J}(w+1)}{\left[\sqrt{\pi/2} e^{x^2/2} w + Z_1(x)\right]^{n+2}} dw, \tag{11}$$

where C is a small loop about $w = w^*(x)$ in the complex plane, with

$$w^*(x) = -\sqrt{\frac{2}{\pi}} e^{-x^2/2} Z_1(x).$$

To expand (11) for $n \to \infty$ with a fixed $x \in (0, \infty)$, we employ singularity analysis. The function $\mathfrak{J}(w)$ has singularities at $w = \pm 1$. By (1), we have

$$w = \frac{2}{\sqrt{\pi}} \int_0^{\mathfrak{J}} e^{-t^2} dt = 1 - e^{-\mathfrak{J}^2} \left[\frac{1}{\sqrt{\pi}\mathfrak{J}} + O\left(\mathfrak{J}^{-3}\right) \right], \quad \mathfrak{J} \to \infty,$$

so that

$$\mathfrak{J}(w) \sim \sqrt{-\ln(1-w)}, \quad w \to 1^-,$$

and by symmetry, we have

$$\mathfrak{J}(w) \sim -\sqrt{-\ln(1+w)}, \quad w \to -1^+.$$

The integrand in (11) thus has singularities at $w = 0$ and $w = -2$, but for $x > 0$, the former is closer to $w^*(x)$. We expand (11) around $w = 0$ by setting $w = \delta/n$ and using

$$\left[Z_1(x) + \sqrt{\frac{\pi}{2}} e^{x^2/2} \frac{\delta}{n} \right]^{-(n+2)} \sim \left[Z_1(x)\right]^{-(n+2)} \exp\left[-\sqrt{\frac{\pi}{2}} e^{x^2/2} \frac{\delta}{Z_1(x)} \right]. \tag{12}$$

Then, we deform the contour C in (11) to a new contour C_1 that encircles the branch points at $w = 0$ and at $w = -2$ (see Figure 1) and write

$$P_n(x) = \sqrt{\pi} e^{x^2/2} \frac{(n+1)!}{2\pi \mathrm{i}} \oint_{C_1} \frac{\mathfrak{J}\left(w+1\right)}{\left[\sqrt{\pi/2}e^{x^2/2}w + Z_1(x)\right]^{n+2}} \, dw. \tag{13}$$

In order to estimate the contribution of the integrand in (11) around the large semi-circular arcs of C_1 (see Figure 1), we use the following lemma.

Lemma 1. *The function $\mathfrak{J}(z)$ has the asymptotic behavior*

$$|\mathfrak{J}(z)| = O\left(\sqrt{\ln|z|}\right), \quad |z| \to \infty.$$

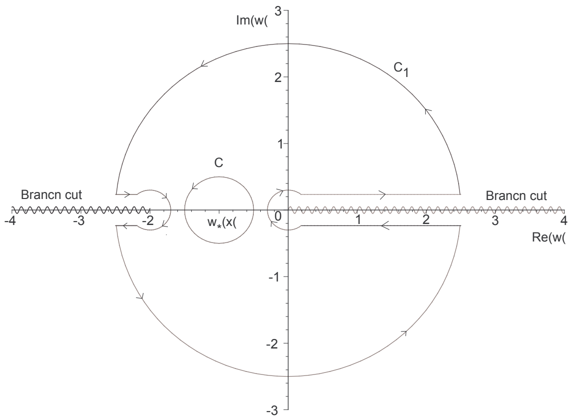

FIGURE 1. A sketch of the contours C and C_1.

Proof. We have

$$\frac{|\Im(z)|}{\sqrt{\ln|z|}} = \frac{|s|}{\sqrt{\ln|\mathrm{erf}\,(s)|}}, \quad |z| > 1,$$

where $z = \mathrm{erf}\,(s)$. Now [**1**],

$$|\mathrm{erf}\,(s)| \longrightarrow \infty$$

if and only if $|s| \to \infty$ and

$$\left| \arg(s) \pm \frac{\pi}{2} \right| < \frac{\pi}{4}.$$

In this case,

$$|\mathrm{erf}\,(s)| \sim \frac{1}{\sqrt{\pi} r} \exp\left[-r^2 \cos(2\theta)\right], \quad r \to \infty,$$

where $s = re^{\mathrm{i}\theta}$. So,

$$
\begin{aligned}
\lim_{|z|\to\infty} \frac{|\mathfrak{I}(z)|}{\sqrt{\ln|z|}} &= \lim_{\substack{|s|\to\infty \\ |\arg(s)\pm\pi/2|<\pi/4}} \frac{|s|}{\sqrt{\ln|\operatorname{erf}(s)|}} \\
&= \lim_{\substack{r\to\infty \\ |\theta\pm\pi/2|<\pi/4}} \frac{r}{\sqrt{-r^2\cos(2\theta) - \ln(r) - 1/2\ln(\pi)}} \\
&= \frac{1}{\sqrt{-\cos(2\theta)}}, \quad 2\theta \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)
\end{aligned}
$$

and the result follows. □

In particular, we see that

$$|\mathfrak{I}(z)| = o(|z|), \quad |z| \to \infty,$$

and therefore the contribution of the integrand in (13) around the large semi-circular arcs of C_1 can be neglected.

