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SOME OPTIMAL ELEMENTARY INEQUALITIES ON THE UNIT DISC

RENJIANG ZHANG

ABSTRACT. Let z be a complex number satisfying $|z| \le 1$. Then the following inequalities

$$\sin 1 \cdot |z| \le |\sin z| \le \frac{1}{2}\left(e - \frac{1}{e}\right)|z|,$$

$$\cos 1 \le |\cos z| \le \frac{1}{2}\left(e + \frac{1}{e}\right),$$

$$\frac{e^2 - 1}{e^2 + 1}|z| \le |\tan z| \le \tan 1 \cdot |z|,$$

and

$$\left(1 - \frac{1}{e}\right)|z| \le |e^z - 1| \le (e-1)|z|$$

hold; moreover, the constants in the above inequalities are optimal.

1. Introduction. References [**3, 4**] cited the following inequalities which can be found in the literature [**2**]:

Let z be a complex number satisfying $|z| \le 1$. Then inequalities

$$|\cos z| < 2, \qquad |\sin z| \le \frac{6}{5}|z|$$

and

$$\frac{1}{4}|z| < |e^z - 1| < \frac{7}{4}|z|$$

hold.

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We find that the constants in the above inequalities are not optimal; they can be improved. We have obtained some new optimal inequalities. The main results are as follows:

Theorem 1. *Let z be a complex number satisfying $|z| \leq 1$. Then inequalities*

$$\sin 1 \cdot |z| \leq |\sin z| \leq \frac{1}{2}\left(e - \frac{1}{e}\right)|z|, \tag{1}$$

$$\cos 1 \leq |\cos z| \leq \frac{1}{2}\left(e + \frac{1}{e}\right), \tag{2}$$

$$\frac{e^2-1}{e^2+1}|z| \leq |\tan z| \leq \tan 1 \cdot |z| \tag{3}$$

hold; *moreover, the constants in the above inequalities* (1)–(3) *are optimal.*

Theorem 2. *Let z be a complex number satisfying $0 < |z| < 1$. Then inequalities*

$$\left(1 - \frac{1}{e}\right)|z| < |e^z - 1| < (e-1)\,|z| \tag{4}$$

hold; *moreover, both constants $1-(1/e)$ and $e-1$ in* (4) *are optimal.*

The above results are a bundle of new elegant inequalities.

2. The proof of Theorem 1. To prove Theorem 1, we need a preliminary lemma.

Lemma 1. *Suppose a positive sequence $\{ka_k\}$ $(k = 1, 2, \dots)$ is monotonically decreasing and tends to zero as k tends to ∞. Then the following function series*

$$f(x) = \sum_{k=1}^{\infty}(-1)^k a_k x^k$$

is monotonically decreasing on $[0,1]$; *thus*

$$\sum_{k=1}^{\infty}(-1)^k a_k \le \sum_{k=1}^{\infty}(-1)^k a_k x^k \le 0.$$

Proof. For a fixed x, $x \in (0,1]$, the function series

$$\sum_{k=1}^{\infty}(-1)^k a_k x^k$$

is the alternating series with the negative leading term. So it converges to a non-positive number. Since the derivative of the series $f(x)$ exists according to the condition of Lemma 1, so

$$f'(x) = \sum_{k=1}^{\infty}(-1)^k k a_k x^{k-1} \le 0.$$

This completes the proof of the Lemma. $\square$

Now we turn to prove the Theorem.

