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SPECIAL EFFECT VARIETIES AND (-1)-CURVES

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To my mother for her 60th birthday

ABSTRACT. Here we introduce the concept of special effect curve which permits to study, from a different point of view, special linear systems in $\mathbf{P}^2$, i.e., linear system with general multiple base points whose effective dimension is strictly greater than the expected one. In particular we study two different kinds of special effect: the α-*special effect* is defined by requiring some numerical conditions, while the definition of h^1-*special effect* concerns cohomology groups. We state two new conjectures for the characterization of special linear systems and we prove they are equivalent to the Segre and the Harbourne-Hirschowitz ones.

1. Introduction.

Let X be a smooth, irreducible, complex projective variety of dimension n. Let $\mathcal{L}$ be a complete linear system of divisors on X. Fix points $P_1, \dots, P_h$ on X in general position and positive integers $m_1, \dots, m_h$. We denote by $\mathcal{L}(-\sum_{i=1}^h m_i P_i)$ the subsystem of $\mathcal{L}$ given by all divisors having multiplicity at least m_i at P_i, $i = 1, \dots, h$. Since a point of multiplicity m imposes $\binom{m+n-1}{n}$ conditions we can define the *virtual dimension* of the system $\mathcal{L}(-\sum_{i=1}^h m_i P_i)$ as

$$\nu\bigg(\mathcal{L}\bigg(-\sum_{i=1}^h m_i P_i\bigg)\bigg) := \operatorname{virtdim}\bigg(\mathcal{L}\bigg(-\sum_{i=1}^h m_i P_i\bigg)\bigg) = \dim(\mathcal{L}) - \sum_{i=1}^h \binom{m_i+n-1}{n}.$$

This virtual dimension can be negative: in this case we expect that the system $\mathcal{L}(-\sum_{i=1}^h m_i P_i)$ is empty. We can then define the *expected*

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dimension of $\mathcal{L}(-\sum_{i=1}^{h} m_iP_i)$ as

$$\begin{aligned}\varepsilon\Big(\mathcal{L}\Big(-\sum_{i=1}^{h} m_iP_i\Big)\Big) &:= \exp\dim\Big(\mathcal{L}\Big(-\sum_{i=1}^{h} m_iP_i\Big)\Big)\\ &= \max\Big\{\nu\Big(\mathcal{L}\Big(-\sum_{i=1}^{h} m_iP_i\Big)\Big), -1\Big\}.\end{aligned}$$

The conditions imposed by the multiple points m_iP_i can be dependent, so, in general we have

$$\dim\Big(\mathcal{L}\Big(-\sum_{i=1}^{h} m_iP_i\Big)\Big) \geq \varepsilon\Big(\mathcal{L}\Big(-\sum_{i=1}^{h} m_iP_i\Big)\Big)$$

and we can state the following

Definition 1.1. A system $\mathcal{L}(-\sum_{i=1}^{h} m_iP_i)$ is *special* if

$$\dim\Big(\mathcal{L}\Big(-\sum_{i=1}^{h} m_iP_i\Big)\Big) > \varepsilon\Big(\mathcal{L}\Big(-\sum_{i=1}^{h} m_iP_i\Big)\Big),$$

otherwise $\mathcal{L}(-\sum_{i=1}^{h} m_iP_i)$ is said to be *non-special*.

By definition a system which is empty is nonspecial. For a nonempty system nonspeciality means that the imposed conditions are independent.

Since we expect that most systems are nonspecial, we can pose the following classification problem: *classify all special systems*.

The dimensionality problem is quite hard if we consider a general variety X, so we fix our attention on particular varieties and linear systems. As a first choice we can take $X = \mathbf{P}^n$ and $\mathcal{L} = \mathcal{L}_{n,d} := |\mathcal{O}_{\mathbf{P}^n}(d)|$, the system of hypersurfaces of degree d in $\mathbf{P}^n$. In this case we have

$$\nu\Big(\mathcal{L}_{n,d}\Big(-\sum_{i=1}^{h} m_iP_i\Big)\Big) = \binom{d+n}{n} - 1 - \sum_{i=1}^{h}\binom{m_i+n-1}{n}.$$

Starting with the case $X = \mathbf{P}^2$, we have some precise conjectures about the characterization of special linear systems and a rich series of results on the conjectures. The main conjectures are the following.

