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NICE BASES AND THICKNESS IN PRIMARY ABELIAN GROUPS

PETER V. DANCHEV AND PATRICK W. KEEF

ABSTRACT. All groups considered will be abelian p-groups. Several results pertaining to nice bases and Ulm subgroups are established. For example, it is shown that a separable group A is thick if and only if for all reduced groups G with $G/p^\omega G = A$, if G has a nice basis, then $p^\omega G$ is a direct sum of cyclics. We also consider the class of separable groups A such that, for every group G with $G/p^\omega G = A$, if $p^\omega G$ has a nice basis, then G has a nice basis. It is shown that if A is a direct sum of cyclics then it has this latter property, and that the converse of this statement is related to the continuum hypothesis. Finally, it is shown that any group of length $\omega \cdot 2$ is a summand of a group with a nice basis, and in particular, that a summand of a group with a nice basis need not retain that property.

0. Introduction. By the term *group* we will mean an abelian p-group, where p is a prime fixed for the duration. Our terminology and notation will generally follow [**3**]. We say a group G is Σ-*cyclic* if it is isomorphic to a direct sum of cyclic groups. We will also utilize the language of *valuated vector spaces* (see [**4**]).

A classical result of Hill (see [**5**]) states that if a group G is the ascending union of a sequence of pure subgroups $\{B_n\}_{n<\omega}$ and each B_n is Σ-cyclic, then G will also be Σ-cyclic. Note that since $p^\omega B_n = p^\omega G \cap B_n = \{0\}$, B_n will actually be an *isotype* subgroup of G (in other words, $p^\alpha B_n = p^\alpha G \cap B_n$ for all ordinals α). Since isotype and *nice* subgroups are, in some sense, dual to one another (where a subgroup N of G is nice if $p^\alpha(G/N) = [p^\alpha G+N]/N$ for every ordinal α), it is natural to consider the following definition from [**1, 2**]: G has a *nice basis* if it is the union of an ascending sequence $\{N_n\}_{n<\omega}$ of nice subgroups such that each N_n is Σ-cyclic. The class of groups having a nice basis is quite extensive; for example, it contains the totally projective groups and all separable groups. In addition, it is closed with respect to direct sums (e.g., [**1, 2**]).

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A recurring theme in the study of abelian groups is the relationship between a group G, its first Ulm subgroup $p^\omega G$ and its corresponding Ulm factor $G/p^\omega G$. In this direction, we begin with two statements from [**1, 2**] that give, respectively, a sufficient and a necessary condition for a group G to have a nice basis:

(1) If $p^\omega G$ is Σ-cyclic, then G has a nice basis.

(2) If G has a nice basis, then $p^\omega G$ has a nice basis.

Our first goal is to describe precisely for which separable groups $A = G/p^\omega G$, the converse of (1) always holds. We will say A has the *nice basis restriction property* if for all reduced groups G with $A = G/p^\omega G$, if G has a nice basis, then $p^\omega G$ is Σ-cyclic. Note that in this definition, it is only the groups G which are not separable which are of interest, and in fact, only those for which $p^\omega G$ is unbounded, since a bounded group is always Σ-cyclic. Our most significant result (Theorem 1.3) is that the separable group A has the nice basis restriction property if and only if it is *thick*. Recall that a group A is thick if and only if whenever C is Σ-cyclic and $f : A \to C$ is a homomorphism, then f must be *small*, i.e., the kernel of f contains a *large* subgroup of A. This class of groups, originally identified by Pierce (see [**7**]), has been studied by a number of authors. We will use the characterization that a group G is thick if and only if for every homomorphism $f : G \to C$ where C is Σ-cyclic, there is an $n < \omega$ such that $f((p^n G)[p]) = \{0\}$ (see, for instance, [**6**]).

We next consider the class of separable groups A for which the converse of (2) always holds. We say A has the *nice basis extension property* if for all groups G with $A = G/p^\omega G$, if $p^\omega G$ has a nice basis, then G has a nice basis. As above, in this definition, only those G for which $p^\omega G$ is unbounded are of interest because it was proved in [**2**] that each group of length $< \omega \cdot 2$ possesses a nice basis. This condition is intimately related to the behavior of the pure subgroups and the closed subgroups of A (where all topological notions are with respect to the p-adic topology). In particular, we will say that A is *countably-closed* if whenever A is the ascending union of a sequence of pure subgroups $\{B_n\}_{n<\omega}$, then A is also the ascending union of a sequence of closed subgroups $\{C_n\}_{n<\omega}$, such that for all n, $C_n \subseteq B_n$. Observe that there is a corresponding notion for valuated vector spaces V, which are *separable* in the sense that $V(\omega) = \{0\}$ (in this context, the condition of purity is superfluous). We show (Theorem 2.3) that if A is Σ-cyclic, then it is countably-closed; if it is countably-closed, then it has the nice basis extension property; and if it has the nice basis extension property, then $A[p]$ is a countably-closed valuated vector space. This suggests the question of whether any countably-closed valuated vector space is necessarily free. We are able to verify that the continuum hypothesis (CH) guarantees that this is the case whenever V has a countable basic subspace (Corollary 2.12). It follows that CH implies that when A has a countable basic subgroup, A is Σ-cyclic if and only if it has the nice basis extension property (Corollary 2.13).

We next turn to the question posed in [**2**] of whether the class of groups with nice bases is closed with respect to direct summands. We show that this fails in a surprisingly general way: If G is any group with $p^{\omega\cdot 2}G = \{0\}$, then there is a *dsc group* (i.e., a direct sum of countable groups) H of length $\omega \cdot 2$ such that $G \oplus H$ has a nice basis (Theorem 3.1). In particular, Theorem 1.3 implies that if A is any unbounded, separable thick group, and G is a group such that $G/p^\omega G = A$ and $p^\omega G$ is a separable group which is not Σ-cyclic, then G is a group of length $\omega \cdot 2$ that does not have a nice basis, but it is a summand of a group that does have a nice basis.

1. Nice bases and thick groups. To begin, recall that if G is a group and S is a subgroup of $G[p]$ which is dense in $G[p]$ in the p-adic topology, then any subgroup M of G which is maximal with respect to the property that $M[p] = S$ will be pure and dense in G (see, for example, Theorem 66.3 of [**3**]). Since we will be concerned with the relationship between subgroups and factor-groups of G, $p^\omega G$ and $G_\omega = G/p^\omega G$, we include the following list of essentially well-known results:

Lemma 1.1. *Suppose G is a group, X is a subgroup of G, $A = G/X$ and Y is a subgroup of G that is maximal with respect to the property $Y \cap X = \{0\}$. Then*

(a) *$\pi(g + X) = pg + pX$ gives a well-defined homomorphism $A[p] \to X/pX$ with kernel $[G[p] + X]/X \cong Y[p]$;*

(b) *$X \subseteq p^\omega G$ if and only if for every $n < \omega$, $\pi((p^n A)[p]) = X/pX$;*

(c) *$X = p^\omega G$ if and only if A is separable and for every $n < \omega$, $\pi((p^n A)[p]) = X/pX$.*

Supposing now that $X = p^\omega G$, *we have*:

(d) *If* X' *is a pure subgroup of* X, *then there is a pure and dense subgroup* G' *of* G *containing* Y *such that* $X' = p^\omega G' = G' \cap X$ (*note that* $A' = G'/X'$ *maps to a pure and dense subgroup of* $A = G/X$ *under the usual homomorphism*);