Proof of Theorem 1. We first prove the left-hand inequality of (1). Since the inequalities (1) hold obviously for $z=0$, we assume that $z \neq 0$ in the following. Let $\overline{z}$ denote the complex conjugate of z. Using basic properties of $\mathbf{C}$ [**1**], we have

$$(5) \qquad |\sin z| = |z|\left|\frac{\sin z}{z}\right| = |z|\sqrt{\frac{\sin z}{z}\cdot\overline{\left(\frac{\sin z}{z}\right)}} = |z|\sqrt{\frac{\sin z\cdot\sin\overline{z}}{z\overline{z}}}.$$

Writing $z = r(\cos\theta + i\sin\theta)$ $(0 < r \le 1)$ and applying the trigonometric identity

$$\sin A\cdot\sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)],$$

by Euler's formula we have

$$(6) \qquad \sin z\cdot\sin\overline{z} = -\frac{1}{2}[\cos(2r\cos\theta) - \cos(2ir\sin\theta)].$$

Using the Taylor expansion of $\cos x$, we have

$$\cos(2r\cos\theta) = \sum_{k=0}^{\infty}(-1)^k\frac{(2r\cos\theta)^{2k}}{(2k)!} \tag{7}$$

and

$$\cos(2ir\sin\theta) = \sum_{k=0}^{\infty}\frac{(2r\sin\theta)^{2k}}{(2k)!}. \tag{8}$$

The expressions (6), (7) and (8) yield

$$\sin z\cdot\sin\overline{z} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(2r)^{2k}}{(2k)!}[\sin^{2k}\theta+(-1)^{k+1}\cos^{2k}\theta].$$

From Lemma 1, we obtained the following inequalities

$$\begin{aligned}\frac{\sin z\cdot\sin\overline{z}}{z\overline{z}} &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{2^{2k}r^{2(k-1)}}{(2k)!}[\sin^{2k}\theta+(-1)^{k+1}\cos^{2k}\theta]\\ &\geq 1+2\cos^2\theta\sum_{k=2}^{\infty}(-1)^{k+1}\frac{2^{2(k-1)}}{(2k)!}(r\cos\theta)^{2(k-1)}\\ &\geq 1+2\sum_{k=2}^{\infty}(-1)^{k+1}\frac{2^{2(k-1)}}{(2k)!}(r\cos\theta)^{2(k-1)}\\ &\geq 1+2\sum_{k=2}^{\infty}(-1)^{k+1}\frac{2^{2(k-1)}}{(2k)!}\\ &= \frac{1-\cos 2}{2} = \sin^2 1\end{aligned} \tag{9}$$

where we have used the inequality $[\sin^{2k}\theta+(-1)^{k+1}\cos^{2k}\theta]\leq\sin^2\theta+\cos^2\theta=1$. Combining (5) and (9), we obtain the desired inequality. Now we turn to the right-hand inequality of (1).

Using the Taylor expansion of e^x, we have

$$e=\sum_{k=0}^{\infty}\frac{1}{k!}\quad\text{and}\quad e^{-1}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!};$$

then

$$\begin{aligned}\left|\frac{\sin z}{z}\right| &= \left|\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k}}{(2k+1)!}\right| \le \sum_{k=0}^{\infty}\left|\frac{z^{2k}}{(2k+1)!}\right| \\ &\le \sum_{k=0}^{\infty}\frac{1}{(2k+1)!} = \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{1}{k!} - \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\right) \\ &= \frac{1}{2}\left(e - \frac{1}{e}\right).\end{aligned}$$

So the right-hand inequality of (1) holds.

Both equal signs of inequalities (1) hold for $z = 1$ and $z = i$ respectively, so inequalities (1) are optimal.

We now move to the proofs of the inequalities (2). Using the basic properties of $\mathbf{C}$, we have

(10)

$$|\cos z| = \sqrt{\cos z \cdot \overline{\cos z}} = \sqrt{\cos z \cdot \cos \overline{z}} = \sqrt{\frac{1}{2}[\cos(z+\overline{z}) + \cos(z-\overline{z})]}.$$

Using a method similar to that used in the proof of (1), we can easily get

$$\text{(11)}\quad \frac{1}{2}[\cos(z+\overline{z})+\cos(z-\overline{z})] \ge 1+\frac{1}{2}\sum_{k=1}^{\infty}\frac{2^{2k}}{(2k)!} = \frac{1+\cos 2}{2} = \cos^2 1.$$