Conjecture 1.2 [(SC), **14**]. *If a linear system of plane curves with general multiple base points $\mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ is special, then its general member is nonreduced, i.e., the linear system has, according to Bertini's theorem, some multiple fixed component.*

Conjecture 1.3 [(HHC), **9, 10**]. *A linear system of plane curves with general multiple base points $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ is special if and only if is (-1)-special, i.e., its strict transform on the blow-up along the points $P_1, \dots, P_h$ splits as $\widetilde{\mathcal{L}} = \sum_{i=1}^{k} N_i C_i + \widetilde{\mathcal{M}}$ where the C_i, $i = 1, \dots, k$, are (-1)-curves such that $C_i \cdot \widetilde{\mathcal{L}} = -N_i < 0$, $\nu(\widetilde{\mathcal{M}}) \geq 0$ and there is at least one index j such that $N_j > 2$.*

In [**7**], Ciliberto and Miranda proved that the Harbourne-Hirschowitz and Segre conjectures are equivalent. Although the Harbourne-Hirschowitz conjecture is still unproved, it is important to notice that, in more than a century of research, no special system has been discovered except (-1)-special systems. For an overview on these results the reader may consult [**2, 4, 5, 12**].

When we pass to $\mathbf{P}^n$, $n \geq 3$, very little is known about special linear systems. One of the most important result is the classification of the homogeneous special systems for double points:

Theorem 1.4 [**2**]. *The system $\mathcal{L}_{n,d}(2^h)$ is nonspecial unless*:

$$\begin{array}{cccccc} n & \textit{any} & 2 & 3 & 4 & 4 \\ d & 2 & 4 & 4 & 4 & 3\,. \\ h & 2,\dots,n & 5 & 9 & 14 & 7 \end{array}$$

Continuing with $\mathbf{P}^n$, $n \geq 3$, we notice that there is not a precise conjecture. Although the Segre conjecture can be generalized in every ambient variety using the statement concerning $H^1 \neq 0$ (see, for example, [**2**] or [**7**]) there is nothing that characterizes the special systems from a geometric point of view as, for example, in the case of $(-1)-$curves in $\mathbf{P}^2$.

A worthy goal would be "find a conjecture (C) in $\mathbf{P}^n$, [or in a generic variety X] such that, when we read (C) in $\mathbf{P}^2$, (C) is equivalent to the Segre (1.2) and Harbourne-Hirschowitz (1.3) conjectures."

In Sections 3 and 4 we state two potential candidates for the above-mentioned goal: the *Numerical Special Effect Conjecture* and the *Cohomological Special Effect Conjecture*. In fact, in these sections, we define the concepts of "α" and "h^1" special effect curves which permit the introduction of a different approach in the study of special linear systems in $\mathbf{P}^2$. Moreover, in Section 5 we prove that these conjectures are equivalent to the Segre and the Harbourne-Hirschowitz ones.

In Section 6 we present some examples of special effect varieties in $\mathbf{P}^n$, $n \geq 3$. Due to its complexity, the generalization of the "Numerical" and "Cohomological" conjectures to the higher-dimensional case is presented in [**3**] where we prove also that these conjectures hold for every special system listed in Theorem 4.

Finally, in Section 7 we show some results on special effect varieties when the ambient variety is a Hirzebruch surface or a $K3$ surface.

2. Preliminaries. We collect some facts about linear systems that will be useful in the next sections.

Consider the blow-up $\pi : \widetilde{\mathbf{P}}^n \to \mathbf{P}^n$ at points $P_1, \dots, P_h$, and let E_i, $i = 1, \dots, h$ be the exceptional divisors corresponding to the blow-up of the points P_i, $i = 1, \dots, h$. If we denote by H the pull-back of a general hyperplane of $\mathbf{P}^n$ via π, then we can write the strict transform of the system $\mathcal{L} := \mathcal{L}_{n,d}(\sum_{i=1}^h m_i P_i)$ as $\widetilde{\mathcal{L}} = |dH - \sum_{i=1}^h m_i E_i|$. In the future, if confusion cannot arise, we will indicate both $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ by $\mathcal{L}$.