(e) *If* X' *is pure and dense in* X *and* G' *is as in* (d), *then we have* $A' = G'/X' \cong G/X = A$;

(f) *If* X *is a pure and dense subgroup of* X'' *and* $G'' = [G \oplus X'']/\{(x, -x) : x \in X\}$, *then we can view* X'' *and* G *as subgroups of* G'', *and we have* $G'' = G + X''$, $X = G \cap X''$, $G''/X'' \cong G/X$, G *is pure in* G'' *and* $p^\omega G'' = X''$;

(g) *If* N *is a subgroup of* G, *then* N *is nice in* G *if and only if* $N(\omega) = N \cap X$ *is nice in* X *and* $N' = [N + X]/X$ *is a closed subgroup of* $A = G/X$. *This latter condition is equivalent to requiring that* $G/[N + X] \cong A/N'$ *is separable.*

Proof. As to (a), it can be directly verified, but a more conceptual approach is to note that π is the connecting homomorphism $A[p] \cong \mathrm{Tor}\,(A, \mathbf{Z}_p) \to X \otimes \mathbf{Z}_p \cong X/pX$. The last isomorphism follows easily from the maximality of Y with respect to $X \cap Y = \{0\}$.

Regarding (b), if $X \subseteq p^\omega G$, then for all $x \in X$ and $n < \omega$, there is a $z \in p^n G$ such that $pz = x$; therefore, $z + X \in (p^n A)[p]$ with $\pi(z + X) = x + pX$ gives one direction. Conversely, a simple induction using the given condition shows that $X \subseteq p^n G$ for all $n < \omega$, so that $X \subseteq p^\omega G$.

As to (c), since A is separable, we must have $p^\omega G \subseteq X$; the other inclusion follows from (b).

Turning to (d), note that $Y[p] \oplus X'[p]$ is a dense subgroup of $G[p]$, so if we choose $G' \subseteq G$ containing $Y \oplus X'$ which is maximal with respect to the condition that $G'[p] = Y[p] \oplus X'[p]$, then G' is pure in G. This implies that $X' \subseteq G' \cap X = p^\omega G'$. On the other hand, the conditions (i) X' is pure in X; (ii) $X' \subseteq p^\omega G' \subseteq X$; and (iii) $X'[p] = (p^\omega G')[p]$ can readily be seen to imply that $X' = p^\omega G'$.

Regarding (e), note that we have injections $Y[p] \to A'[p] \to A[p]$. In addition, since X' is pure and dense in X, we can conclude $X'/pX' \to$

X/pX is an isomorphism. Considering the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y[p] \arrow[r] \arrow[d, equal] & A'[p] \arrow[r] \arrow[d] & X'/pX' \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & Y[p] \arrow[r] & A[p] \arrow[r] & X/pX \arrow[r] & 0 \end{tikzcd}$$

we can conclude that $A'[p] = A[p]$ (where we have identified A' with its image under the above embedding). But since A' is pure in A, it follows that $A' = A$.

Turning to (f), the statements $G'' = G + X''$, $X = G \cap X''$ and $G''/X'' \cong G/X$ are elementary. To verify the last two as well, note that $X = p^\omega G \subseteq p^\omega G''$, so the map $G'' \to G''/X \cong (G/X) \oplus (X''/X)$ necessarily preserves all finite heights. Since X''/X is divisible, it follows that $X'' \subseteq p^\omega G''$. On the other hand, since $G''/X'' \cong G/X$ is separable, we have $p^\omega G'' \subseteq X''$; hence $X'' = p^\omega G''$. This also means that $G/p^\omega G = G/X \cong G''/X'' = G''/p^\omega G''$ is an isomorphism, which certainly implies that $G \subseteq G''$ preserves all finite heights, and so in particular, G is pure in G''.

Finally, as to (g), if $g \in G$ and we consider the coset $g + N$, then either there is an element $x \in N$ such that $g + x$ has infinite height, or not. In the first case, the niceness of $N(\omega)$ in X implies the existence of an $g + x_0 \in g + N$ of maximal height, and in the second case, the closure of N' in A implies the existence of such an $g + x_0 \in g + N$ of maximal height. □

The following is also well known (see [**4**]):

Lemma 1.2. *Suppose V and W are valuated vector spaces, W is free and separable (i.e., $W(\omega) = \{0\}$) and $f : V \to W$ is a valuated homomorphism with kernel K. Then there is an isometry $V \cong K \oplus W'$, where W' is free and separable.*

Proof. Since W is separable and free, it is the ascending union of a sequence of subspaces W_n, such that $W_n(n) = \{0\}$. If $W' = V/K$ is the quotient valuated vector space, then W' is the ascending union of the subspaces $W'_n = f^{-1}(W_n)/K$, and $W'_n(n) = \{0\}$. It follows that W' is also free and separable, so that the required isometry must exist. $\square$

Theorem 1.3. *If A is a separable group, then A has the nice basis restriction property if and only if it is thick.*

Proof. It is easy to check that a bounded group has both the nice basis restriction property and is thick, so we may assume A is unbounded. Suppose first that A is thick, that G is a reduced group with $G/p^\omega G = A$, and that G is the ascending union of the nice subgroups N_n which are Σ-cyclic; we need to show that $p^\omega G$ is Σ-cyclic. Note $N'_n = [N_n + p^\omega G]/p^\omega G$ is closed in A and $A = \cup_n N'_n$.

Claim 1. *There is an $n < \omega$ such that $(p^n A)[p] \subseteq N'_n$.*

Supposing that this fails, we will show that A is actually not thick. Now, for each $n < \omega$, let $x_n \in (p^n A)[p] - N'_n$. Observe that A/N'_n is a separable group and $x_n + N'_n$ is a non-zero element of it. It follows that there is a cyclic group C_n and a composite homomorphism $\phi_n : A \to A/N'_n \to C_n$ such that $\phi_n(x_n) \neq 0$.

Define $\phi : A \to \prod_{n<\omega} C_n$ by the equation $\phi(x) = (\phi_n(x))_{n<\omega}$. Note that if $x \in N'_m$ and $n \geq m$, then $x \in N'_m \subseteq N'_n$ and it follows that $\phi_n(x) = 0 \in C_n$. Since every x is in N'_m for some m, it follows that the image of ϕ is actually contained in $\oplus_{n<\omega} C_n$, which is Σ-cyclic. Since for each $n < \omega$, $\phi(x_n) \neq 0$, it follows by [**6**] that A is not thick, as required.

By the last Claim, fix some $n < \omega$ such that $(p^n A)[p] \subseteq N'_n$.

Claim 2. $p^\omega G/N_n(\omega) = p^\omega G/(N_n \cap p^\omega G)$ *is divisible.*

Let $x \in p^\omega G$, and choose $y \in G$ such that $py = x$ and $ht_G(y) \geq n$. So $y + p^\omega G \in (p^n A)[p] \subseteq N'_n$. It follows that there is a $z \in N_n$ such that $y + p^\omega G = z + p^\omega G$. Therefore, $y = z + w$, where $w \in p^\omega G$, so that $x = py = pz + pw \in N_n(\omega) + p(p^\omega G)$. Therefore, we have $p^\omega G = N_n(\omega) + p(p^\omega G)$, which establishes the claim.

Since G is reduced and $N_n(\omega)$ is nice in $p^\omega G$, it follows that $p^\omega G = N_n(\omega) \subseteq N_n$, and in particular, $p^\omega G$ must be Σ-cyclic. So, A has the nice basis restriction property, as required.