Thus the left-hand inequality of (2) holds by combining (10) and (11). The right-hand inequality of (2) can be derived directly as follows:

$$|\cos z| = \left|\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k}}{(2k)!}\right| \le \sum_{k=0}^{\infty}\left|\frac{z^{2k}}{(2k)!}\right| \le \sum_{k=0}^{\infty}\frac{1}{(2k)!} = \frac{1}{2}\left(e+\frac{1}{e}\right).$$

Both equal signs of inequalities (1) hold for $z = 1$ and $z = i$ respectively, so the inequalities are optimal.

Finally, we need to prove inequalities (3). Since inequalities (3) hold obviously for $z = 0$, we suppose $z \neq 0$ in the following. From

(12)

$$\left|\frac{\tan z}{z}\right| = \sqrt{\frac{\tan z}{z}\cdot\overline{\left(\frac{\tan z}{z}\right)}} = \sqrt{\frac{\tan z}{z}\cdot\frac{\tan \overline{z}}{\overline{z}}} = \sqrt{\frac{\sin z \cdot \sin \overline{z}}{z\overline{z}\cos z \cdot \cos \overline{z}}}$$

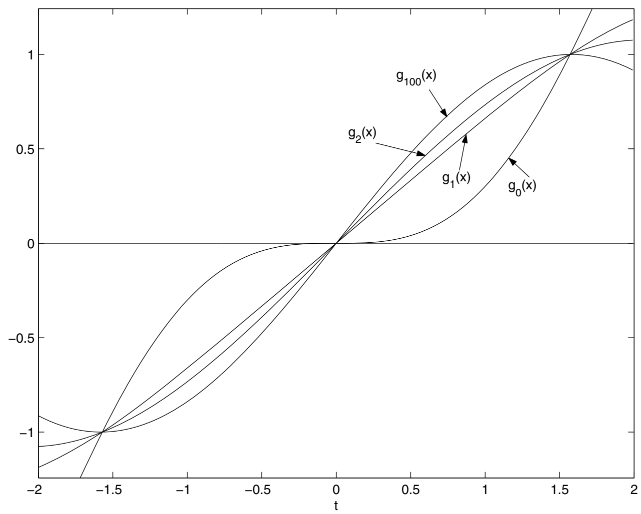

FIGURE 1. Graphs of $g_0(x)$, $g_1(x)$, $g_2(x)$ and $g_{100}(x)$.

we can see that it is sufficient to show inequalities (3) hold for a complex number z lying in the half closed unit disc above the real axis except the origin. That is, we need only to consider the case in which a complex number z lies on the region $\{0 < r \leq 1, 0 \leq \theta \leq \pi\}$. By trigonometric identities and Euler's formula, we have

$$\begin{aligned} \frac{\sin z \cdot \sin \overline{z}}{\cos z \cdot \cos \overline{z}} &= \frac{\cos(z-\overline{z}) - \cos(z+\overline{z})}{\cos(z-\overline{z}) + \cos(z+\overline{z})} \\ &= \frac{\cos(2ir\sin\theta) - \cos(2r\cos\theta)}{\cos(2ir\sin\theta) + \cos(2r\cos\theta)}. \end{aligned} \tag{13}$$

Write

$$f(r,\theta) := \frac{\cos(2ir\sin\theta) - \cos(2r\cos\theta)}{\cos(2ir\sin\theta) + \cos(2r\cos\theta)}. \tag{14}$$