It is an easy application of the (generalized) Riemann-Roch theorem to observe that

$$\nu(\mathcal{L}) = \chi(\widetilde{\mathcal{L}}) - 1. \tag{2.1}$$

Consider now the case of $\mathbf{P}^2$, and let $\mathcal{L} := \mathcal{L}_{2,d}(\sum_{i=1}^h m_i P_i)$. By

Riemann-Roch, remembering that $h^2(\widetilde{\mathbf{P}}^2, \widetilde{\mathcal{L}}) = 0$, we obtain

$$\begin{aligned} \dim(\mathcal{L}) = \dim(\widetilde{\mathcal{L}}) &= \frac{\widetilde{\mathcal{L}} \cdot (\widetilde{\mathcal{L}} - \widetilde{K})}{2} + h^1(\widetilde{\mathbf{P}}^2, \widetilde{\mathcal{L}}) - h^2(\widetilde{\mathbf{P}}^2, \widetilde{\mathcal{L}}) \\ &= \widetilde{\mathcal{L}}^2 - g_{\mathcal{L}} + 1 + h^1(\widetilde{\mathbf{P}}^2, \widetilde{\mathcal{L}}) = \nu(\mathcal{L}) + h^1(\widetilde{\mathbf{P}}^2, \widetilde{\mathcal{L}}) \end{aligned} \tag{2.2}$$

where g is the arithmetic genus p_a of a curve in $\widetilde{\mathcal{L}}$ and $\widetilde{K}$ is the canonical class on $\widetilde{\mathbf{P}}^2$.

Hence, by the previous formula, we have

(2.3) $\mathcal{L}$ is nonspecial if and only if $h^0(\widetilde{\mathbf{P}}^2, \widetilde{\mathcal{L}}) \cdot h^1(\widetilde{\mathbf{P}}^2, \widetilde{\mathcal{L}}) = 0$.

Remark 2.1. The reducible curve $C = \sum_{i=1}^k N_i C_i$ in Conjecture 1.3 is called a (-1)-**configuration** on $\widetilde{\mathbf{P}}^2$.

Whenever not otherwise specified, we work over the field $\mathbf{C}$.

3. α-special effect curves. Let $P_1, \dots, P_h$ be points in $\mathbf{P}^2$ in general position, and fix positive integers $m_1, \dots, m_h$. Consider the system $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^h m_i P_i)$ of planar curves of degree d passing through the points P_i with multiplicity at least m_i.

Definition 3.1. Let $\mathcal{L}$ and $P_i, \dots P_h$ be as above. An irreducible curve Y, of degree e, has the *α-special effect property* for $\mathcal{L}$ on $\mathbf{P}^2$ if there exist nonnegative integers $\alpha, c_{j_1}, \dots c_{j_s}$, with $\alpha e \leq d$ and $1 \leq \alpha \leq \min\{\lfloor \frac{m_{j_i}}{c_{j_i}} \rfloor, i = 1, \dots, s\}$, such that

(i) Y contains the point P_{j_i} with multiplicity at least c_{j_i} for $j = 1, \dots, s$, where $P_{j_i} \in \{P_1, \dots, P_h\}$;

(ii) $\nu(\mathcal{L} - \alpha Y) > \nu(\mathcal{L})$.

Moreover we require that α is the maximum admissible value for the α-special effect property and, if $\beta > \alpha$ then $\nu(\mathcal{L} - \beta Y) < \nu(\mathcal{L} - \alpha Y)$.

In the following, we will mainly ask for a condition stronger than (i):

(i*) $\nu(|Y|) \geq 0$,

where $|Y|$ represents the linear system $|Y| = |eH - \sum_{i=1}^s c_{j_i} P_{j_i}|$. It is clear that condition (i*) implies condition (i).

Condition (ii) is surely the most interesting. As a matter of fact it tells us that the number of conditions imposed on the system of curves of degree d by imposing a multiple curve αY and the points P_{j_i} with multiplicity $m_{j_i} - \alpha c_{j_i}$ (such that the final multiplicity at the point P_{j_i} is at least m_{j_i}, $i = 1, \dots, s$) plus eventually the other multiple points $m_t P_t$, $t \notin \{j_1, \dots, j_s\}$ is less than the number of conditions imposed to the same system $|dH|$ only imposing each P_i with multiplicity at least m_i, $i = 1, \dots, h$. This sounds like a crazy requirement because, in general, we expect that a positive dimensional variety imposes more conditions than a zero-dimensional variety. It is important to notice the similarity with the "strange" requirement in the case of (-1)-curves in [**5**]: we asked there for a curve C whose double is not expected to exist!