We denote by $\Psi_1(x,n)$ the contribution from the branch point at $w = 0$. To leading order, this is

$$
\begin{aligned}
\Psi_1(x,n) &\sim (n+1)!\sqrt{\pi}e^{x^2/2}\left[Z_1(x)\right]^{-(n+2)}\frac{1}{n} \\
&\times \frac{1}{2\pi\mathrm{i}}\int_0^\infty \left(\Upsilon^+ - \Upsilon^-\right)\exp\left[-\sqrt{\frac{\pi}{2}}\frac{e^{x^2/2}}{Z_1(x)}\delta\right]d\delta,
\end{aligned}
\tag{14}
$$

where

$$\Upsilon^\pm(\delta,n) = \sqrt{\pm\mathrm{i}\pi - \ln(\delta) + \ln(n)}.$$

Here $\Upsilon^\pm(\delta,n)$ corresponds to the approximation of $\mathfrak{I}(w+1)$ for $w \to 0$, above or below the right branch cut in Figure 1.

For n large we have

$$\Upsilon^+(\delta,n) - \Upsilon^-(\delta,n) \sim \frac{\pi\mathrm{i}}{\sqrt{\ln(n)}}$$

and evaluating the elementary integral in (14) leads to

$$
\begin{aligned}
\Psi_1(x,n) &\sim \frac{n!}{\sqrt{2\ln(n)}}\left[Z_1(x)\right]^{-(n+1)} \\
&= \frac{n!}{\sqrt{2\ln(n)}}\left[\frac{e^{-x^2/2}}{\zeta(x)}\right]^{n+1}, \quad n \to \infty
\end{aligned}
\tag{15}
$$

with

$$\begin{aligned}\zeta(x) &= \sqrt{\frac{\pi}{2}}\left[1-\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right] \\ &\sim \exp\left(-\frac{x^2}{2}\right)\left[x^{-1}+O\left(x^{-3}\right)\right], \quad x\to\infty.\end{aligned} \tag{16}$$

By using the symmetry relation

$$\sqrt{\frac{\pi}{2}}\exp\left(\frac{x^2}{2}\right)w+Z_1(-x)=\sqrt{\frac{\pi}{2}}\exp\left(\frac{x^2}{2}\right)(w+2)-Z_1(x),$$

we see that the integrand in (11) is antisymmetric with respect to the map $(x,w)\to(-x,-w-2)$. Thus, the branch point at $w=-2$ gives the contribution $(-1)^n\Psi_1(-x,n)$, and hence

$$P_n(x)\sim\frac{n!}{\sqrt{2\ln(n)}}\left\{[Z_1(x)]^{-(n+1)}+(-1)^n\left[Z_1(-x)\right]^{-(n+1)}\right\}. \tag{17}$$

This formula applies for $n\to\infty$ and all fixed $x\geq 0$. For $x>0$ the second term in the right-hand side of (17) is asymptotically negligible, since $Z_1(-x)>Z_1(x)$. However, for $x=O(n^{-1})$ this is no longer true.

We let $x=y/n$ with $y=O(1)$, and use

$$Z_1(x)=Z_1\left(\frac{y}{n}\right)=\sqrt{\frac{\pi}{2}}-\frac{y}{n}+O\left(n^{-2}\right), \quad n\to\infty \tag{18}$$

so that (17) simplifies to

$$\begin{aligned}P_n(x)\sim\Psi_2(x,n)&=\frac{n!}{\sqrt{2\ln(n)}}\left(\sqrt{\frac{2}{\pi}}\right)^{n+1} \\ &\times\left[\exp\left(y\sqrt{\frac{2}{\pi}}\right)+(-1)^n\exp\left(-y\sqrt{\frac{2}{\pi}}\right)\right].\end{aligned} \tag{19}$$

Note that, for $x=O(n^{-1})$, the singularities at $w=0$ and $w=-2$ in (11) are nearly equidistant from $w^*(x)$.

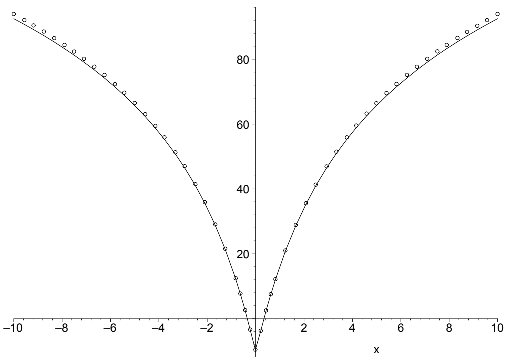

FIGURE 2. A plot of $\ln[P_{40}(x)/40!]$ (solid line) and $\ln[\Psi_1(x, 40)/40!]$ (ooo).

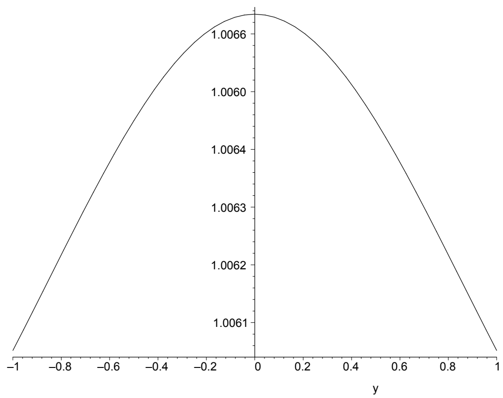

FIGURE 3. A plot of the ratio $\ln[P_{40}(y/40)/40!]/\ln[\Psi_2(y, 40)/40!]$.

In Figure 2 we plot $\ln[P_{40}(x)/40!]$ and $\ln[\Psi_1(x, 40)/40!]$. We see that the approximation is very good for $x = O(1)$, but it becomes less precise as $x \to \infty$. This is because our previous analysis assumes that $n \to \infty$ with $0 < x < \infty$. If $x \to \infty$, we must modify it.