First, for a fixed r, $f(r,\theta)$ can be regarded as a scalar function with regard to θ. The partial differentiation of $f(r,\theta)$ is

$$\frac{\partial f(r,\theta)}{\partial \theta}$$
$$= \frac{-4r[\sin(2ir\sin\theta)\cdot i\cos\theta\cdot\cos(2r\cos\theta)+\cos(2ir\sin\theta)\cdot\sin\theta\cdot\sin(2r\cos\theta)]}{(\cos(2ir\sin\theta)+\cos(2r\cos\theta))^2}$$
$$= \frac{-4r\left[\sum_{k=0}^{\infty}(-1)^k\frac{(2ir\sin\theta)^{2k+1}}{(2k+1)!}\cdot i\cos\theta\cdot\cos(2r\cos\theta)+\sum_{k=0}^{\infty}\frac{(2r\sin\theta)^{2k}}{(2k)!}\sin\theta\cdot\sin(2r\cos\theta)\right]}{(\cos(2ir\sin\theta)+\cos(2r\cos\theta))^2}$$
$$= \frac{-4r\sum_{k=0}^{\infty}\left[\left(-\frac{1}{2k+1}2r\cos\theta\cos(2r\cos\theta)+\sin(2r\cos\theta)\right)\frac{1}{(2k)!}(2r)^{2k}\sin^{2k+1}\theta\right]}{(\cos(2ir\sin\theta)+\cos(2r\cos\theta))^2}.$$

We now consider the function

$$g_k(x) := -\frac{1}{2k+1}x\cos x+\sin x,\ x = 2r\cos\theta,\ x\in[-2,2],\ k = 0,1,2,\ldots.$$

It is not difficult to prove that (see Figure 1)

$$g_k(x) > 0, x \in (0,2]; \quad g_k(x) = 0, x = 0; \quad g_k(x) < 0, \quad x \in [-2,0).$$

Thus we can easily get

$$\frac{\partial f(r,\theta)}{\partial \theta} < 0, \quad \theta \in \left[0, \frac{\pi}{2}\right); \quad \frac{\partial f(r,\theta)}{\partial \theta}\Bigg) = 0,$$
$$\theta = \frac{\pi}{2}; \quad \frac{\partial f(r,\theta)}{\partial \theta} > 0, \quad \theta \in \left(\frac{\pi}{2}, \pi\right].$$

We derive easily that $\partial f(r,\theta)/\partial\theta$ has three zeros $\theta = 0$, $(\pi/2)$, π and $f(r,\theta)$ attains its minimum value at point $\theta = \pi/2$ and attains its maximum value at points $\theta = 0$ or $\theta = \pi$ on the interval $[0,\pi]$. Thus we obtain for a fixed r that

$$\frac{\cos(2ir)-1}{\cos(2ir)+1} = f\left(r, \frac{\pi}{2}\right) \le f(r,\theta) \le f(r,0) = \frac{1-\cos(2r)}{1+\cos(2r)}. \tag{15}$$

Expression (15) gives

$$\frac{f(r,\theta)}{r^2} \ge \frac{f\left(r,\frac{\pi}{2}\right)}{r^2} = \frac{\cos(2ir)-1}{r^2(\cos(2ir)+1)} := g(r).$$

The derivative of $g(r)$ is

$$
\begin{aligned}
g'(r) &= \frac{-4ir^2 \sin(2ir) - 2r\cos^2(2ir) + 2r}{(r^2(\cos(2ir)+1))^2} \\
&= \frac{-4ir^2 \sin(2ir) - r\cos(4ir) + r}{(r^2(\cos(2ir)+1))^2} \\
&= \frac{-4ir^2 \sum\limits_{k=1}^{\infty} (-1)^{k-1} \frac{(2ir)^{2k-1}}{(2k-1)!} - r \sum\limits_{k=0}^{\infty} (-1)^k \frac{(4ir)^{2k}}{(2k)!} + r}{(r^2(\cos(2ir)+1))^2} \\
&= \frac{4r^2 \sum\limits_{k=1}^{\infty} \frac{(2r)^{2k-1}}{(2k-1)!} - r \sum\limits_{k=1}^{\infty} \frac{(4r)^{2k}}{(2k)!}}{(r^2(\cos(2ir)+1))^2} \\
&= \frac{\sum\limits_{k=1}^{\infty} \left(\frac{2^{2k+1}}{(2k-1)!} - \frac{4^{2k}}{(2k)!} \right) r^{2k+1}}{(r^2(\cos(2ir)+1))^2}.
\end{aligned}
$$