As for the converse, suppose A is not thick. To complete the argument, we need to construct a group G with a nice basis such that $G/p^\omega G = A$ and $p^\omega G$ is not Σ-cyclic (in fact, in our example, $p^\omega G$ will be an unbounded torsion complete group).

Since A is not thick, we can find a subgroup U of A such that $A/U \cong \oplus_{i\in I} C_i$ where each C_i is a cyclic group and, for every $n < \omega$, $(p^n A)[p]$ is not contained in U. By Lemma 1.2, there is a valuated decomposition $A[p] = U[p] \oplus F$, where F is a separable and free valuated vector space with $F(n) \neq \{0\}$ for every $n < \omega$. Fix some valuated decomposition $F = \oplus_{b\in\mathcal{B}} \langle b \rangle$.

We claim that we can find a sequence of subgroups, $U_n, U'_n \subseteq A$, satisfying the following:

(a) $\cup_{n<\omega} U_n = A$;

(b) $U \subseteq U_n \subseteq U_{n+1}$;

(c) $U_n/U \cong \oplus_{i\in I_n} C_i$ for some subset $I_n \subseteq I$, so in particular, U_n is closed in A;

(d) U'_n is an unbounded Σ-cyclic group such that $U'_n \cap U_n = \{0\}$ and $U'_n \subseteq U_{n+1}$.

Let b_1 be some element of $\mathcal{B}$ with $\mathrm{ht}\,(b_1) \geq 1$; let $z_1 \in A$ satisfy $pz_1 = b_1$ and $\mathrm{ht}\,(z_1) + 1 = \mathrm{ht}\,(b_1)$, and let J_1 be the support of $z_1 + U$ in the decomposition $\oplus_{i\in I} C_i$. For $j > 1$, we continue this construction, inductively defining basis elements $b_j \in \mathcal{B}$, elements $z_j \in A$ and finite subsets J_j of I such that

(1) $\mathrm{ht}\,(b_j) > \exp(C_i) + j$ for all $i \in J_{j-1}$ (where if C is a cyclic group of order p^k, then $\exp(C) = k$);

(2) $p^j z_j = b_j$;

(3) $\mathrm{ht}\,(z_j) = \mathrm{ht}\,(b_j) - j$;

(4) J_j is the support of $z_j + U$ in the decomposition $\oplus_{i\in I} C_i$.

It is clear that this can be done. Note that if we select any sequence of $i_j \in J_j$, then these facts imply that

$$\begin{aligned}\cdots < \mathrm{ht}\,(b_{j-1}) - (j-1) = \mathrm{ht}\,(z_{j-1}) \leq \mathrm{ht}_{A/U}(z_{j-1} + U) < \exp(C_{i_{j-1}})\\ < \mathrm{ht}\,(b_j) - j = \mathrm{ht}\,(z_j) \leq \mathrm{ht}_{A/U}(z_j + U) < \exp(C_{i_j})\\ < \mathrm{ht}\,(b_{j+1}) - (j+1) = \mathrm{ht}\,(z_{j+1}) \leq \cdots .\end{aligned}$$

In particular, since $j < k$ always implies $\exp(C_{i_j}) < \exp(C_{i_k})$, it follows that all the J_j's are pairwise disjoint. Finally, we let $J_0 = I - \cup_{1 \le j} J_j$.

Next, for all $j < \omega$, $z_j + U \in \oplus_{i \in J_j} C_i$ and $p^j(z_j + U) = x_j + U$ is a non-zero element of $(A/U)[p]$, we can conclude that $x_j + U$ is a non-zero element of $\oplus_{i \in J_j} C_i$. Since the J_j are pairwise disjoint, we can conclude that the composition $\phi : \oplus_{j<\omega} \langle z_j \rangle \to A \to A/U \subset \oplus_{i \in I} C_i$ is injective.

Now, for each $n < \omega$, we let $I_n = J_0 \cup [\cup\{J_j : 2^n$ does not divide $j\}]$, and define U_n by the isomorphism in (c). Since $I = \cup_{n<\omega} I_n$ and $I_n \subseteq I_{n+1}$, (a) and (b) are clear. Regarding (d), if we let $U_n' = \langle z_j : j = 2^n \ell$ where ℓ is odd $\rangle$, then since $\circ(z_j) = p^{j+1}$ and the collection of j of the form $j = 2^n \ell$ with ℓ odd is unbounded, we can conclude that U_n' is unbounded; and since ϕ is injective, we can conclude U_n' is Σ-cyclic. Also, for such a $j = 2^n \ell$, $2^n | j$ and $2^{n+1} \not| j$; from this we can conclude $J_j \cap I_n = \varnothing$ and $J_j \subseteq I_{n+1}$. Therefore, $U_n' \cap U_n = \{0\}$ and $U_n' \subseteq U_{n+1}$, so that (d) follows.

Let Z_n be a group such that $Z_n/p^\omega Z_n = U_n'$ and $p^\omega Z_n \cong \mathbf{Z}_{p^{n+1}}$, and let $W_0 = \{0\}$, and for $n \ge 1$, $W_n = \oplus_{m<n} Z_m$. We now inductively construct an ascending chain of groups $H_0 \subseteq H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$, such that $W_n \subseteq H_n$, $p^\omega W_n = p^\omega H_n$ and $H_n/p^\omega W_n \cong U_n$. We begin by letting $H_0 = U_0$; if H_n has been constructed, then to construct H_{n+1}, we begin by letting $H_{n+1}' = H_n \oplus Z_n$, so that $W_{n+1} = W_n \oplus Z_n \subseteq H_{n+1}'$, and $H_{n+1}'/p^\omega W_{n+1} \cong (H_n/p^\omega W_n) \oplus (Z_n/p^\omega Z_n) \cong U_n \oplus U_n'$.

Using the surjectivity of the map

$$\operatorname{Ext}(U_{n+1}, p^\omega W_{n+1}) \longrightarrow \operatorname{Ext}(U_n \oplus U_n', p^\omega W_{n+1}),$$

we can define H_{n+1} so that it fits into a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & p^\omega W_{n+1} \arrow[r] \arrow[d, equal] & H_{n+1}' \arrow[r] \arrow[d] & U_n \oplus U_n' \arrow[r] \arrow[d, "\subseteq"] & 0 \\
0 \arrow[r] & p^\omega W_{n+1} \arrow[r] & H_{n+1} \arrow[r] & U_{n+1} \arrow[r] & 0
\end{tikzcd}$$

It follows that (interpreting injective maps as inclusions) $H_n \subseteq H_{n+1}' \subseteq H_{n+1}$, $W_{n+1} \subseteq H_{n+1}' \subseteq H_{n+1}$ and that there is a commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & p^\omega W_n \arrow[r] \arrow[d, "\subseteq"] & H_n \arrow[r] \arrow[d, "\subseteq"] & U_n \arrow[r] \arrow[d, "\subseteq"] & 0 \\
0 \arrow[r] & p^\omega W_{n+1} \arrow[r] & H_{n+1} \arrow[r] & U_{n+1} \arrow[r] & 0
\end{tikzcd}
$$

Since $W_{n+1} \subseteq H_{n+1}$, we have $p^\omega W_{n+1} \subseteq p^\omega H_{n+1}$, and since $H_{n+1}/p^\omega W_{n+1} \cong U_{n+1}$ is separable, it follows that $p^\omega H_{n+1} = p^\omega W_{n+1}$. If $H = \cup_{n<\omega} H_n$ and $W = \cup_{n<\omega} W_n = \oplus_{m<\omega} Z_m$, then it follows that $p^\omega H = p^\omega W$ and $H/p^\omega H \cong A$.