Example 3.2. Let $\mathcal{L}$ be the system $\mathcal{L}_{2,9}(-6P_1 - 6P_2 - 6P_3)$. This system is special since $\nu(\mathcal{L}) = -9$ but its effective dimension is 0 since it contains $3Y$, with $Y = L_{12} + L_{13} + L_{23}$, where L_{ij} is the line through P_i and P_j. We claim that each of the lines L_{ij} has the 3-special effect property. We prove this for L_{12}. Obviously one has $\nu(|L_{12}|) \geq 0$; indeed, it is a (-1)-curve. Moreover $\mathcal{L} - L_{12}$ is the system $\mathcal{L}' := \mathcal{L}_{2,8}(-5P_1 - 5P_2 - 6P_3)$ and its virtual dimension is

$$\nu(\mathcal{L}') := \frac{8 \cdot 11}{2} - 2\frac{5 \cdot 6}{2} - \frac{6 \cdot 7}{2} = 44 - 30 - 21 = -7.$$

Going further we can observe that

$$\nu(\mathcal{L} - 2L_{12}) = \nu(\mathcal{L} - 3L_{12}) = -6$$

while

$$\nu(\mathcal{L} - 4L_{12}) = -7.$$

So the claim follows.

Example 3.3. Let $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ and consider a (-1)-curve E such that $\mathcal{L} \cdot E = -N < 0$. Thus $\mathcal{L} = NE + \mathcal{M}$, where $E \cdot \mathcal{M} = 0$. Using Riemann-Roch it is easy to prove $\nu(\mathcal{L} - NE) = \nu(\mathcal{L}) + \binom{N}{2}$ and $\nu(\mathcal{L} - (N+1)E) = \nu(\mathcal{L} - NE) - 1$. Hence E has the N-special effect property if $N \geq 2$.

Going back to the definition of α-special effect curves, we now see how the conditions (i)–(ii) give some numerical information about the intersection $\mathcal{L} \cdot Y$. We will also work on the blow-up of $\mathbf{P}^2$ at the points $P_1, \dots, P_h$ and, as in the case of (-1)-curves, we will consider the strict transform $\widetilde{Y}$ of the α-special effect curve Y, but in general we will denote both Y and $\widetilde{Y}$ by Y.

Lemma 3.4. *Let Y be an irreducible curve having the α-special effect property for a system $\mathcal{L}$. Then $\mathcal{L} \cdot Y < [(\alpha+1)/2]Y^2$.*

Proof. Let $\mathcal{L}$ be the system $\mathcal{L}_{2,d}(-\sum_{i=1}^h m_i P_i)$, and suppose Y has degree e and passes through P_{j_i}'s with multiplicity at least c_{j_i}. From conditions $(i*)$ and (ii) of the α-special effect property we have, respectively,

$$e^2 + 3e \geq \sum_{i=1}^{s}(c_{j_i}^2 + c_{j_i}) \text{ then } -3e + \sum_{i=1}^{s} c_{j_i} \leq e^2 - \sum_{i=1}^{s} c_{j_i}^2 \tag{3.1}$$

$$\frac{1}{2}\Big(-de + \sum_{i=1}^{s} m_{j_i}c_{j_i} + \alpha e^2 - 3e - \sum_{i=1}^{s}(\alpha c_{j_i}^2 + c_{j_i})\Big) > 0. \tag{3.2}$$

Since $\mathcal{L} \cdot Y := \widetilde{\mathcal{L}} \cdot \widetilde{Y} = de - \sum_{i=1}^{s} m_{j_i}c_{j_i}$, we obtain (by using (3.2) and (3.1)):

$$\begin{aligned}\mathcal{L} \cdot Y = de - \sum_{i=1}^{s} m_{j_i}c_{j_i} &< \frac{1}{2}\Big(\alpha e^2 - 3e - \sum_{i=1}^{s}(\alpha c_{j_i}^2 + c_{j_i})\Big)\\ &\leq \frac{(\alpha+1)}{2}(e^2 - \sum_{i=1}^{s} c_{j_i}^2)\end{aligned}$$

so that $\mathcal{L} \cdot Y < [(\alpha+1)/2]Y^2$. $\square$

By the previous lemma we can also obtain some information about Y^2.