In Figure 3 we plot the ratio $\ln[P_{40}(y/40)/40!]/\ln[\Psi_2(y, 40)/40!]$ and verify the accuracy of (19).

Letting $x \to \infty$ in (17) using (16) yields

$$P_n(x) \sim \frac{n!\, x^{n+1}}{\sqrt{2\ln(n)}},$$

which differs from (7). This suggests that another scale must be analyzed, where x and n are both large. Thus, we consider the case of $x \to \infty$, with $x = O(\sqrt{\ln n})$. Now the singularity at $w = 0$ in (11) becomes close to $w^*(x)$, since

$$w^*(x) \sim -\frac{\sqrt{2}}{x\sqrt{\pi}}, \quad x \to \infty.$$

We use the form (9) and expand $S[N(x) + zN'(x)]$ for $z \to 0$ and $x \to \infty$. Setting $z = \xi/x$ with $\xi = O(1)$, we obtain

$$\begin{aligned} S[N(x) + zN'(x)] &= \sqrt{2}\Im\left\{1 + \sqrt{\frac{2}{\pi}}\left[ze^{-x^2/2} - \zeta(x)\right]\right\} \\ &= \sqrt{2}\Im\left[1 + \sqrt{\frac{2}{\pi}}e^{-x^2/2}\left(z - \frac{1}{x} + O\left(x^{-3}\right)\right)\right] \\ &\sim \sqrt{2}\sqrt{-\ln\left[\sqrt{\frac{2}{\pi}}e^{-x^2/2}\left(\frac{1}{x} - z\right)\right]} \\ &\sim \sqrt{2}\sqrt{\frac{x^2}{2} + \ln(x) - \frac{1}{2}\ln\left(\frac{2}{\pi}\right) - \ln(1-\xi)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} P_n(x) &\sim \frac{\sqrt{2}\,(n+1)!x^{n+1}}{2\pi\mathrm{i}} \\ &\quad\times \oint_{C_2}\sqrt{\frac{x^2}{2} + \ln\left(\frac{x}{1-\xi}\right) - \frac{1}{2}\ln\left(\frac{2}{\pi}\right)}\frac{d\xi}{\xi^{n+2}}. \end{aligned} \tag{20}$$

Here the contour C_2 is a small loop about $\xi = 0$. Now we again employ singularity analysis, with the branch point at $\xi = 1$ determining the

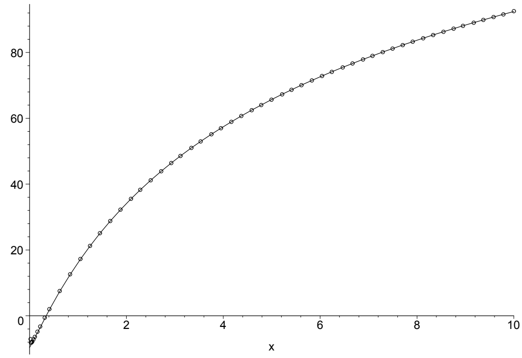

FIGURE 4. A plot of $\ln[P_{40}(x)/40!]$ (solid line) and $\ln[\Psi_3(x, 40)/40!]$ (ooo).

asymptotic behavior for $n \to \infty$. A deformation similar to that in Figure 1 leads to

$$P_n(x) \sim \frac{n!x^{n+1}}{\sqrt{2}} \frac{1}{\sqrt{(x^2/2) + \ln(nx) - (1/2)\ln(2/\pi)}}. \tag{21}$$

For $x \gg \sqrt{\ln(n)}$ this reduces to (7).

By examining (15) and (21), we can obtain the following approximation

$$P_n(x) \sim \Psi_3(x, n) = \frac{n!}{\sqrt{x^2 + 2\ln(nx) - \ln(2/\pi)}} \left[\frac{e^{-x^2/2}}{\zeta(x)}\right]^{n+1}, \tag{22}$$

which is more uniform in x, since it holds both for $x = O(1)$ and $x = O(\sqrt{\ln n})$ for n large and for $x \to \infty$ with n fixed. However, we must still use (19) if n is large and x is small. In Figure 4 we plot $\ln[P_{40}(x)/40!]$ and $\ln[\Psi_3(x, 40)/40!]$ and confirm that (22) is a better approximation than (15) for large values of x.

3. WKB analysis.

We shall now re-derive the results in the previous section by using only the recurrences (6) and (7). We apply the

WKB method to (5), seeking solutions of the form $P_n(x) = n!\overline{P}_n(x)$, with

$$\overline{P}_n(x) \sim \exp\left[(n+1)\, A(x)\right] B(x,n), \quad n \to \infty. \tag{23}$$

Thus, we are assuming an exponential dependence on n and an additional weaker (e.g., algebraic) dependence that arises from the function $B(x,n)$. Use of (23) in (6) leads to

$$e^{A(x)}\left[B(x,n) + \frac{\partial}{\partial n}B(x,n)\right] + O\left(\frac{\partial^2 B}{\partial n^2}\right) = \left[x + A'(x)\right] B(x,n) + \frac{1}{n+1}\frac{\partial}{\partial x}B(x,n).$$

Expecting that $\partial B/\partial n = o(B)$ and $\partial^2 B/\partial n^2 = o(\partial B/\partial n)$, we set

$$e^{A(x)} = x + A'(x) \tag{24}$$

and

$$e^{A(x)}\frac{\partial}{\partial n}B(x,n) = \frac{1}{n+1}\frac{\partial}{\partial x}B(x,n) \sim \frac{1}{n}\frac{\partial}{\partial x}B(x,n). \tag{25}$$