Finally, as in Lemma 1.1 (f), let G be the sum of $\overline{p^\omega H}$ and H along $p^\omega H$, where $\overline{p^\omega H}$ is the torsion-completion of $p^\omega H$, so that $G = \overline{p^\omega H} + H$ and $p^\omega H = H \cap \overline{p^\omega H}$. It follows that $p^\omega G = \overline{p^\omega H}$ is torsion-complete and $G/p^\omega G \cong H/p^\omega H \cong A$. Our aim, then, is to construct a nice basis for this G.

Now, given $n < \omega$, let $M_n \subseteq H_n$ be the subgroup containing $p^\omega W_n$ such that $M_n/p^\omega W_n = U_n[p^n]$, and define $N_n = M_n + (p^\omega G)[p^n]$. Note that $U_n[p^n]$ and $p^\omega W_n$ are bounded by p^n, so M_n is bounded by p^{2n}, and $(p^\omega G)[p^n]$ is trivially bounded, so N_n is bounded and hence Σ-cyclic. Next, observe that $p^\omega G$ is contained in $\cup_{n<\omega} N_n$, and

$$\cup_{n<\omega} N_n' = \cup_{n<\omega} \frac{[N_n + p^\omega G]}{p^\omega G} \cong \cup_{n<\omega} U_n[p^n] = A,$$

and these two facts imply $\cup_{n<\omega} N_n = G$.

Finally, we will be done if we can show N_n is nice in G. To that end, observe that $M_n \cap p^\omega G = p^\omega W_n \subseteq (p^\omega G)[p^n]$, so that $N_n(\omega) = N_n \cap p^\omega G = (M_n + (p^\omega G)[p^n]) \cap (p^\omega G)[p^n] = (p^\omega G)[p^n]$ which is certainly closed in $p^\omega G$, and $N_n' = [N_n + p^\omega G]/p^\omega G \cong M_n/p^\omega W_n = U_n[p^n] = A[p^n] \cap U_n$ is the intersection of two closed subgroups of A, and hence is closed in A, as well. It follows, therefore, by Lemma 1.1 (g) that N_n is nice in G, as required. $\square$

In [**2**], a group G of length $\omega \cdot 2$ is constructed that does not have a nice basis. In fact, that group has the property that both $p^\omega G$ and $A = G/p^\omega G$ are unbounded torsion-complete groups; in particular, A

is thick and $p^\omega G$ is not Σ-cyclic, so the fact that G fails to have a nice basis is also a consequence of Theorem 1.3.

Recall that (imitating [**2**]) a group G has a *bounded nice basis* if it is the ascending union of nice subgroups $\{N_n\}_{n<\omega}$ such that each N_n is bounded (and hence Σ-cyclic). Note also that a group G is thick if and only if its Ulm factor $G/p^\omega G$ is thick (see [**6**]). The following, then, answers Conjectures 1 and 2 from [**2**].

Corollary 1.4. *Let G be a reduced thick group. Then*

(a) *G has a nice basis if and only if $p^\omega G$ is Σ-cyclic*;

(b) *G has a bounded nice basis if and only if $p^\omega G$ is bounded.*

Proof. (a) follows immediately from Theorem 1.3.

Regarding (b), suppose $A = G/p^\omega G$. If $p^\omega G$ is bounded, then if we define N_n to be the subgroup containing $p^\omega G$ such that $N_n/p^\omega G = A[p^n]$, then N_n is bounded and nice, and G is their ascending union (this argument does not require G to be thick). Conversely, if G (and hence A) is thick, and N_n is a bounded nice basis for G, then by the proof of Theorem 1.3, for some $n < \omega$, $p^\omega G \subseteq N_n$, so $p^\omega G$ is bounded. $\square$

2. Nice bases and Σ-cyclic groups. Recall, the separable group A has the *nice basis extension property* if for every group G such that $G/p^\omega G = A$, if $p^\omega G$ has a nice basis, then G has a nice basis. We will also say A has the *strong nice basis extension property* if for every such G, if $\{M_n\}_{n<\omega}$ is a nice basis for $p^\omega G$, then there is a nice basis $\{N_n\}_{n<\omega}$ for G such that $M_n = N_n \cap p^\omega G$.

Again, recall the separable group A is *countably-closed* if whenever A is the ascending union of a sequence of pure subgroups B_n, then it is also the ascending union of a sequence of closed subgroups C_n, where for all $n < \omega$, $C_n \subseteq B_n$. Correspondingly, the separable valuated vector space V is *countably-closed* if whenever V is the ascending union of a sequence of the subspaces T_n, then it is also the ascending union of a sequence of closed subspaces S_n, where for all $n < \omega$, $S_n \subseteq T_n$; and so the separable group A *has a countably-closed socle* if $A[p]$ is countably-closed as a valuated vector space.

We pause for the following observation:

Lemma 2.1. *The separable group A is countably-closed if and only if it satisfies the following condition*: *whenever A is the ascending* *union of a sequence of pure and dense subgroups D_n, then it is also the ascending union of a sequence of closed subgroups C_n, where for all $n < \omega$, $C_n \subseteq D_n$. In other words, we may restrict our attention to ascending chains of pure subgroups which are also dense.*

Proof. Necessity being trivial, suppose A satisfies this condition for all ascending sequences of pure and dense subgroups, and A is the ascending union of the pure subgroups B_n. For each n, let Y_n be a basic subgroup of B_{n+1}/B_n. Since Y_n is Σ-cyclic and B_n is a pure subgroup of A, and hence of B_{n+1}, it follows easily that there is a subgroup X_n of B_{n+1} which maps isomorphically onto Y_n under the natural map $B_{n+1} \to B_{n+1}/B_n$, and that $B_n \oplus X_n$ is a pure and dense subgroup of B_{n+1}. Note that $\bigoplus_{m<\omega} X_m$ will be a basic subgroup of A, and if we let

$$D_n = B_n \oplus \left(\bigoplus_{m \geq n} X_m \right),$$

then D_n will be a pure and dense subgroup of A. It also follows that $D_n \subseteq D_{n+1}$, and since A is the union of the B_n, it is also the union of the D_n.

By hypothesis, then, we can express A as the ascending sequence of closed subgroups C_n such that $C_n \subseteq D_n$ for all n. Let $C'_n = C_n \cap B_n \subseteq B_n$; since the B_n and C_n are ascending sequences whose union is A, the same can be said of the C'_n. The important point to verify is that C'_n is actually closed in A. Note that if x_i is a sequence in C'_n which converges (in the p-adic topology) to $y \in A$, then since C_n is closed, we must have $y \in C_n \subseteq D_n = B_n \oplus (\bigoplus_{m \geq n} X_m)$. Since D_n is a pure subgroup of A, we have that x_n converges to y in the p-adic topology on D_n. However, C'_n is a closed subgroup of D_n, since it is the intersection of two closed subgroups, so that $y \in C'_n$, as required. □

An analogous argument shows the following:

Lemma 2.2. *The separable valuated vector space V is countably-closed if and only if it satisfies the following condition: whenever V is the ascending union of a sequence dense subspaces U_n, then it is also the ascending union of a sequence of closed subspaces S_n, where for all $n < \omega$, $S_n \subseteq U_n$. In other words, we may restrict our attention to ascending chains of dense subspaces.*

Theorem 2.3. *Suppose A is a separable group.*

(a) *If A is Σ-cyclic, then it is countably-closed;*

(b) *If A is countably-closed, then it has the strong nice basis extension property;*

(c) *If A has the strong nice basis extension property, then it has the nice basis extension property;*

(d) *If A has the nice basis extension property, then it has a countably-closed socle.*

Proof. For the implication in (a), note that if $A = \oplus_{i\in I}\langle x_i\rangle$, and $\{B_n\}_{n<\omega}$ is an ascending chain of pure subgroups whose union is A, then let $C_n = \langle x_i : x_i \in B_n\rangle$. Since C_n is a summand of A, it is clearly closed, $C_n \subseteq C_{n+1}$ and their union is all of A.

