ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 6, 2010

ALTERNATING SUBSETS AND PERMUTATIONS

AUGUSTINE O. MUNAGI

ABSTRACT. We give new proofs of theorems on alternating subsets of integers by means of bijective transformations. It is shown that all known results are consequences of a simple result on the residue class of an integer. The notion of alternating subset is extended to permutations of $\{1, 2, \dots, n\}$. In particular, we obtain solutions to the problems of Terquem and Skolem's generalization for permutations.

1. Introduction. A finite, increasing, sequence of natural numbers $(x_1, x_2, \dots)$ is called *alternating* [**5**] if it fulfills the condition

$$x_i \not\equiv x_{i-1} \pmod 2, \quad i > 1. \tag{1}$$

The empty sequence and the 1-term sequence are also alternating sequences by convention.

Such sequences are known as *alternating subsets of integers* (see for example [**1, 4, 10**]). In particular, we recall the fundamental result [**1, 2**]:

The number $h(n,k)$ of alternating k-subsets of $\{1, 2, \dots, n\}$ is given by

$$h(n,k) = \binom{\lfloor \frac{n+k}{2} \rfloor}{k} + \binom{\lfloor \frac{n+k-1}{2} \rfloor}{k}, \tag{2}$$

where $\lfloor N \rfloor$ denotes the greatest integer $\leq N$. It is known that $\sum_{k>0} h(n,k) = F_{n+3} - 2$, where F_N is the Nth Fibonacci number. We will adopt the notation $[n] = \{1, 2, \dots, n\}$.

We consider generalizations of (2) and show that practically all known results are consequences of the following simple lemma on the residue

2010 AMS *Mathematics subject classification.* Primary 05A05, 11B50, 05A15.
Keywords and phrases. Alternating subset, Terquem problem, permutation.
Received by the editors on February 25, 2008, and in revised form on June 16, 2008.

DOI:10.1216/RMJ-2010-40-6-1965

class of an integer. Apart from simpler proofs of alternating subset theorems, a novel contribution of this paper is the extension of the alternating sequence concept to permutations of $[n]$. It turns out that this lemma also plays a critical role in obtaining generalizations of the permutations.

Lemma 1.1. *Given positive integers n, m, $m > 1$, let c be a fixed element of $[m]$. The cardinality of the set $\{c,\, m+c,\, 2m+c, \dots\} \subset [n]$ is given by $\lfloor (n+m-c)/m \rfloor$.*

Proof. By Euclid's algorithm $n = mq + r$, where q, r are integers, $q \geq 0$, $0 \leq r \leq m-1$. Let the residue classes of m be $V(1), V(2), \dots, V(m)$, where $V(c) = \{c,\, m+c,\, 2m+c, \dots\} \subset [n]$, $1 \leq c \leq m$. Then the cardinalities give a unique partition of n, $n = |V(1)| + |V(2)| + \cdots + |V(m)|$, such that $|V(c)| = q+1$ if $r \geq c$ and $|V(c)| = q$ if $r < c$. But $|V(c)| = \lfloor (n+m-c)/m \rfloor$ since

$$\left\lfloor \frac{n+m-c}{m} \right\rfloor = \left\lfloor \frac{n-r+m+r-c}{m} \right\rfloor = q + \left\lfloor \frac{m+r-c}{m} \right\rfloor$$
$$= \begin{cases} q+1 & r \geq c, \\ q & r < c. \end{cases} \qquad \square$$

The classical problem of Terquem, according to Riordan [**7**, page 17, Example 15], is to enumerate increasing k-combinations of $[n]$ with odd elements in odd positions and even elements in even positions. This was generalized by Skolem (see [**6**, pages 313–314]) as follows: find the number of increasing k-combinations in which the jth element is congruent to j modulo m, where m is a fixed modulus ≥ 2.

Skolem's generalization of the Terquem problem has attracted the attention of several authors. Church and Gold [**3**] obtained a proof by counting lattice paths in a rectangular array, while Abramson and Moser [**1**] employed binary digits to prove a more general result. Goulden and Jackson [**4**] rediscovered the result of Abramson and Moser by means of generating functions.