To solve (24) we let

$$A(x) = -\frac{x^2}{2} + a(x)$$

to find that

$$a'(x) = e^{-x^2/2}e^{a(x)}.$$

Solving this separable ODE leads to

$$A(x) = -\frac{x^2}{2} - \ln\left[\zeta\left(x\right) + k\right], \tag{26}$$

where k is a constant of integration. To fix k, we assume that expansion (23), as $x \to \infty$, will asymptotically match to (7), when this is expanded for $n \to \infty$. In view of (23) this implies that

$$\overline{P}_n(x) \sim x^n = \exp\left[n\ln(x)\right], \quad x \to \infty,$$

so that

$$A(x) \sim \ln(x), \quad x \to \infty.$$

In view of (26) this is possible only if $k = 0$ and then from (16) we have

$$A(x) = -\ln\left[e^{x^2/2}\zeta(x)\right] \sim \ln(x), \quad x \to \infty. \tag{27}$$

We next analyze (25). Using (27) to compute $e^{A(x)}$, we obtain

$$\frac{e^{-x^2/2}}{\zeta(x)} \frac{\partial}{\partial n} B(x,n) = \frac{1}{n} \frac{\partial}{\partial x} B(x,n).$$

Solving this first order PDE by the method of characteristics, we obtain

$$B(x,n) = b\left[\frac{n}{\zeta(x)}\right],$$

where $b(\cdot)$ is at this point an arbitrary function. However, since n is large and $x = O(1)$, we need only the behavior of $b(\cdot)$ for large values of its argument. We again argue that by matching to (7) we have

$$\exp\left[(n+1)A(x)\right] B(x,n) \sim x^n, \quad x \to \infty,$$

and using (27) we get

$$B(x,n) \sim e^{-A(x)} \sim \frac{1}{x}, \quad x \to \infty,$$

and thus

$$b\left(nxe^{x^2/2}\right) \sim \frac{1}{x}, \quad x \to \infty$$

so that

$$b(z) \sim \frac{1}{\sqrt{2\ln(z)}}, \quad z \to \infty.$$

Combining our results, we have found that

$$P_n(x) \sim \frac{n!}{\sqrt{2}\sqrt{\ln(n) - \ln\left[\zeta(x)\right]}} \left[\frac{e^{-x^2/2}}{\zeta(x)}\right]^{n+1}, \quad n \to \infty. \tag{28}$$

This applies for $x = O(1)$ and $n \to \infty$, where we can regain the results of the singularity analysis (15) by simply using

$$\sqrt{\ln(n) - \ln\left[\zeta\left(x\right)\right]} \sim \sqrt{\ln(n)}, \quad n \to \infty.$$

Formula (28) is also valid for $n = O(1)$ and $x \to \infty$, where it reduces to (7). However, (23) breaks down for $x \to 0$.

We thus consider the scale $x = y/n$, $y = O(1)$ and set

$$P_n(x) = n!\widetilde{P}_n(nx) = n!\widetilde{P}_n(y), \tag{29}$$

with which (6) becomes

$$\widetilde{P}_{n+1}\left(y + \frac{y}{n}\right) = \frac{n}{n+1}\widetilde{P}_n'(y) + \frac{y}{n}\widetilde{P}_n(y). \tag{30}$$

For fixed y, we seek an asymptotic solution of (30) in the form

$$\widetilde{P}_n(y) \sim e^{\alpha n} q\left(y, n\right), \quad n \to \infty, \tag{31}$$

where $q(y, n)$ will have a weaker (e.g., algebraic or logarithmic) dependence on n. From (30) we obtain, using (31),

$$\begin{aligned} e^{\alpha}\left[q(y,n) + \frac{\partial}{\partial n}q(y,n) + \frac{y}{n}\frac{\partial}{\partial y}q(y,n) + O\left(n^{-2}\right)\right] \\ = \left[1 - \frac{1}{n} + O\left(n^{-2}\right)\right]\frac{\partial}{\partial y}q(y,n) + \frac{y}{n}q(y,n). \end{aligned} \tag{32}$$

If $q(y, n)$ has an algebraic dependence on n, then $(\partial/\partial n)q(y, n)$ should be roughly $O(n^{-1})$ relative to $q(y, n)$, $(\partial^2/\partial n^2)q(y, n)$ roughly $O(n^{-2})$, and so on. Thus, we expand $q(y, n)$ as

$$q\left(y, n\right) = q_0(y, n) + \frac{1}{n}q_1(y, n) + O\left(n^{-2}\right), \tag{33}$$

where $q_0(y, n)$, $q_1(y, n)$ have a very weak (e.g., logarithmic) dependence on n and balance terms in (32) of order $O(1)$ and $O(n^{-1})$ to obtain

$$e^{\alpha}q_0(y, n) = \frac{\partial}{\partial y}q_0(y, n) \tag{34}$$

and

$$e^{\alpha}\left[q_1(y,n)+\frac{\partial}{\partial n}q_1(y,n)+y\frac{\partial}{\partial y}q_0(y,n)\right] = \frac{\partial}{\partial y}q_1(y,n)-\frac{\partial}{\partial y}q_0(y,n)-yq_0(y,n). \tag{35}$$

The solution of (34) yields

$$q_0(y,n)=\exp\left(e^{\alpha}y\right)\mathfrak{q}(n), \tag{36}$$

where $\mathfrak{q}(n)$ must be determined. We could solve (35), using (36), but its solution would involve another arbitrary function of n. Thus, considering higher order terms will not help in determining $\mathfrak{q}(n)$. Instead, we employ asymptotic matching to (28). Expanding (28) for $x \to 0$ and comparing the result to (29) as $y \to \infty$, with (31), (33) and (36), we conclude that

$$\alpha=\frac{1}{2}\ln\left(\frac{2}{\pi}\right), \qquad \mathfrak{q}(n)=\frac{1}{\sqrt{\pi\ln(n)}}. \tag{37}$$