For the implication in (b), suppose A is countably-closed and G is a group with $G/p^\omega G = A$ and $p^\omega G$ has a nice basis $\{M_n\}_{n<\omega}$; without loss of generality, assume $M_0 = \{0\}$. We need to construct a nice basis $\{N_n\}_{n<\omega}$ for G such that $M_n = N_n \cap p^\omega G$.

Let B be a basic subgroup of $p^\omega G$, and assume $B = \oplus_{i\in I}\langle b_i\rangle$. Suppose Y is a high subgroup of G (i.e., maximal with respect the property that $Y \cap p^\omega G = \{0\}$); so Y is pure and dense in G, and $B \cap Y = \{0\}$.

Given $n < \omega$, let $I_n = \{i \in I : b_i \in M_n\}$ and $B_n = \oplus_{i\in I_n}\langle b_i\rangle$; since $B \subseteq \cup_{n<\omega} M_n$, we can conclude $I = \cup_{n<\omega} I_n$, so that $B = \cup_{n<\omega} B_n$. Using the proof of Lemma 1.1 (d), construct pure and dense subgroups $H = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq G$ such that $p^\omega E_n = B_n$. If we let $G' = \cup_{n<\omega} E_n$, then G' is also pure and dense in G, $H \subseteq G'$ and $B = p^\omega G'$, and by Lemma 1.1 (e) the natural map $G'/p^\omega G' \to G/p^\omega G = A$ is an isomorphism. Therefore, if we let $D_n = [E_n + p^\omega G]/p^\omega G \subseteq A$, then D_n is pure and dense in A, $D_n \subseteq D_{n+1}$ and $A = \cup_{n<\omega} D_n$. Now, since A is countably-closed, there is an ascending sequence C_n of closed subgroups such that $A = \cup_{n<\omega} C_n$ and $C_n \subseteq D_n$. Replacing C_n by $C_n[p^n]$ does not affect these observations, so we may assume $p^n C_n = \{0\}$.

Next, note that $E_n/B_n = E_n/[E_n \cap p^\omega G] \cong [E_n + p^\omega G]/p^\omega G = D_n$, and we let $L_n \subseteq E_n$ be defined by the equation $L_n/B_n \cong C_n$. Since $E_n \subseteq E_{n+1}$ and $B_n \subseteq B_{n+1}$, and $C_n \subseteq C_{n+1}$, it readily follows that $L_n \subseteq L_{n+1}$, and in addition, since $p^n C_n = \{0\}$, we have $p^n L_n \subseteq B_n$. Let $N_n = M_n + L_n$; note that $p^n N_n = p^n M_n + p^n L_n \subseteq M_n + B_n = M_n$, which is Σ-cyclic, so that N_n is Σ-cyclic. In addition, since $M_n \subseteq M_{n+1}$ and $L_n \subseteq L_{n+1}$, it follows that $N_n \subseteq N_{n+1}$.

Next, note that $M_n \subseteq p^\omega G$ implies that $N_n(\omega) = M_n + L_n(\omega) = M_n + B_n = M_n$; and this implies that (1) $N_n(\omega)$ is nice in $p^\omega G$, (2) the N_n extend the M_n and (3) $p^\omega G = \cup_{n<\omega} N_n(\omega)$. Also, observe that

$$N'_n = [N_n + p^\omega G]/p^\omega G = [L_n + p^\omega G]/p^\omega G = C_n,$$

and since C_n is closed in A, in view of Lemma 1.1 (g) we have that N_n is nice in G. Finally, since $A = \cup_{n<\omega} C_n$, it follows that $G = \cup_{n<\omega} N_n$, which shows that the N_n's form a nice basis for G, as required.

Clearly, the implication (c) is trivial.

Finally, to prove (d), assume A has the nice basis extension property, and $A[p]$ is the ascending union of dense socles T_n. Let E_n be an ascending sequence of pure and dense subgroups of A such that $E_n[p] = T_n$ (such a sequence clearly exists) and let $Y = E_0$. Since $\cup_{n<\omega} E_n$ will be a pure subgroup containing $A[p]$, it follows that $A = \cup_{n<\omega} E_n$. For $0 < i < \omega$, let κ_i be the rank of T_i/T_{i-1}, and let U_i be a direct sum of κ_i copies of $\mathbf{Z}_{p^i}$; next, let $B_0 = \{0\}$ and for $0 < n < \omega$, let $B_n = \oplus_{0<i\leq n} U_i$ and $B = \oplus_{0<i<\omega} U_i$. Let D_n be an ascending chain of divisible groups containing the B_n as essential subgroups, and let $D = \cup_{n<\omega} D_n$. Fix an isomorphism $D/B \to A/Y$ which takes $D_n/B_n \cong [D_n + B]/B$ to E_n/Y. We define G' by the pull-back diagram:

$$\begin{tikzcd}
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & G' \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] & A \arrow[r] & A/Y \arrow[r] & 0,
\end{tikzcd}$$

and we identify B and Y with subgroups of G'. We next claim that $B = p^\omega G'$, or in light of Lemma 1.1 (c) and the separability of A, that $\pi((p^n A)[p]) = B/pB$ for all $n < \omega$: The purity of Y in A together with the fact that $B \subseteq p^\omega D$ (and Lemma 1.1 (b)) implies that we can factor $\pi|_{(p^n A)[p]}$ into two surjective homomorphisms

$$(p^n A)[p] \longrightarrow (p^n(A/Y))[p] \longrightarrow B/pB,$$

so that $B = p^\omega G'$, as desired.

It can be seen that in the construction in the last paragraph, $\pi(T_n) = \pi(E_n[p]) = [B_n + p^{\omega+1}G']/p^{\omega+1}G'$.

Let G be the sum of G' and $\overline{B}$ along B (see Lemma 1.1 (f)). In particular, $p^\omega G = \overline{B}$ and $A \cong G/p^\omega G$. Since $\overline{B}$ is separable, it has a nice basis, and since A has the nice basis extension property, it follows that G has a nice basis, which we denote by N_n, where we will assume $N_0 = \{0\}$. Note next that $M_n = N_n(\omega)$ will be a nice basis for $p^\omega G = \overline{B}$. It follows that M_n is closed in $\overline{B}$, so that it is also torsion-complete (see, for example, Exercise 6 of Section 70 of [**3**]). Since, in addition, it is Σ-cyclic, if follows that M_n must be bounded. If we add terms to the nice basis by simply repeating them if necessary, we may assume that for all $n < \omega$, $p^n M_n = \{0\}$.

For each $n < \omega$, let $S_n = N_n'[p] = ([N_n + p^\omega G]/p^\omega G)[p] \subseteq A[p]$. Note first that since N_n is nice, N_n' is closed in A, so that S_n is closed in $A[p]$. Since $\cup_{n<\omega} N_n' = A$ and $N_n' \subseteq N_{n+1}'$, we have $A[p] = \cup_{n<\omega} S_n$ and $S_n \subseteq S_{n+1}$.