In Section 2 we give a new proof of Skolem's theorem which is then used to obtain its own best-known generalization (Theorem 2.4).

In Section 3 we consider k-permutations of $[n]$ in linear order satisfying the parity condition (1), to be called *permutations with parity-*

alternating entries. We obtain the permutation analogue of (2) and deduce an important specialization with a combinatorial application. Section 4 deals with extensions of permutations with parity-alternating entries, leading to a Skolem-type theorem for permutations (Theorem 4.1). Lastly, in Section 5 we obtain a generalization of the Skolem-type theorem for permutations.

2. Alternating subsets.

All results in this section rely on the following proposition which is derived from Lemma 1.1. Note that m represents a fixed integer > 1.

Proposition 2.1. *Given an integer $c \in [m]$, the number $b(c,n,k,m)$ of k-combinations, with repetitions allowed, of $\{c,\ m+c,\ 2m+c, \dots\} \subset [n]$, is given by*

$$b(c,n,k,m) = \binom{\lfloor \frac{n+mk-c}{m} \rfloor}{k}. \tag{3}$$

The recurrence relation is

$$\begin{gathered} b(c,n,k,m) = b(c,n,k-1,m) + b(c,n-m,k,m), \\ k > 0,\ 2 < m < n, \\ b(c,n,0,m) = 1, \qquad b(c,n,k,n) = 1. \end{gathered} \tag{4}$$

Proof. Equation (3) follows at once from Lemma 1.1 and the formula for the number of k-combinations of $[n]$ with repetitions, namely $\binom{n+k-1}{k}$.

Let $n = mq + r$, $q \geq 0$, $0 \leq r \leq m-1$. If $r \geq c$, then $0 \leq r - c < m$ and $n = mq + c + (r-c)$ or $n - (r-c) = mq + c$. Similarly, if $r < c$, then $0 \leq m + r - c < m$ and $n - (m + r - c) = m(q-1) + c$. Thus, the greatest element of $\{c,\ m+c,\ 2m+c, \dots\}$ is $n - s$, where $s = r - c$ or $s = m + r - c$ depending upon whether $r \geq c$ or not. This gives a recurrence relation for $b(c,n,k,m)$ according to whether $n - s$ is or is not selected:

$$b(c,n,k,m) = b(c,n-s,k-1,m) + b(c,n-s-m,k,m).$$

Since $0 \leq s < m$, this relation agrees with (4). $\square$

An immediate consequence of Proposition 2.1 is the following general theorem by Skolem. Note that Terquem's problem is the case $m = 2$.

Theorem 2.2. *The number $f(n,k,m)$ of combinations $(x_1, \dots, x_k)$ of $[n]$ which satisfy*

$$x_j \equiv j \pmod{m}, \quad 1 \le j \le k, \tag{5}$$

is given by

$$f(n,k,m) = \binom{\lfloor \frac{n+(m-1)k}{m} \rfloor}{k}. \tag{6}$$

The following relation holds:

$$\begin{gathered} f(n,k,m) = f(n-1,k-1,m) + f(n-m,k,m) \\ f(n,0,m) = 1, \qquad f(n,k,n) = 1. \end{gathered} \tag{7}$$

Proof. Let $F(n,k,m)$ and $B(c,N,k,m)$ denote the sets of objects enumerated by $f(n,k,m)$ and $b(c,N,k,m)$, respectively, for a fixed $c \in [N]$. We give a bijection ϕ between $F(n,k,m)$ and $B(1,n-k+1,k,m)$. Let $(x_1, x_2, \dots, x_k) \in F(n,k,m)$, then

$$\phi : (x_1, x_2, \dots, x_k) \longmapsto (x_1, x_2 - 1, \dots, x_k - k + 1). \tag{8}$$

Clearly $\phi((x_1, x_2, \dots, x_k)) \in B(1, n-k+1, k, m)$ and ϕ^{-1} is readily obtained.