It is easy to check that

$$
\frac{2^{2k+1}}{(2k-1)!} - \frac{4^{2k}}{(2k)!} \le 0, \quad (k \ge 1 \text{ is a integer}).
$$

Thus we have that $g'(r) < 0$ and $g(r)$ is a monotonically decreasing function on $[0, 1]$. We conclude that

$$
\begin{aligned}
\frac{f(r,\theta)}{r^2} \ge g(r) \ge g(1) = \frac{\cos(2i) - 1}{\cos(2i) + 1} &= \frac{\sum\limits_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} - 1}{1 + \sum\limits_{k=0}^{\infty} \frac{2^{2k}}{(2k)!}} \\
= \frac{\frac{e^2+e^{-2}}{2} - 1}{\frac{e^2+e^{-2}}{2} + 1} &= \left(\frac{e^2 - 1}{e^2 + 1} \right)^2.
\end{aligned} \tag{16}
$$

Combining (12), (13), (14), (16) and the above expression, we have

$$
\left| \frac{\tan z}{z} \right| = \sqrt{\frac{\sin z \cdot \sin \overline{z}}{z \cos z \cdot \cos \overline{z}}} = \sqrt{\frac{f(r,\theta)}{r^2}} \ge \frac{e^2 - 1}{e^2 + 1}. \tag{17}
$$

This completes the proof of the left-hand inequality of (3).

Next we prove the right-hand inequality of (3).

Expression (15) gives

$$\frac{f(r,\theta)}{r^2} \le \frac{f(r,0)}{r^2} = \frac{1}{r^2}\frac{1-\cos(2r)}{1+\cos(2r)}.$$

We denote

$$h(r) := \frac{1}{r^2}\frac{1-\cos(2r)}{1+\cos(2r)};$$

then

$$h'(r) = \frac{2r\sin(2r)(2r-\sin(2r))}{\left(r^2+r^2\cos(2r)\right)^2} > 0, \quad 0 < r \le 1.$$

We know that $h(t)$ is monotonically increasing function on $(0,1]$. This yields

$$\frac{f(r,\theta)}{r^2} \le h(r) \le h(1) = \frac{1-\cos 2}{1+\cos 2} = \tan^2 1.$$

Recalling (17) we have immediately

$$\left|\frac{\tan z}{z}\right| = \sqrt{\frac{f(r,\theta)}{r^2}} \le \tan 1.$$

This completes the proof of inequalities (3) and thus completes the proof of Theorem 1.

3. The proof of Theorem 2. To prove Theorem 2, we need some preliminary inequalities.

Lemma 2. *If $0 < x < 1$, then $e^x - 1 > x$. Furthermore, if $-1 < x < 0$, then*

$$e^x - 1 < \left(1-\frac{1}{e}\right)x + \frac{1}{2e}x(1+x).$$

Proof. The first assertion is well known, so we just prove the second one. Let the real function $f(x)$ be given by

$$f(x) := e^x - 1 - \left(1-\frac{1}{e}\right)x - \frac{1}{2e}x(1+x);$$

then

$$f''(x) = e^x - \frac{1}{e} > 0, \quad \text{for all } -1 < x < 0.$$

So $f(x)$ cannot attain its maximal value in $-1 < x < 0$. Then

$$f(x) < \max\left\{\lim_{x\to 0} f(x), \lim_{x\to -1} f(x)\right\} = 0.$$

The above expression gives

$$e^x - 1 < \left(1 - \frac{1}{e}\right)x + \frac{1}{2e}x(1+x), \quad \text{for all } -1 < x < 0. \qquad \square$$

Lemma 3. *If $-1 < x < 0$, then*

$$(e^x - 1)^2 > \left(1 - \frac{1}{e}\right)^2 x^2 + \frac{1}{e}\left(1 - \frac{1}{e}\right)x^2(1+x).$$

Lemma 4. *The real function $f(x) = (\sin x)/x$ is monotonically decreasing on the interval $(0, (\pi/2))$.*

Lemma 3 follows directly from Lemma 2, while Lemma 4 can be obtained by direct calculation.