Lemma 3.5. *Suppose Y has the α-special effect property for a system $\mathcal{L}$. If $h^0(\mathcal{L} - \alpha Y) \geq 1$, then $Y^2 \leq -1$.*

Proof. By Lemma 3.4 we have

$$(\mathcal{L} - \alpha Y) \cdot Y = \mathcal{L} \cdot Y - \alpha Y^2 < \frac{(1-\alpha)}{2} Y^2. \tag{3.3}$$

Consider first the case $\alpha = 1$; then Y splits from $\mathcal{L} - Y$ and we can compute

$$(\mathcal{L} - 2Y) \cdot Y = \mathcal{L} \cdot Y - 2Y^2 = \mathcal{L} \cdot Y - Y^2 - Y^2 < -Y^2$$

Hence, if $Y^2 \geq 0$, then Y is a fixed component of $\mathcal{L} - 2Y$. But at this point we can iterate the procedure and we would obtain

$$(\mathcal{L} - NY) \cdot Y = \mathcal{L} \cdot Y - Y^2 - (N-1)Y^2 < -(N-1)Y^2.$$

Thus, if $Y^2 \geq 0$, Y appears with multiplicity ∞ in $\mathcal{L} - Y$; but this is a contradiction, hence $Y^2 \leq -1$.

Consider now the case $\alpha \geq 2$ in (3.3). If $Y^2 \geq 0$, then Y is a fixed component of $\mathcal{L} - \alpha Y$. Moreover, for $N > \alpha$, we have

$$(\mathcal{L} - NY) \cdot Y = \mathcal{L} \cdot Y - NY^2 < \frac{(\alpha + 1 - 2N)}{2} Y^2 < 0.$$

Thus we can conclude again that if $Y^2 \geq 0$, then we obtain a contradiction. Hence, $Y^2 \leq -1$. □

Definition 3.6. Let $\mathcal{L}$ and $P_1, \ldots, P_h$ be as above. An irreducible curve Y, of degree e, is an *α-special effect curve* for $\mathcal{L}$ on $\mathbf{P}^2$ if Y has the α-special effect property for $\mathcal{L}$, and moreover, $\nu(\mathcal{L} - \alpha Y) \geq 0$.

We recall that the existence of a (-1)-configuration $C = \sum_{i=1}^t N_i C_i$ such that $\mathcal{L} := \sum_{i=1}^t N_i C_i + \mathcal{M}$ leads us to the inequality

$$\dim(\mathcal{L}) = \dim(\mathcal{M}) \geq \nu(\mathcal{M}) = \nu(\mathcal{L}) + \sum_{i=1}^{t} \binom{N_i}{2}, \tag{3.4}$$

which, under the assumption of (-1)-speciality of $\mathcal{L}$, i.e., $\nu(\mathcal{M}) \geq 0$ and $N_i \geq 2$ for at least one index i, implies that $\mathcal{L}$ is special. Observe that the existence of an α-special effect curve Y for a system $\mathcal{L}$ forces the system itself to be special. In fact we have the following chain of inequalities

$$\dim(\mathcal{L}) \geq \dim(\mathcal{L} - \alpha Y) \geq \nu(\mathcal{L} - \alpha Y) > \nu(\mathcal{L})$$

and, together with condition $\nu(\mathcal{L} - \alpha Y) \geq 0$, one has $\dim(\mathcal{L}) > \varepsilon(\mathcal{L})$.

Example 3.7. Let $\mathcal{L} := \mathcal{L}_{2,2}(-2P_1 - 2P_2)$ be the linear system of conics with two double points. Let Y be a line through P_1 and P_2, i.e, $Y = H - P_1 - P_2$. Obviously condition (i) is satisfied. Since

$$\nu(\mathcal{L} - Y) = \nu(\mathcal{L} - 2Y) = 0$$

while $\nu(\mathcal{L}) = -1$, one has that condition (ii) is satisfied. From the positivity of $\nu(\mathcal{L} - 2Y)$ we conclude that the line through P_1 and P_2 is a 2-special effect curve for $\mathcal{L}$ and so $\mathcal{L}$ is special.

Example 3.8. We want to show how the problem of the existence of an α-special effect curve can turn into a pure combinatorial problem and its solution is more or less difficult according to the initial data.

For example, we can look for an irreducible smooth α-special effect curve Y of degree e for a generic homogeneous system $\mathcal{L} := \mathcal{L}_{2,d}(m^h)$. Moreover, we require that Y passes through all points $P_1, \ldots, P_h$. The smoothness of Y means $c_1 = \cdots = c_h = 1$.

The conditions for the existence of Y are:

(i) $P_i \in Y$ for $i = 1, \ldots, h$,

(ii) $\nu(|(d - \alpha e)H - \sum_{i=1}^{h}(m - \alpha)P_i|) > \nu(|dH - \sum_{i=1}^{n} mP_i|)$.