Finally, we need to show that $S_n \subseteq T_n$. Note that

$$\pi : A[p] \longrightarrow p^\omega G/p^{\omega+1}G = [B + p^{\omega+1}G]/p^{\omega+1}G$$

has kernel $T_0 = E_0[p] = Y[p]$ and

$$\pi(T_n) = [B_n + p^{\omega+1}G]/p^{\omega+1}G \cong [B_n + p^{\omega+1}G']/p^{\omega+1}G'.$$

Suppose $x + p^\omega G \in S_n$ where $x \in N_n$. Observe first that $px \in p^\omega G = \overline{B}$, so using the standard representation of torsion complete groups in terms of their basic subgroups, we can express $px = (u_i)_{0<i<\omega} \in \prod_{0<i<\omega} U_i$. Note further that $px \in N_n \cap p^\omega G = M_n$, and $p^n M_n = \{0\}$, so that $p^n u_i = 0$ for all $0 < i < \omega$. Therefore, for all $i > n$, $u_i \in pU_i$. If y is the image of px under the projection $\overline{B} \to B_n$ (i.e., y has coordinates u_i for $i \leq n$ and 0 for $i > n$), then $px - y \in p^{\omega+1}G$. If $z + p^\omega G \in T_n$ is chosen such that $\pi(z + p^\omega G) = y + p^{\omega+1}G$, then $\pi(x + p^\omega G) = px + p^{\omega+1}G = y + p^{\omega+1}G = \pi(z + p^\omega G)$. Consequently, $(x + p^\omega G) - (z + p^\omega G) \in T_0 \subseteq T_n$, so that $x + p^\omega G \in T_n$. This shows that $S_n \subseteq T_n$, and completes the proof. $\square$

Corollary 2.4. *If A is an unbounded separable group with the nice basis extension property, then A is not thick.*

Proof. Let G be some group such that $p^\omega G$ is separable, but not Σ-cyclic and $G/p^\omega G = A$. Then $p^\omega G$ has a nice basis, so by our hypothesis on A, G also has a nice basis. Therefore, by Theorem 1.3, A is not thick. $\square$

Corollary 2.5. *If A is an unbounded separable group which is countably-closed, then A is not thick.*

We now show that the various properties listed in Theorem 2.3 are preserved under arbitrary subgroups.

Proposition 2.6. *If A is a separable countably-closed group and B is an arbitrary subgroup of A, then B is countably-closed.*

Proof. Suppose $B = \cup_{n<\omega} E_n$, where E_n is a pure dense subgroup of B. Let S be a subgroup of $A[p]$ such that there is a vector space decomposition $A[p] = B[p] \oplus S$ (we do not assume this preserves values). Note that for all $n < \omega$, $A[p] = B[p] + S \subseteq \overline{E_n[p] + S}$, so that $E_n[p] + S$ is dense in $A[p]$. We will inductively choose pure and dense subgroups D_n of A such that the following are satisfied:

(a) $D_{n-1} \subseteq D_n$;

(b) $E_n = D_n \cap B$;

(c) $D_n[p] = E_n[p] \oplus S$.

Set $D_{-1} = \{0\}$; assume D_n has been constructed and we want to define D_{n+1}. Our objective is to choose it to be maximal with respect to the following:

(1) $D_n + E_{n+1} \subseteq D_{n+1}$;

(2) $D_{n+1}[p] = E_{n+1}[p] \oplus S$.

Once this has been accomplished, the $n+1$-st case of (a) and (c) follow immediately. In addition, (1) implies $E_{n+1} \subseteq D_{n+1} \cap B$ and (2) implies $E_{n+1}[p] = (D_{n+1} \cap B)[p]$, and since E_{n+1} is a maximal subgroup of B

supported by $E_{n+1}[p]$, we can conclude that condition (b) holds for $n+1$ as well. To make this definition work, it will suffice to show that $(D_n + E_{n+1})[p] = E_{n+1}[p] + S$. Note that the containment $\supseteq$ is straightforward, so we need to show the reverse. Let $x \in D_n$, $y \in E_{n+1}$ and $0 = p(x+y)$. Now, $py = -px \in E_{n+1} \cap D_n \subseteq B \cap D_n = E_n$, which is pure in B. Therefore, there is a $z \in E_n$ such that $py = pz$. Note that $y - z \in E_{n+1}[p]$ and $x + z \in D_n[p]$, which by induction equals $E_n[p] + S$. It follows that

$$x + y = (x+z) + (y-z) \in (E_n[p] + S) + E_{n+1}[p] = E_{n+1}[p] + S,$$

as required.

Since A is countably-closed, it follows that $A = \cup_{n<\omega} C_n$ where C_n is closed in A, $C_n \subseteq C_{n+1}$ and $C_n \subseteq D_n$. If we let $C_n' = B \cap C_n$, then since A/C_n is separable and B/C_n' embeds in A/C_n, C_n' is closed in B. Finally, $B = \cup_{n<\omega} C_n'$, $C_n' \subseteq C_{n+1}'$ and $C_n' \subseteq B \cap D_n = E_n$, so B is countably-closed, as required. $\square$

A nearly identical proof to the above, which will therefore be omitted, can be used to show the following:

Proposition 2.7. *Suppose V is a separable valuated vector space which is countably-closed and $Z \subseteq V$ is a subspace with a valuation v_Z such that for all $z \in Z$, $v_Z(z) \leq v_V(z)$. Then Z is also countably-closed.*

Corollary 2.8. *Suppose A is a separable group with a countably-closed socle and B is an arbitrary subgroup of A. Then B also has a countably-closed socle.*

Proposition 2.9. *If A is a separable group with the (strong) nice basis extension property and B is an arbitrary subgroup of A, then B also has the (strong) nice basis extension property.*

Proof. We prove this for the nice basis extension property, and essentially the same proof will work for the other statement, as well. Suppose G is a group such that $G/p^\omega G = B$, and $p^\omega G$ has a nice basis. By the surjectivity of the homomorphism $\mathrm{Ext}\,(A, p^\omega G) \to \mathrm{Ext}\,(B, p^\omega G)$, there is a group H containing G such that $H/p^\omega G = A$. Now, since A is separable, it follows that $p^\omega H = p^\omega G$. Since A has the nice basis extension property, there is a nice basis N_n of H such that $M_n = p^\omega H \cap N_n$ is a nice basis for $p^\omega H$. We claim that $P_n = N_n \cap G$ forms a nice basis for G: Certainly, since N_n is Σ-cyclic, so is P_n, and since H is the ascending union of the N_n, G is the ascending union of the P_n. Next, $P_n \cap p^\omega G = N_n \cap p^\omega H = M_n$ is nice in $p^\omega H = p^\omega G$. Finally, since $N_n' = [N_n + p^\omega H]/p^\omega H$ is closed in A, A/N_n' is separable. Since $P_n' = [P_n + p^\omega G]/p^\omega G = N_n' \cap B$, B/P_n' embeds in A/N_n', so it is separable, as well. It follows that P_n' is closed in B, so that the P_n's form a nice basis for G, as required. $\square$

Clearly, Theorem 2.3 suggests the following:

Problem 1. *Suppose V is a separable valuated vector space. If V is countably-closed, is V free?*

Since a separable group A is Σ-cyclic if and only if its socle $A[p]$ is free as a valuated vector space, an affirmative answer to this problem would imply the equivalence of all the conditions stated in Theorem 2.3. We turn to a consideration of some partial results in this direction:

Proposition 2.10. *Suppose V is an unbounded separable valuated vector space. If V is countably-closed, then V is not essentially finitely indecomposable* (efi)*, i.e., V has a valuated direct summand W which is free and unbounded.*

Proof. Let $V = \cup_{n<\omega} T_n$, where each T_n is dense in V, $T_n \subseteq T_{n+1}$ and V/T_n is unbounded. By hypothesis, $V = \cup_{n<\omega} S_n$, where each S_n is closed in V and $S_n \subseteq T_n$, so that V/S_n is not bounded. As in the proof of Theorem 1.3, for each $n < \omega$, let $x_n \in V(n) - S_n$. Note that V/S_n is a separable valuated vector space and $x_n + S_n$ is non-zero element of it. It follows that there is a valuated isomorphism $C_n \cong \mathbf{Z}_p$ and a composite homomorphism $\phi_n : V \to V/S_n \to C_n$ such that $\phi_n(x_n) \neq 0$; and we define $\phi : A \to \oplus_{n<\omega} C_n$ as before. It follows from Lemma 1.2 that V is not efi. $\square$

Theorem 2.11. *Suppose V is a separable valuated vector space of final rank κ and the generalized continuum hypothesis* (GCH) *holds for all cardinals $\gamma < \kappa$ (i.e., $2^\gamma = \gamma^+$). If V is countably-closed, then V is a Q-space (i.e., for all infinite subspaces $W \subseteq V$, $|\overline{W}| = |W|$).*

Proof. By Proposition 2.7, any subspace of V (with the induced valuation) is countably-closed. We will be done then if we can show that whenever V is a separable valuated vector space of final rank $\lambda \leq \kappa$ with a basic subspace B of final rank $\gamma < \lambda$, then V cannot be countably-closed. In fact, splitting off summands of bounded value, we may assume ranks and final ranks agree. Since we are assuming GCH works up to $\lambda \leq \kappa$, $\lambda = |V| \leq |B|^{\aleph_0} = \gamma^{\aleph_0}$ implies that γ has countable cofinality and $\lambda = \gamma^{\aleph_0} = \gamma^+ = 2^\gamma$.

Let $\mathcal{C}$ be the collection of all closed subspaces of V of final rank λ. We claim $|\mathcal{C}| \leq \lambda$: Note that if $C \in \mathcal{C}$, and B_C is a basic subspace of C, then $C = \overline{B_C}$ and $|B_C| = \gamma$. It follows that C is determined by B_C, and so:

$$|\mathcal{C}| \leq \lambda^\gamma = (2^\gamma)^\gamma = 2^\gamma = \lambda,$$

proving the claim.

Let $\mathcal{C} = \{C_i : i < \lambda\}$ be an enumeration of $\mathcal{C}$. We will inductively define subspaces W_i, for $i < \lambda$ satisfying the following:

(a) $W_i \subseteq C_i$;

(b) W_i is countably infinite;

(c) $W_i \cap [\oplus_{k<i} W_k] = \{0\}$ (i.e., the W_i's are linearly independent).

To make this work, suppose W_k has been constructed for all $k < i < \lambda$. Note that $\oplus_{k<i} W_k$ has cardinality at most $\gamma + |i| < \lambda$, and C_i has final rank λ, so we can find a countable subspace W_i of C_i such that $W_i \cap [\oplus_{k<i} W_k] = \{0\}$, completing the construction.

Note that (b) implies that $W_i = \oplus_{j<\omega} \langle x_{i,j} \rangle$ for some $x_{i,j} \in W_i$ (here we do not care about valuations). Let U be a subspace of V such that there is a vector space decomposition

$$V = U \oplus (\oplus_{i<\lambda} W_i).$$

For $n < \omega$, let

$$T_n = U \oplus (\oplus_{i<\lambda} \oplus_{j<n} \langle x_{i,j} \rangle).$$

It is easily seen that $T_n \subseteq T_{n+1}$ and $\cup_{n<\omega} T_n = V$. If V were, in fact, countably-closed then it would be the ascending union of a sequence of closed subspaces S_n with $S_n \subseteq T_n$. Since the cofinality of λ is uncountable (in fact, $\lambda = \gamma^+$ is clearly regular), it follows that for some $n < \omega$, S_n must have final rank λ, i.e., $S_n \in \mathcal{C}$. Suppose $i < \lambda$ is the value for which $S_n = C_i$, so that $x_{i,n} \in W_i \subseteq C_i = S_n \subseteq T_n$. Since by construction, $x_{i,n} \notin T_n$, this contradiction implies the result. $\square$

Corollary 2.12. *Suppose V is a separable valuated vector space with a countable basic subspace. Assuming the continuum hypothesis, if V is countably-closed, then it is free.*

Proof. Assuming B is a countable basic subspace of V, then $|V| \leq 2^{\aleph_0} = \aleph_1$. If $|V| = \aleph_1$, then by Theorem 2.11, V is not countably-closed. It follows that V must be countable, and hence free. $\square$

Corollary 2.13. *Assuming the continuum hypothesis, if A is a separable group with a countable basic subgroup, then the following are equivalent*:

(a) *A is Σ-cyclic*;

(b) *A is countably-closed*;

(c) *A has the strong nice basis extension property*;

(d) *A has the nice basis extension property*;

(e) *A has a countably-closed socle.*

3. Summands of groups with nice bases. In [**2**], it was proved that if $G \oplus H$ has a nice basis and H is separable, then G also has a nice basis. The next result demonstrates that the separability of H is necessary. Moreover, in [**2**] it was established that any group of length strictly less than $\omega \cdot 2$ possesses a nice basis. Nevertheless, as was noted in the introduction, the following result, together with Theorem 1.3, shows that there are many groups of length $\omega \cdot 2$ which have nice bases, but which have summands that fail to have that property.

Theorem 3.1. *If G is any group with $p^{\omega \cdot 2}G = \{0\}$, then there is a totally projective group H with $p^{\omega \cdot 2}H = \{0\}$ such that $K = G \oplus H$ has a nice basis.*

Proof. For each n, let U_n be the subgroup of G containing $p^\omega G$ such that $U_n/p^\omega G = (G/p^\omega G)[p^n]$, and let $U_n = \{x_{n,i}\}_{i\in I_n}$ be an enumeration of U_n. Note $G[p^n] \subseteq U_n$. Next, let H be a dsc group such that

$$p^\omega H = \oplus_{n<\omega} \oplus_{i\in I_n} \langle y_{n,i}\rangle$$

where $\langle y_{n,i}\rangle \cong \langle x_{n,i}\rangle$. Let M_n be the subgroup of $K = G \oplus H$ defined by

$$M_n = G[p^n] + \langle(-x_{m,i}, y_{m,i}) : m \le n, i \in I_m\rangle.$$

Claim 1. $M_n \subseteq M_{n+1}$.

Proof of Claim 1. This is fairly obvious.

Claim 2. $\cup_{n<\omega} M_n = G \oplus p^\omega H$.

Proof of Claim 2. The inclusion $\subseteq$ is clear, so consider the reverse inclusion. Note first that $G = \cup_{n<\omega} G[p^n] \subseteq \cup_{n<\omega} M_n$. In addition, if $a \in p^\omega H$, then suppose $a = k_1 y_{m_1,i_1} + \cdots + k_j y_{m_j,i_j}$, where each k and m is an integer and each $i \in I_m$. If $g = k_1 x_{m_1,i_1} + \cdots + k_j x_{m_j,i_j}$ then choose an n such that $n \ge m$ for all these ms and such that $g \in G[p^n]$; it follows that $(0,a) = (g,0) + (-g,a) \in M_n$, as required.