Proof of Theorem 2. We first prove the left-hand inequality of (4). Using basic properties of $\mathbf{C}$

$$|z|^2 = z\overline{z} \quad \text{and} \quad \overline{e^z} = e^{\overline{z}};$$

we obtain

$$|e^z - 1| = |z|\left|\frac{e^z - 1}{z}\right| = |z|\sqrt{\frac{(e^z - 1)(e^{\overline{z}} - 1)}{z\overline{z}}}.$$

Writing $z = r(\cos\theta + i\sin\theta)$ $(0 < r < 1)$, we have by Euler's formula

$$\begin{aligned} f(r,\theta) := \frac{(e^z - 1)(e^{\overline{z}} - 1)}{z\overline{z}} &= \frac{e^{2r\cos\theta} - 2e^{r\cos\theta}\cos(r\sin\theta) + 1}{r^2} \\ &= \frac{(e^{r\cos\theta} - 1)^2 + 2e^{r\cos\theta}(1 - \cos(r\sin\theta))}{r^2} \\ &= \frac{(e^{r\cos\theta} - 1)^2 + 4e^{r\cos\theta}\sin^2\left(\frac{r\sin\theta}{2}\right)}{r^2}. \end{aligned}$$

To prove the left-hand inequality of (4), it is sufficient to show that

$$f(r,\theta) > \left(1-\frac{1}{e}\right)^2. \tag{18}$$

In the following we derive the proof of (18) by distinguishing three cases:

Case 1. $\cos\theta > 0$. Applying Lemma 1, noting that the inequality $\sin x/x > (2/\pi)$ $(0 < x < (\pi/2))$ and $4/\pi^2 > (1-(1/e))^2$, we can deduce that

$$\begin{aligned} f(r,\theta) &> \frac{r^2\cos^2\theta + 4\sin^2\left(\frac{r\sin\theta}{2}\right)}{r^2} > \frac{r^2\cos^2\theta + 4\left(\frac{2}{\pi}\frac{r\sin\theta}{2}\right)^2}{r^2} \\ &> \frac{r^2\cos^2\theta + \frac{4}{\pi^2}r^2\sin^2\theta}{r^2} > \left(1-\frac{1}{e}\right)^2. \end{aligned}$$

We see that (18) holds in this case.

Case 2. $\cos\theta < 0$. Applying Lemma 3, we have

$$f(r,\theta) > \frac{\left(1-\frac{1}{e}\right)^2 r^2\cos^2\theta + \frac{1}{e}\left(1-\frac{1}{e}\right) r^2(1+r\cos\theta)\cos^2\theta + 4e^{r\cos\theta}\sin^2\left(\frac{r\sin\theta}{2}\right)}{r^2}.$$

To prove the desired inequality, it is sufficient to show that

$$\frac{1}{e}\left(1-\frac{1}{e}\right)r^2(1+r\cos\theta)\cos^2\theta + 4e^{r\cos\theta}\sin^2\left(\frac{r\sin\theta}{2}\right) > \left(1-\frac{1}{e}\right)^2 r^2\sin^2\theta,$$

which is equivalent to

$$\frac{1}{e}\left(1-\frac{1}{e}\right)(1+r\cos\theta)\cot^2\theta + e^{r\cos\theta}\left(\sin\frac{r\sin\theta}{2}\Big/\frac{r\sin\theta}{2}\right)^2 > \left(1-\frac{1}{e}\right)^2.$$