But then our approximation for $y=O(1)$ is not consistent with $P_n(0)=0=\widetilde{P}_n(0)$ for odd n. We return to (30) and observe that the equation also admits an asymptotic solution of the form

$$\widetilde{P}_n(y)\sim(-1)^n e^{\beta n}\overline{q}(y,n), \quad n\to\infty$$

where, analogously to (34), we find that

$$-e^{\beta}\overline{q}_0(y,n)=\frac{\partial}{\partial y}\overline{q}_0(y,n),$$

so that another asymptotic solution to (30) is

$$\widetilde{P}_n(y)\sim(-1)^n e^{\beta n}\exp\left(e^{\beta}y\right)\overline{\mathfrak{q}}(n). \tag{38}$$

We argue that any linear combination of (31) and (38) is also a solution, and that the combination which vanishes at $y=0$ for odd n has $\beta=\alpha$

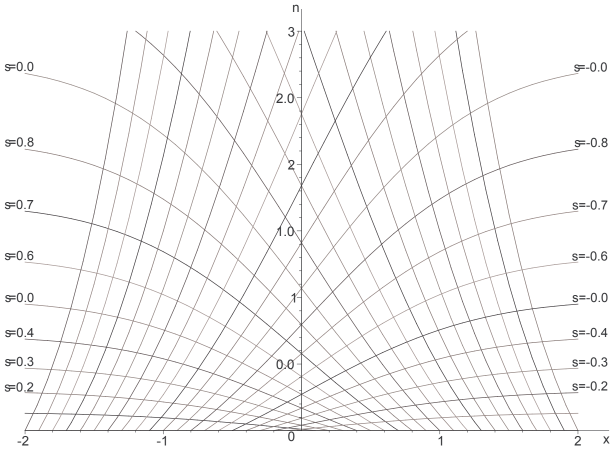

FIGURE 5. A plot of the rays $x(t,s)$, $n(t,s)$ with $s=-0.9,-0.8,\dots,0.8,0.9$.

and $\overline{\mathfrak{q}}(n)=\mathfrak{q}(n)$, as in (37). We have thus obtained, for $y=O(1)$ and $n\to\infty$, that

$$P_n(x)\sim\frac{n!}{\sqrt{\pi\ln(n)}}\left(\frac{2}{\pi}\right)^{n/2}\left[\exp\left(y\sqrt{\frac{2}{\pi}}\right)+(-1)^n\exp\left(-y\sqrt{\frac{2}{\pi}}\right)\right].$$

This agrees with (19), obtained by singularity analysis in Section 2.

To summarize, we have shown how to infer the asymptotics of $P_n(x)$ using only recursion (6) and the large x behavior (7). Our analysis does need to make some assumptions about the forms of various expansions and the asymptotic matching between different scales.

4. The discrete ray method. We shall now find a uniform asymptotic approximation for $P_n(x)$ using a discrete form of the ray method [**12**]. This approximation will apply for x and/or n large. We seek an approximate solution for (6) of the form

$$P_n(x)=\exp\left[f(x,n)+g(x,n)\right], \tag{39}$$

where $g=o(f)$ as $n\to\infty$. Since $P_n(x)=x^n$, $n=0,1$, we see that we must have

$$f(x,n)\sim n\ln(x)\quad\text{and}\quad g(x,n)\to 0 \tag{40}$$

as $n \to 0$. Using (39) in (6), we have

$$\exp\left(\frac{\partial f}{\partial n} + \frac{1}{2}\frac{\partial^2 f}{\partial n^2} + \frac{\partial g}{\partial n}\right) \sim \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) + (n+1)\,x \tag{41}$$

for $n \to \infty$, where we have used

$$f(x, n+1) = f(x,n) + \frac{\partial f}{\partial n}(x,n) + \frac{1}{2}\frac{\partial^2 f}{\partial n^2}(x,n) + \cdots .$$

From (41) we obtain, to leading order, the *eikonal* equation

$$\frac{\partial f}{\partial x} + (n+1)\,x - \exp\left(\frac{\partial f}{\partial n}\right) = 0, \tag{42}$$

and the *transport* equation

$$\frac{1}{2}\frac{\partial^2 f}{\partial n^2} + \frac{\partial g}{\partial n} - \frac{\partial g}{\partial x}\exp\left(-\frac{\partial f}{\partial n}\right) = 0. \tag{43}$$

To solve (42), we use the method of characteristics, which we briefly review. Given the first order partial differential equation

$$F\,(x, n, f, p, q) = 0, \quad \text{with} \quad p = \frac{\partial f}{\partial x}, \qquad q = \frac{\partial f}{\partial n},$$

we search for a solution $f(x,n)$ by solving the system of "characteristic equations"

$$\begin{aligned}
\frac{dx}{dt} &= \frac{\partial F}{\partial p}, \qquad \frac{dn}{dt} = \frac{\partial F}{\partial q}, \\
\frac{dp}{dt} &= -\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial f}, \\
\frac{dq}{dt} &= -\frac{\partial F}{\partial n} - q\frac{\partial F}{\partial f}, \\
\frac{df}{dt} &= p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q},
\end{aligned}$$

with initial conditions

$$F\,[x(0,s), n(0,s), f(0,s), p(0,s), q(0,s)] = 0, \tag{44}$$

and

$$\frac{d}{ds} f(0,s) = p(0,s) \frac{d}{ds} x(0,s) + q(0,s) \frac{d}{ds} n(0,s), \tag{45}$$

where we now consider $\{x, n, f, p, q\}$ all to be functions of the variables s and t.