For example, $(1,5,6,13,17,18,22,29) \in F(30,8,3)$ maps to $(1,4,4,10, 13,13,16,22) \subset B(1,23,8,3)$. It follows from Proposition 2.1, with $c = 1$ and replacing n by $n-k+1 \equiv r \pmod{m}$, that $f(n,k,m)$ is given by

$$\begin{aligned} f(n,k,m) &= b(1, n-k+1, k, m) \\ &= \binom{\lfloor \frac{n-k+1+mk-1}{m} \rfloor}{k} = \binom{\lfloor \frac{n+(m-1)k}{m} \rfloor}{k}. \end{aligned}$$

The recurrence relation (7) is correspondingly deduced from (4). $\square$

A natural extension of Skolem's theorem is the following corollary which is a special case of the next, general theorem (Theorem 2.4), which in turn happens to be a consequence of Theorem 2.2.

Corollary 2.3. *Let c be a fixed element of $[m]$. The number $f_c(n,k,m)$ of combinations $(x_1,\dots,x_k)$ of $[n]$ which satisfy*

$$x_j \equiv j + c - 1 \pmod{m}, \quad 1 \le j \le k, \ 2 \le m \le n, \tag{9}$$

is given by

$$f_c(n,k,m) = \binom{\lfloor \frac{n-c+1+(m-1)k}{m} \rfloor}{k}. \tag{10}$$

Combinations satisfying (9) have the property that, when reduced modulo m, each object gives a finite number of repetitions of the sequence $c, c+1, \dots, m, 1, 2, \dots, c-1$, of the first m residues followed by the first i residues in order, $0 \le i < m$.

For example, an object enumerated by $f_3(50,11,4)$ is $A = (7,8,13,18,19,24,25,34,39,40,45)$, and $A \bmod 4 = (3,4,1,2,3,4,1,2,3,4,1)$.

Notice that $h(n,k)$ (see equation (2)) is given by $h(h,k) = f_1(n,k,2) + f_2(n,k,2)$.

The most general extension of Skolem's theorem which is explicitly known is the following result by Abramson and Moser [**1**]. Even though Goulden and Jackson [**4**] discovered a generating function that properly contains the same result, no further explicit generalization was stated.

Theorem 2.4 (Abramson and Moser). *Given integers m_j, $0 \le m_j \le m-1$, $j \in [k]$, the number $f(n,k,m \mid m_1,\dots,m_k)$ of combinations $(x_1,\dots,x_k)$ of $[n]$ which satisfy*

$$x_1 \equiv 1 + m_1 \pmod{m}, \qquad x_j \equiv x_{j-1} + 1 + m_j \pmod{m}, \qquad 1 \le j \le k, \tag{11}$$

is given by

$$f(n,k,m \mid m_1,\dots,m_k) = \binom{\lfloor \frac{n+(m-1)k-(m_1+\cdots+m_k)}{m} \rfloor}{k}. \tag{12}$$

Moreover,

$$\begin{aligned} f(n,k,m \mid m_1,\dots,m_k) &= f(n-1-m_k, k-1, m \mid m_1,\dots,m_{k-1}) \\ &\quad + f(n-m,k,m \mid m_1,\dots,m_k). \\ f(n,0,m \mid m_1,\dots,m_k) &= 1, \\ f(n,k,n \mid m_1,\dots,m_k) &= 1 \end{aligned}$$

Proof. The condition (11) is equivalent to

$$x_j \equiv j + m_1 + m_2 + \cdots + m_j \pmod{m}, \quad 1 \le j \le k, \tag{13}$$

since $x_j - x_{j-1} \equiv (j + m_1 + \cdots + m_j) - ((j-1) + m_1 + \cdots + m_{j-1}) \equiv 1 + m_j$, and conversely. Now each object $(x_1,\dots,x_k)$ satisfying (13) corresponds to a unique object enumerated by $f(n - m_1 - m_2 - \cdots - m_k, k, m)$ via the transformation

$$(x_1,\dots,x_k) \longrightarrow (x_1 - m_1,\, x_1 - m_1 - m_2, \dots, x_k - m_1 - \cdots - m_k).$$