Noting that $\cos\theta < 0$ and $0 < r < 1$, we need only to show that

$$I := \frac{1}{e}\left(1-\frac{1}{e}\right)(1+\cos\theta)\cot^2\theta + e^{\cos\theta}\left(\sin\frac{\sin\theta}{2}\Big/\frac{\sin\theta}{2}\right)^2 > \left(1-\frac{1}{e}\right)^2. \tag{19}$$

To prove (19), we need to distinguish two cases again.

a. $\cos\theta \geq -0.75$. In this case, it is easy to show $|\sin\theta| < 0.67$, $e^{\cos\theta} > 0.47$. Applying Lemma 4, we have

$$\left(\sin\frac{\sin\theta}{2}\Big/\frac{\sin\theta}{2}\right)^2 = \left|\sin\left|\frac{\sin\theta}{2}\right|\Big/\left|\frac{\sin\theta}{2}\right|\right|^2 > \left(\frac{\sin 0.335}{0.335}\right)^2 > 0.96.$$

Thus

$$I > e^{\cos\theta}\left(\sin\frac{\sin\theta}{2}\Big/\frac{\sin\theta}{2}\right)^2 > 0.47\times 0.96 = 0.4512 > \left(1-\frac{1}{e}\right)^2 \approx 0.39958.$$

We see that (19) holds in this case.

b. $\cos\theta < -0.75$. Again applying Lemma 4, we have

$$\begin{aligned} I &> \frac{1}{e}\left(1-\frac{1}{e}\right)\frac{\cos^2\theta}{1-\cos\theta} + \frac{1}{e}\left(\frac{\sin\frac{1}{2}}{\frac{1}{2}}\right)^2 > \frac{1}{e}\left(1-\frac{1}{e}\right)\frac{0.75^2}{2} + \frac{1}{e}\left(2\sin\frac{1}{2}\right)^2 \\ &> 0.36788 \times (0.63212 \times 0.28125 + 0.91940) > 0.40 > \left(1-\frac{1}{e}\right)^2. \end{aligned}$$

Combining **a** with **b**, we obtain (19) for the case $\cos\theta < 0$.

It remains to show

Case 3. $\cos\theta = 0$. In this case $\sin\theta = \pm 1$, we see that

$$f(r,\theta) := g(r) = \frac{4\sin^2\left(\frac{r\sin\theta}{2}\right)}{r^2} = \left(\sin\frac{r}{2}\Big/\frac{r}{2}\right)^2 > \left(\sin\frac{1}{2}\Big/\frac{1}{2}\right)^2 > \left(1-\frac{1}{e}\right)^2.$$

Together with cases **1, 2** and **3**, we obtain the left-hand inequality of (4).

Second, we prove the right-hand inequality of (4). We have

$$\begin{aligned} |e^z - 1| &= \left|\sum_{k=1}^{\infty}\frac{z^k}{k!}\right| = |z|\left|\sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}\right| \leq |z|\sum_{k=0}^{\infty}\frac{|z|^k}{(k+1)!} \\ &< |z|\sum_{k=0}^{\infty}\frac{1}{(k+1)!} = (e-1)|z|. \end{aligned}$$

Finally, we show the optimality of (4). Considering $z = x \to -1$ (x is a real number), we see that the constant $1 - (1/e)$ in inequality (4) is optimal. Considering $z = x \to 1$ (x is a real number), we see that the constant $e - 1$ in the inequalities (4) is also optimal. This completes the proof.

From the procedure of the proof of Theorem 2, we immediately get

Corollary. *Let z be a complex number satisfying $|z| \le 1$. Then the inequalities*

$$\left(1 - \frac{1}{e}\right)|z| \le |e^z - 1| \le (e-1)\,|z|$$

hold. Moreover, both constants $1 - (1/e)$ and $e - 1$ in the above inequalities are optimal.

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Department of Mathematics, Zhejiang Gongshang University, Hangzhou, Zhejiang, 310018 P.R. China
Email address: renjiang@mail.hz.zj.cn; zrj@cjlu.edu.cn