For the eikonal equation (42), we have

$$F(x, n, f, p, q) = p - e^q + (n+1) x, \tag{46}$$

and therefore the characteristic equations are

$$\frac{dx}{dt} = 1, \qquad \frac{dn}{dt} = -e^q, \qquad \frac{dp}{dt} = -(n+1), \qquad \frac{dq}{dt} = -x, \tag{47}$$

and

$$\frac{df}{dt} = p - qe^q. \tag{48}$$

Solving (47) subject to the initial conditions

$$x(0,s) = s, \quad n(0,s) = 0, \quad p(0,s) = A(s), \quad q(0,s) = B(s),$$

we obtain

(49)

$$\begin{aligned}
&x = t + s, \quad n = -\sqrt{\frac{\pi}{2}} \exp\left(\frac{s^2}{2} + B\right) \left[\operatorname{erf}\left(\frac{t+s}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{s}{\sqrt{2}}\right)\right], \\
&p = \sqrt{\frac{\pi}{2}} \exp\left(\frac{s^2}{2} + B\right) (t+s) \left[\operatorname{erf}\left(\frac{t+s}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{s}{\sqrt{2}}\right)\right] \\
&\qquad + \exp\left(-\frac{1}{2}t^2 - st + B\right) - t - e^B + A, \\
&q = -\frac{1}{2}t^2 - st + B.
\end{aligned}$$

From (40) we have

$$A(s) = 0 \quad \text{and} \quad B(s) = \ln(s), \tag{50}$$

which is consistent with (44). Therefore,

$$x = t + s, \quad n = -\sqrt{\frac{\pi}{2}} s \exp\left(\frac{s^2}{2}\right)\left[\operatorname{erf}\left(\frac{t+s}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{s}{\sqrt{2}}\right)\right], \tag{51}$$

$$\begin{aligned} p &= \sqrt{\frac{\pi}{2}} s \exp\left(\frac{s^2}{2}\right)(t+s)\left[\operatorname{erf}\left(\frac{t+s}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{s}{\sqrt{2}}\right)\right] \\ &\quad + s \exp\left(-\frac{1}{2}t^2 - st\right) - (t+s), \\ q &= -\frac{1}{2}t^2 - st + \ln(s). \end{aligned} \tag{52}$$

In Figure 5 we sketch some of the rays $x(t,s)$, $n(t,s)$ for $s = -0.9, -0.8, \dots, 0.8, 0.9$.

Using (52) in (48) we have

$$\begin{aligned} \frac{df}{dt} &= \sqrt{\frac{\pi}{2}} s \exp\left(\frac{s^2}{2}\right)(t+s)\left[\operatorname{erf}\left(\frac{t+s}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{s}{\sqrt{2}}\right)\right] \\ &\quad + s\left[1 + \frac{1}{2}t^2 + st - \ln(s)\right]\exp\left(-\frac{1}{2}t^2 - st\right) - (t+s). \end{aligned} \tag{53}$$

Using (50) in (45), we get

$$f(0,s) = f_0, \tag{54}$$

and solving (53) subject to (54), we obtain

$$\begin{aligned} f(t,s) &= \sqrt{\frac{\pi}{2}} \exp\left(\frac{s^2}{2}\right)\left[\operatorname{erf}\left(\frac{t+s}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{s}{\sqrt{2}}\right)\right] \\ &\quad \times s\left[1 + \frac{1}{2}t^2 + st - \ln(s)\right] - \left(\frac{1}{2}t^2 + st\right) + f_0 \end{aligned} \tag{55}$$

or, using (51),

$$f = [\ln(s) - 1]\, n + \frac{1}{2}\left(s^2 - x^2\right)(n+1) + f_0. \tag{56}$$

To solve the transport equation (43), we need to compute $(\partial^2 f)/(\partial n^2), (\partial g/\partial n)$ and $(\partial g/\partial x)$ as functions of t and s. Use of the chain rule gives

$$\begin{bmatrix} \partial x/\partial t & \partial x/\partial s \\ \partial n/\partial t & \partial n/\partial s \end{bmatrix} \begin{bmatrix} \partial t/\partial x & \partial t/\partial n \\ \partial s/\partial x & \partial s/\partial n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and hence,

$$\begin{bmatrix} \partial t/\partial x & \partial t/\partial n \\ \partial s/\partial x & \partial s/\partial n \end{bmatrix} = \frac{1}{J(t,s)} \begin{bmatrix} \partial n/\partial s & -\partial x/\partial s \\ -\partial n/\partial t & \partial x/\partial t \end{bmatrix}, \tag{57}$$

where the Jacobian $J(t,s)$ is defined by

$$J\left(t,s\right) = \frac{\partial x}{\partial t}\frac{\partial n}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial n}{\partial t} = \frac{\partial n}{\partial s} - \frac{\partial n}{\partial t}. \tag{58}$$

Using (51), we can show after some algebra that

$$J = \left(s + \frac{1}{s}\right)n + s. \tag{59}$$

Using $q = \partial f/\partial n$ in (43), we have

$$\frac{1}{2}\frac{\partial q}{\partial n} + \frac{\partial g}{\partial n} - \frac{\partial g}{\partial x}e^{-q} = 0$$

or

$$\frac{\partial}{\partial n}\left(\frac{1}{2}e^q\right) = \frac{\partial g}{\partial x} - \frac{\partial g}{\partial n}e^q$$

and, using (47), we obtain

$$\frac{\partial}{\partial n}\left(\frac{1}{2}e^q\right) = \frac{\partial g}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial g}{\partial n}\frac{\partial n}{\partial t} = \frac{\partial g}{\partial t}.$$