Claim 3. M_n is Σ-cyclic.

Proof of Claim 3. In fact, it can be checked that the terms in the definition of M_n are linearly independent, i.e.,

$$M_n \cong G[p^n] \oplus [\oplus_{m\le n} \oplus_{i\in I_m} \langle(-x_{m,i}, y_{m,i})\rangle].$$

The right terms in the sum are explicitly written as cyclic groups, and $G[p^n]$ is bounded, and hence Σ-cyclic, as well.

Claim 4. $K/[M_n + p^\omega K] \cong p^n(G/p^\omega G) \oplus (H/p^\omega H)$, and in particular, it is separable.

Proof of Claim 4. Note that $M_n + p^\omega K = U_n \oplus p^\omega H$, and it follows that

$$\begin{aligned} K/[M_n + p^\omega K] &\cong (G/U_n) \oplus (H/p^\omega H) \\ &\cong [(G/p^\omega G)/(U_n/p^\omega G)] \oplus (H/p^\omega H) \\ &\cong ((G/p^\omega G)/(G/p^\omega G)[p^n]) \oplus (H/p^\omega H) \\ &\cong p^n(G/p^\omega G) \oplus (H/p^\omega H). \end{aligned}$$

We now let $L = \oplus_{m \le n} \oplus_{i \in I_m} \langle (0, y_{m,i}) \rangle$, $L' = \oplus_{m > n} \oplus_{i \in I_m} \langle (0, y_{m,i}) \rangle$ and $L'' = \oplus_{m \le n} \oplus_{i \in I_m} \langle (-x_{m,i}, y_{m,i}) \rangle$.

Claim 5. $[M_n + p^\omega K]/M_n \cong p^\omega G \oplus L'$, and in particular, it is separable.

Proof of Claim 5. If $\phi : G \oplus p^\omega H = G \oplus L \oplus L' \to G \oplus L'$ is the identity on $G \oplus L'$ and $\phi(y_{m,i}) = x_{m,i}$ for $y_{m,i} \in L$, then it is fairly clear that $L'' \subseteq M_n$ is the kernel of ϕ, and that $\phi(M_n) = G[p^n]$ and $\phi(M_n + p^\omega K) = U_n \oplus L'$. It follows that

$$\begin{aligned} [M_n + p^\omega K]/M_n &\cong ([M_n + p^\omega K]/L'')/(M_n/L'') \\ &\cong (U_n \oplus L')/G[p^n] \cong (U_n/G[p^n]) \oplus L'. \end{aligned}$$

Finally, note that multiplication by p^n gives a surjective homomorphism $U_n \to p^\omega G$ whose kernel is $G[p^n]$, giving the required isomorphism.

Suppose $\{z_j\}_{j \in J}$ is a collection of elements of H such that $H/p^\omega H = \oplus_{j \in J} \langle z_j + p^\omega H \rangle$, and let $z_j + p^\omega H$ have order p^{e_j}. For $n < \omega$, let $J_n = \{j \in J : e_j \le n \text{ and } p^{e_j} z_j \in M_n\}$ and

$$N_n = M_n + \langle z_j : j \in J_n \rangle.$$

We now verify that these subgroups form a nice basis for $K = G \oplus H$.

Claim 1′. $N_n \subseteq N_{n+1}$.

Proof of Claim 1′. Since $J_n \subseteq J_{n+1}$, this easily follows from Claim 1.

Claim 2′. $\cup_{n<\omega} N_n = K$.

Proof of Claim $2'$. First $\cup_{n<\omega} M_n = G \oplus p^\omega H \subseteq \cup_{n<\omega} N_n$. Further, if $j \in J$ and we choose $n < \omega$ such that $e_j \leq n$ and $p^{e_j} z_j \in M_n$, we find that each $z_j \in \cup_{n<\omega} N_n$, so that $K = \cup_{n<\omega} N_n$.

Claim 3$'$. N_n is Σ-cyclic.

Proof of Claim $3'$. Note that for all $j \in J_n$, $e_j \leq n$ and $p^{e_j} z_j \in M_n$ implies $p^n z_j \in M_n$. Therefore, $p^n N_n \subseteq M_n$. Since M_n is Σ-cyclic, so is N_n.

Claim 4$'$. $K/[N_n + (p^\omega K)] \cong p^n(G/p^\omega G) \oplus [\oplus_{j \notin J_n} \langle z_j + p^\omega H\rangle]$, *and in particular, it is separable.*

Proof of Claim $4'$. In the isomorphism in Claim 4, $\langle z_j : j \in J_n\rangle$ maps to $\oplus_{j\in J_n} \langle z_j + p^\omega H\rangle$, and since

$$H/p^\omega H \cong [\oplus_{j\in J_n} \langle z_j + p^\omega H\rangle] \oplus [\oplus_{j\notin J_n} \langle z_j + p^\omega H\rangle],$$

we have the isomorphism.

Claim 5$'$. $p^\omega K/(N_n \cap p^\omega K) \cong p^\omega G \oplus L'$ *is separable, and in particular,* $N_n(\omega) = N_n \cap p^\omega K$ *is nice in* $p^\omega K$.

Proof of Claim $5'$. We claim that $N_n \cap p^\omega K = M_n \cap p^\omega K$: The inclusion $\supseteq$ being obvious, suppose $a \in M_n$, $j_1, \dots , j_k \in J_n$, $\ell_1, \dots , \ell_k$ are integers and $b = a + \ell_1 z_{j_1} + \cdots \ell_k z_{j_k} \in N_n \cap p^\omega K$. It follows that $\ell_1 z_{j_1} + \cdots \ell_k z_{j_k} = b - a \in H \cap ((N_n \cap p^\omega K) + M_n) \subseteq H \cap (p^\omega K + M_n) = H \cap (U_n \oplus p^\omega H) = p^\omega H$. Therefore, $p^{e_{j_1}} | \ell_1, \dots , p^{e_{j_k}} | \ell_k$, and so $\ell_1 z_{j_1}, \dots , p^{\ell_k} z_{j_k} \in M_n$, which implies that $b \in M_n$, i.e., $N_n \cap p^\omega K \subseteq M_n \cap p^\omega K$.

By Claim 5 and the last paragraph, $p^\omega K/(N_n \cap p^\omega K) = p^\omega K/(M_n \cap p^\omega K) \cong [M_n + p^\omega K]/M_n \cong p^\omega G \oplus L'$, as required.

It follows from Claims $1'$–$5'$ and Lemma 1.1 (g) that the N_n's form a nice basis for K, as required. $\square$

Certainly, there are many questions left unanswered by the above discussion. For example:

Problem 2. *If G is an arbitrary reduced group, does there exist a totally projective group H such that $G \oplus H$ has a nice basis?*

Problem 3. *If $G \oplus H$ has a nice basis, does it follow that either G or H has a nice basis?*

Problem 4. *Suppose G_1 and G_2 are groups such that $p^\omega G_1 \cong p^\omega G_2$ and $G_1/p^\omega G_1 \cong G_2/p^\omega G_2$. If G_1 has a nice basis, does it follow that G_2 has a nice basis?*

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Department of Mathematics, Plovdiv University, "P. Hilendarski", Plovdiv 4000, Bulgaria
Email address: pvdanchev@yahoo.com

Department of Mathematics, Whitman College, Walla Walla, WA 99362
Email address: keef@whitman.edu