Hence, $f(n,k,m \mid m_1,\dots,m_k) = f(n - m_1 - m_2 - \cdots - m_k, k, m)$, and the result follows from (6). The recurrence relation follows from equation (7). □

Remark 2.5. We note some specializations of Theorem 2.4:

- The solution to Terquem's problem is given by $f(n,k,2 \mid 0,\dots,0)$.
- Skolem's generalization (Theorem 2.2): $f(n,k,m) = f(n,k,m \mid 0,\dots,0)$.
- Extension of Skolem's theorem (Corollary 2.3): $f_c(n,k,m) = f(n,k,m \mid c-1,0,\dots,0)$.

3. Permutations with parity-alternating entries. Let n and k be positive integers, $k \le n$. We consider the enumeration of k-permutations of $[n]$ with parity-alternating entries, that is, all permutations $(p_1,p_2,\dots,p_k)$ of $[n]$ satisfying the condition

$$p_i \not\equiv p_{i-1} \pmod{2}, \quad i > 1. \tag{14}$$

Let $pa(n,k)$ denote the number of such permutations.

It is assumed throughout that a product containing an undefined factor is equal to zero.

Theorem 3.1.

$$
\begin{aligned}
pa(n,k) = {} & \frac{\lfloor\frac{n+1}{2}\rfloor!\lfloor\frac{n}{2}\rfloor!}{(\lfloor\frac{n+1}{2}\rfloor - \lfloor\frac{k+1}{2}\rfloor)!(\lfloor\frac{n}{2}\rfloor - \lfloor\frac{k}{2}\rfloor)!} \\
& + \frac{\lfloor\frac{n}{2}\rfloor!\lfloor\frac{n+1}{2}\rfloor!}{(\lfloor\frac{n}{2}\rfloor - \lfloor\frac{k+1}{2}\rfloor)!(\lfloor\frac{n+1}{2}\rfloor - \lfloor\frac{k}{2}\rfloor)!}.
\end{aligned} \tag{15}
$$

Proof. By (14) each object $(p_1, \dots, p_k)$ enumerated by $pa(n,k)$ admits a decomposition into two distinct sequences of elements with the same parity, namely, $p_1, p_3, \dots, p_x$ and $p_2, p_4, \dots, p_y$, where $p_k \in \{p_x, p_y\}$, $|x - y| \in \{0, 1\}$. Since the relative sizes of elements are immaterial, each object $(p_1, \dots, p_k)$ can be obtained by taking an x- (or respectively y-) permutation of $\{1, 3, \dots\} \subset [n]$ and a y- (or respectively x-) permutation of $\{2, 4, \dots\} \subset [n]$, and combining the permutations in a unique way. Hence, the number of such permutations for the p_1 of fixed parity is given by the product of the number of x-permutations of the subset of $[n]$ containing elements with the parity of p_1 and the number of y-permutations of the set of the remaining elements. Since p_1 is either even or odd the full enumeration can be expressed as

$$pa(n,k) = pa(n,k)_{\text{odd}} + pa(n,k)_{\text{even}}, \tag{16}$$

where the two summands denote the numbers of k-permutations in which the first element is odd and even, respectively. Note that $pa(n,k)_{\text{odd}}$ counts k-permutations of $[n]$ with odd numbers in odd positions and hence with even numbers in even positions. $pa(n,k)_{\text{even}}$ is similarly interpreted. If we write $O[n]$ and $E[n]$ for the number of odd and even numbers in $[n]$, respectively, then the explicit result corresponding to (16) is

$$pa(n,k) = p(O[n], O[k])p(E[n], E[k]) + p(E[n], O[k])p(O[n], E[k]),$$

where $p(n,k)$ is the number of k-permutations of $[n]$.