Since $-e^q = \partial n/\partial t$, we have

$$\begin{aligned}\frac{\partial}{\partial n}\left(\frac{1}{2}e^q\right) &= -\frac{1}{2}\frac{\partial}{\partial n}\left(\frac{\partial n}{\partial t}\right) = -\frac{1}{2}\left(\frac{\partial^2 n}{\partial t^2}\frac{\partial t}{\partial n} + \frac{\partial^2 n}{\partial t \partial s}\frac{\partial s}{\partial n}\right) \\ &= -\frac{1}{2J}\left(-\frac{\partial^2 n}{\partial t^2}\frac{\partial x}{\partial s} + \frac{\partial^2 n}{\partial t\partial s}\frac{\partial x}{\partial t}\right) = -\frac{1}{2J}\left(-\frac{\partial^2 n}{\partial t^2} + \frac{\partial^2 n}{\partial t \partial s}\right) \\ &= -\frac{1}{2J}\frac{\partial}{\partial t}\left(\frac{\partial n}{\partial s} - \frac{\partial n}{\partial t}\right) = -\frac{1}{2J}\frac{\partial J}{\partial t},\end{aligned}$$

where we have used (57) and (58). Thus,

$$\frac{\partial g}{\partial t} = -\frac{1}{2J}\frac{\partial J}{\partial t}$$

and therefore

$$g(t,s) = -\frac{1}{2}\ln(J) + C(s)$$

for some function $C(s)$. Since from (40) we have $g(0,s) = 0$, while (59) gives $J(0,s) = s$, we conclude that $C(s) = (1/2)\ln(s)$ and hence

$$g(t,s) = \frac{1}{2}\ln\left[\frac{s}{J(t,s)}\right]. \tag{60}$$

Using (59) we can write (60) as

$$g = \frac{1}{2}\ln\left[\frac{s^2}{(n+1)\,s^2+n}\right]. \tag{61}$$

Replacing f and g in (39) by (56) and (61), we obtain $P_n(x) \sim \Phi(x,n;s)$ as $n \to \infty$, with

$$\Phi\left(x,n;s\right) = \kappa \exp\left[\frac{1}{2}\left(s^2-x^2\right)(n+1) - n\right]\frac{s^n}{\sqrt{n+1+ns^{-2}}}, \tag{62}$$

where $\kappa = e^{f_0}$ is still to be determined.

Eliminating t from (51) we get

$$n + \sqrt{\frac{\pi}{2}}s\exp\left(\frac{s^2}{2}\right)\left[\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{s}{\sqrt{2}}\right)\right] = 0, \tag{63}$$

which defines $s(x,n)$ implicitly. For every $n > 0$ there exist only two solutions $S_m(x,n) < 0$ and $S_p(x,n) > 0$ of (63) (see Figure 5). Since $\operatorname{erf}(x)$ is an odd function, it follows that

$$S_m(x,n) = -S_p(-x,n). \tag{64}$$

Although we have $P_n(x) \sim \Phi[x,n;S_m(x,n)]$ for $x \ll -1$ and $P_n(x) \sim \Phi[x,n;S_p(x,n)]$ for $x \gg 1$, the two approximations are comparable when x is small and therefore we must add their contributions.

We shall now find the constant κ in (62) by using (7). We rewrite (63) as

$$s^2 \exp\left(s^2\right) = n^2 \left[\int_s^x \exp\left(-\frac{\theta^2}{2}\right) d\theta \right]^{-2}, \tag{65}$$

and for a fixed value of n, consider the limit $x \to \infty$. It follows from (65) that $S_p(x,n) \sim x$, and therefore we consider an expansion of the form

$$S_p(x,n) \sim x + s_0 + s_1 x^{-1} + s_2 x^{-2} + s_3 x^{-3}, \quad x \to \infty. \tag{66}$$

Using (66) in (65), we obtain

$$\begin{aligned} s_0 &= s_2 = 0, \quad s_1 = \ln(n+1), \\ s_3 &= 1 - \ln(n+1) - \frac{\ln^2(n+1)}{2} - \frac{1}{n+1}, \end{aligned}$$

and therefore

$$s^2 \exp\left(s^2\right) \sim (n+1)^2 x^2 e^{x^2}, \quad x \to \infty. \tag{67}$$

Solving (67), we have

$$S_p(x,n) \sim \sqrt{\mathrm{W}\left[(n+1)^2 x^2 e^{x^2}\right]}, \quad x \to \infty, \tag{68}$$

where $\mathrm{W}(z)$ denotes the Lambert W function, defined by [**3**]

$$\mathrm{W}(z) \exp\left[\mathrm{W}(z)\right] = z, \quad \text{for all } z \in \mathbf{C}$$

and having the asymptotic behavior

$$\mathrm{W}(z) = \ln(z) - \ln\ln(z) + \frac{\ln\ln(z)}{\ln(z)} + O\left(\left[\frac{\ln\ln(z)}{\ln(z)}\right]^2\right), \quad z \to \infty. \tag{69}$$