Since $O[n] = \lfloor (n+1)/2 \rfloor$ and $E[n] = \lfloor n/2 \rfloor$, we obtain

$$
\begin{aligned}
pa(n,k) = {} & p\left(\lfloor \frac{n+1}{2} \rfloor, \lfloor \frac{k+1}{2} \rfloor\right) p\left(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{k}{2} \rfloor\right) \\
& + p\left(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{k+1}{2} \rfloor\right) p\left(\lfloor \frac{n+1}{2} \rfloor, \lfloor \frac{k}{2} \rfloor\right).
\end{aligned} \tag{17}
$$

The theorem follows upon applying the known formula $p(n,k) = n!/(n-k)!$. $\square$

Equation (15) simplifies to more specific results on fixing the parity of n or k. In particular, the case $n = k$ gives

Corollary 3.2. *The number $pa(n)$ of permutations of $[n]$ with parity-alternating entries is given by*

$$
\begin{aligned}
pa(n) &= 2\left(\left(\frac{n}{2}\right)!\right)^2 && \textit{if } n \textit{ is even,} \\
pa(n) &= \left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)! && \textit{if } n \textit{ is odd.}
\end{aligned}
$$

It is of interest that $pa(n)$ solves a *Traveling Salesman* problem described below. The combinatorial problem, due to David Singmaster, appeared in *Mathematics Magazine* [**8**, pages 321–322] (also reported in [**9**, A092186]):

> *Traveling Salesman*: A salesman's office is located on a straight road. His n customers are all located along this road to the east of the office, with the office of customer i at distance i from the salesman's office. The salesman must make a driving trip whereby he leaves the office, visits each customer exactly once, then returns to the office. Because he makes a profit on his mileage allowance, the salesman wants to drive as far as possible during his trip.
>
> *Find the number of trips in which he covers the maximum distance.*
>
> Assume that if the travel plans call for the salesman to visit customer j immediately after he visits customer i, then he drives directly from i to j.

A little reflection shows that this traveling salesman problem is solved by $pa(n)$.

Indeed since each customer's location is to be visited exactly once, and successive visitations of customers is allowed, a linear arrangement of the n locations which gives the maximum distance corresponds bijectively to a permutation with parity-alternating entries, provided the salesman first visits the locations corresponding to entries with a fixed parity and returns via the locations corresponding to entries of the opposite parity. Thus, if n is odd, $n = 2m + 1$, the $m + 1$ locations corresponding to odd entries can be visited in $(m + 1)!$ orders, and the remaining m locations can be visited in $m!$ orders. Hence, the required number of trips is $(m + 1)!m!$. The enumeration is similar when n is even, $n = 2m$, except that the entry corresponding to the first location may be even or odd. Hence, there are $2(m!)^2$ trips of maximum distance. (Two, longer, solutions of the problem are presented in [**8**]).

In view of this connection, we conclude that Theorem 3.1 is related to the solution of a more general form of Singmaster's problem.

4. The Terquem problem for permutations. We consider the permutation analogues of the problem of Terquem and Skolem's generalization which have been defined for subsets in Section 1.

The Terquem problem for permutations is then to find the number of k-permutations of $[n]$ such that odd elements are in odd positions and even elements are in even positions. The solution may be deduced from equations (15) and (16), and is given by

$$pa(n,k)_{\text{odd}} = \frac{\lfloor \frac{n+1}{2} \rfloor ! \lfloor \frac{n}{2} \rfloor !}{(\lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{k+1}{2} \rfloor)!(\lfloor \frac{n}{2} \rfloor - \lfloor \frac{k}{2} \rfloor)!}, \quad n > k \text{ if } n \not\equiv k \pmod 2. \tag{18}$$

Lastly, we extend permutations with parity-alternating entries to all moduli $m \geq 2$ by defining a Skolem-type generalization for permutations. Consider the problem of enumerating the class of permutations $(p_1, p_2, \dots)$ of $[n]$ in which

$$p_j \equiv j \pmod m, \tag{19}$$

for a given integer $m \geq 2$.

Denote the number of k-permutations of $[n]$ satisfying (19) by $sk(n,k,m)$. Then, invoking Lemma 1.1, we state the general result as follows.

Theorem 4.1. *The number $sk(n,k,m)$ of permutations $(p_1,\dots,p_k)$ of $[n]$ which satisfy*

$$p_j \equiv j \pmod m, \quad 1 \le j \le k,$$

is given by

$$sk(n,k,m) = \prod_{i=0}^{m-1} \frac{\lfloor \frac{n+i}{m} \rfloor !}{(\lfloor \frac{n+i}{m} \rfloor - \lfloor \frac{k+i}{m} \rfloor)!}. \tag{20}$$

(Note that Equation (18) corresponds to the case $m=2$.)

Proof. The proof is obtained by extending one half of the proof of (15). By (19) each object $(p_1,\dots,p_k)$ enumerated by $sk(n,k,m)$ can be split into m distinct sequences of elements with the same parity, namely, $(p_1,p_{1+m},\dots,p_{x_1})$, $(p_2,p_{2+m},\dots,p_{x_2}),\dots(p_m,p_{m+m},\dots,p_{x_m})$, where $p_k \in \{p_{x_1},\dots,p_{x_m}\}$ and $|x_i - x_{i-1}| \in \{0,1,\dots,m-1\}$, $1 \le i \le m$. Thus, each object $(p_1,\dots,p_k)$ can be formed by obtaining an x_i-permutation of the residue class $V(i,n,m) \subset [n]$ of i modulo m, for every $i \in [m]$, and combining the m permutations in a unique way. From Lemma 1.1, $|V(i,n,m)| = \lfloor (n+m-i)/m \rfloor$. Hence, the number of such permutations is

$$p\left(\left\lfloor \frac{n+m-1}{m} \right\rfloor, \left\lfloor \frac{k+m-1}{m} \right\rfloor \right) p\left(\left\lfloor \frac{n+m-2}{m} \right\rfloor, \left\lfloor \frac{k+m-2}{m} \right\rfloor \right) \cdots p\left(\left\lfloor \frac{n}{m} \right\rfloor, \left\lfloor \frac{k}{m} \right\rfloor \right),$$

where

$$p(n,k) = \frac{n!}{(n-k)!}. \qquad \square$$

In particular, the following result holds.

Corollary 4.2. *The number $sk(n,m)$ of permutations $(p_1,\dots,p_n)$ of $[n]$ which satisfy*

$$p_j \equiv j \pmod{m}, \quad 1 \le j \le n,$$

is given by

$$sk(n,m) = \prod_{i=0}^{m-1} \lfloor \frac{n+i}{m} \rfloor !. \tag{21}$$

5. A generalization of Theorem 4.1. In this section we extend Theorem 4.1 by enumerating the permutations of $[n]$ which fulfill the parity condition

$$p_j \equiv j + c - 1 \pmod{m}. \tag{22}$$

As already noted immediately after Corollary 2.3, sequences fulfilling (22) possess a nice underlying periodicity property when reduced modulo m.

Denote the number of permutations $(p_1,\dots,p_k)$ of $[n]$ satisfying (22) by $sk_c(n,k,m)$. Then an obvious special case is

$$sk_c(n,m,m) = sk(n,m,m),$$

since $(p_1,\dots,p_m)$ is counted by $sk(n,m,m)$ if and only if $(p_c,p_{c+1},\dots,p_m,p_1,p_2,\dots,p_{c-1})$ is counted by $sk_c(n,m,m)$.

The general formula is stated below.