Using (68) and (69) in (62), we obtain

$$\Phi(x,n;s) \sim \kappa (n+1)^{n+(1/2)} e^{-n} x^n, \quad x \to \infty.$$

From Stirling's formula,

$$n! = \left[\sqrt{2\pi n} + O\left(n^{-1/2}\right)\right] n^n e^{-n}, \quad n \to \infty, \tag{70}$$

and

$$(n+1)^{n+(1/2)} = \left[e\sqrt{n} + O\left(n^{-1/2}\right)\right] n^n, \quad n \to \infty,$$

we conclude that

$$\kappa = e^{-1}\sqrt{2\pi},$$

and thus

$$\Phi(x,n;s) = s^n \exp\left[\frac{1}{2}\left(s^2 - x^2 - 2\right)(n+1)\right]\sqrt{\frac{2\pi s^2}{(n+1)\,s^2 + n}}. \tag{71}$$

Using (64), we have $P_n(x) \sim \Psi_4(x,n)$ as $n \to \infty$, with

$$\Psi_4(x,n) = \Phi\left[x,n;S_p(x,n)\right] + \Phi\left[x,n;-S_p(-x,n)\right], \quad n \to \infty. \tag{72}$$

In Figure 6 we compare $\ln[P_4(x)/4!]$ and $\ln[\Psi_4(x,4)/4!]$ for $-1 < x < 1$ and in Figure 7 for $0 < x < 10$. We note that the asymptotic approximation (72) is more uniform than (15), (19) and (21), but it is less explicit since $S_p(x,n)$ must be obtained numerically.

Next, we compare the results of this section with those in the previous two sections. We first consider $x > 0$, with $x = O(1)$ and $n \to \infty$. From (63), we have

$$S_p(x,n) \sim \sqrt{\mathrm{W}\left[\frac{(n+1)^2}{\zeta^2(x)}\right]}, \tag{73}$$

where $\zeta(x)$ was defined in (16). Using (73) and (69) in (71), we get

$$P_n(x) \sim n^n e^{-n}\sqrt{\frac{n\pi}{\ln(n)}}\left[\frac{e^{-x^2/2}}{\zeta(x)}\right]^{n+1},$$

which agrees with (15) after taking (70) into account.

Now we consider the limit $n \to \infty$, with $x = y/n$ and $y = O(1)$. From (63), we have

$$S_p\left(\frac{y}{n},n\right) \sim \sqrt{\mathrm{W}\left(\frac{2n^2}{\pi}\right)} + \left(1 + \sqrt{\frac{2}{\pi}}y\right)\frac{1}{\sqrt{\mathrm{W}(2n^2/\pi)}n}. \tag{74}$$

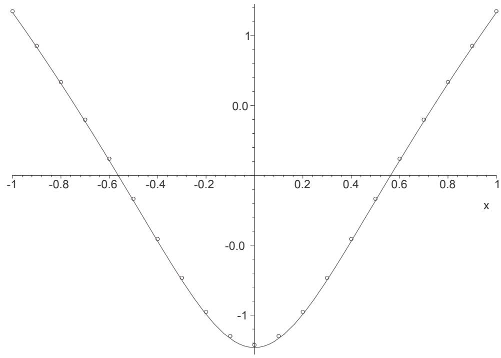

FIGURE 6. A plot of $\ln[P_4(x)/4!]$ (solid line) and $\ln[\Psi_4(x,4)/4!]$ (ooo).

Using (74), (64) and (69) in (71), we find that

$$\Phi\left(y/n,n;S_p\right) \sim n^n e^{-n}\sqrt{\frac{2n}{\ln(n)}}\left(\sqrt{\frac{2}{\pi}}\right)^n \exp\left(\sqrt{\frac{2}{\pi}}y\right),$$

$$\Phi\left(y/n,n;S_m\right) \sim (-1)^n n^n e^{-n}\sqrt{\frac{2n}{\ln(n)}}\left(\sqrt{\frac{2}{\pi}}\right)^n \exp\left(-\sqrt{\frac{2}{\pi}}y\right),$$

and therefore

$$P_n(x) \sim n^n e^{-n}\sqrt{\frac{2n}{\ln(n)}}\left(\sqrt{\frac{2}{\pi}}\right)^n \left[\exp\left(\sqrt{\frac{2}{\pi}}y\right)+(-1)^n \exp\left(-\sqrt{\frac{2}{\pi}}y\right)\right],$$

agreeing with (19).

Finally, we consider the limit $n \to \infty$ with $x = u\sqrt{\ln(n)}$, $u = O(1)$, $u > 0$. From (63), we have

$$S_p\left(u\sqrt{\ln(n)},n\right) \sim \sqrt{\mathrm{W}\left[(n+1)^2 u^2 n^{u^2}\ln(n)\right]}. \tag{75}$$

Using (75) and (69) in (71), we have

$$P_n\left(u\sqrt{\ln(n)}\right) \sim n^n e^{-n}\sqrt{2\pi n}\frac{u^{n+1}}{\sqrt{u^2+2}}\left(\sqrt{\ln(n)}\right)^n, \quad n \to \infty,$$

which agrees with (21).

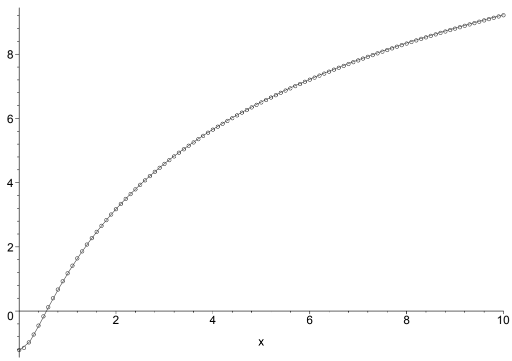

FIGURE 7. A plot of $\ln[P_4(x)/4!]$ (solid line) and $\ln[\Psi_4(x,4)/4!]$ (ooo).

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