Theorem 5.1. *Let c be a fixed element of $[m]$. The number $sk_c(n,k,m)$ of permutations $(p_1,\dots,p_k)$ of $[n]$ which satisfy*

$$p_j \equiv j + c - 1 \pmod{m}, \quad 1 \le j \le k,\ 2 \le m \le n,$$

is given by

$$sk_c(n,k,m) = \prod_{i=1}^{m-c+1} p\left(\left\lfloor \frac{n+m-c+1-i}{m} \right\rfloor, \left\lfloor \frac{k+m-i}{m} \right\rfloor \right) \prod_{s=1}^{c-1} p\left(\left\lfloor \frac{n+m-s}{m} \right\rfloor, \left\lfloor \frac{k+c-1-s}{m} \right\rfloor \right), \tag{23}$$

where $p(n,k) = n!/(n-k)!$.

(Note that $sk_1(n,k,m) = sk(n,k,m)$ as expected).

Proof. Let $P = (p_1, \dots, p_k)$ be an object counted by $sk_c(n,k,m)$, and let $V(c,n,m) \subset [n]$ denote the residue class of c with $v(c,n,m) = |V(c,n,m)| = \lfloor (n+m-c)/m \rfloor$. Then the fixed allocation numbers of the k positions in P is given by the vector $A = (v(1,k,m), v(2,k,m), \dots, v(m,k,m))$, where $v(1,k,m) + \cdots + v(m,k,m) = k$. This means that the residue class of c can occur in P exactly $v(1,k,m)$ times, that of $c+1$ can occur exactly $v(2,k,m)$ times, and so forth. Thus, in particular,

$$sk_c(n,k,m) > 0 \iff v(1,k,m) \le v(c,n,m).$$

Therefore, to obtain $sk_c(n,k,m)$, we proceed as in the proof of Theorem 4.1, but associate $V(c,n,m)$ with $V(1,k,m)$ by taking the number of $v(1,k,m)$-permutations of $v(c,n,m)$ times the number of $v(2,k,m)$-permutations of $v(c+1,n,m)$, and so forth. More precisely, we take the product of $p(B_t, A_t)$ for $1 \le t \le m$, where $A = (A_1, \dots, A_m)$ is defined above and $B = (B_1, \dots, B_m)$ is the vector defined by

$$B = (v(c,n,m), v(c+1,n,m), \dots, v(m,n,m), v(1,n,m), v(2,n,m), \dots, v(c-1,n,m)).$$

Hence,

$$sk_c(n,k,m) = \prod_{i=1}^{m-c+1} p(v(c-1+i,n,m), v(i,k,m)) \prod_{s=1}^{c-1} p(v(s,n,m), v(m-c+1+s,k,m)). \qquad \square$$

Remark 5.2. We are presently unable to obtain the permutation analogue of Theorem 2.4. The interested reader is invited to take a shot at this open question.

REFERENCES

1. M. Abramson and W. Moser, *Generalizations of Terquem's problem*, J. Combin. Theory **7** (1969), 171–180.

2. L. Carlitz, *Alternating sequences*, Discrete Math. **17** (1977), 133–138.

3. C.A. Church and H.W. Gould, *Lattice point solution of the generalized problem of Terquem and an extension of Fibonacci numbers*, Fibonacci Quart. **5** (1967), 59–68.

4. I.P. Goulden and D.M. Jackson, *The enumeration of generalized alternating subsets with congruences*, Discrete Math. **22** (1978), 99–104.

5. F. Hering, *A problem of inequalities*, Amer. Math. Monthly **78** (1971), 275–276.

6. E. Netto, *Lehrbuch der Combinatorik*, 2nd ed., Leipzig, 1927, reprinted by Chelsea, New York, 1958.

7. J. Riordan, *An introduction to combinatorial analysis*, Wiley, New York, 1958.

8. D. Singmaster, *Problem* 1654, Math. Magazine **75** (October 2002), *Solution* in Math. Magazine **76** (October 2003).

9. N.J.A. Sloane, *The on-line encyclopedia of integer sequences*, published electronically at www.research.att.com/ njas/sequences/, 2007.

10. S.M. Tanny, *On alternating subsets of integers*, Fibonacci Quart. **13** (1975), 325–328.

School of Mathematics, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa
Email address: Augustine.Munagi@wits.ac.za