(iii) $\nu(|(d - \alpha e)H - \sum_{i=1}^{h}(m - \alpha)P_i|) \geq 0$,

with the extra conditions $1 \leq \alpha \leq m$ and $\alpha e \leq d$. Using Riemann-Roch we can write the previous conditions as

$$\frac{e(e+3)}{2} \geq h \tag{3.5}$$

(3.6)

$$\frac{(d - \alpha e)(d + \alpha e + 3)}{2} - h\frac{(m - \alpha)(m - \alpha + 1)}{2} > \frac{d(d+3)}{2} - h\frac{m(m+1)}{2}$$

$$\frac{(d-\alpha e)(d+\alpha e+3)}{2} \geq h\frac{(m-\alpha)(m-\alpha+1)}{2}. \tag{3.7}$$

In particular, if we expand condition (3.6), we obtain

$$-d\alpha e + \frac{1}{2}\alpha^2 e^2 - \frac{3}{2}\alpha e + hm\alpha - \frac{1}{2}h\alpha^2 + \frac{1}{2}h\alpha > 0. \tag{3.8}$$

Observe that (3.7) is increasing monotone in d and for $d = \alpha e$, we have

$$0 - h\binom{m-\alpha+1}{2} \geq 0$$

which is satisfied only for $\alpha = m$. Then $d = me$.

We claim that $d \geq me$. The proof of this fact is a long and very tedious study of the equations

$$m^2e^2 - 2met - 2me^2\alpha + 3me + t^2 + 2t\alpha e - 3t + \alpha^2 e^2 - 3\alpha e - hm^2 \\ + 2hm\alpha - hm - h\alpha^2 + h\alpha \geq 0$$

and

$$-2me^2\alpha + 2t\alpha e + \alpha^2 e^2 - 3\alpha e + 2hm\alpha - h\alpha^2 + h\alpha > 0$$

given by (3.7) and (3.8) in which we substitute $d = me - t$, with $t > 0$. Anyway, the previous equations together with $e^2 + 3e \geq 2h$ are verified only if at least one between m, e, t, h and α is equal to zero, but this is not acceptable for our purposes (we can check it by a computer algebra system, e.g., Maple).

Now we show that $e < 3$: using 3.8) we compute

$$d\alpha < \frac{1}{2}\alpha^2 e - \frac{3}{2}\alpha + \frac{h\alpha}{e}\left(m - \frac{1}{2}\alpha + \frac{1}{2}\right)$$

and from $d \geq me$ and (3.5) we obtain

$$m\alpha e \leq d\alpha < \frac{1}{2}\alpha^2 e - \frac{3}{2}\alpha + \alpha\left(m - \frac{1}{2}\alpha + \frac{1}{2}\right)\frac{(e+3)}{2}.$$

Then,

$$me < \frac{1}{2}\alpha e - \frac{3}{2} + \frac{1}{2}me - \frac{1}{4}\alpha e + \frac{1}{4}e + 3m - \frac{3}{4}\alpha + \frac{3}{4},$$

and simplifying, we obtain

$$e\left(\frac{1}{2}m - \frac{1}{4}\alpha - \frac{1}{4}\right) < 3\left(\frac{1}{2} - \frac{1}{4}\alpha - \frac{1}{4}\right)$$

that is $e < 3$.

If we analyze the cases $e = 1$ and $e = 2$, we see that the only possibilities are

- $e = 1$, $h = 2$, $m \leq d < 2m - (1/2) - (1/2)\alpha$
- $e = 2$, $h = 5$, $2m \leq d < (5/2)m - (1/4) - (1/4)\alpha$.

If we substitute $\alpha := \mathcal{L} \cdot Y = de - hm$, we obtain

- $e = 1$, $h = 2$, $m \leq d < 2m - 2$,
- $e = 2$, $h = 5$, $2m \leq d < (5m - 2)/2$.

Then we conclude that the systems

$$\begin{array}{ll} \mathcal{L}_{2,d}(m^2) & m \leq d < 2m - 2 \\ \mathcal{L}_{2,2d}(m^5) & 2m \leq d < \dfrac{5m-2}{2} \end{array}$$

are special. The careful reader can observe that these families of special systems are exactly the first two cases in the classification of the homogeneous (-1)-special systems described in Theorem 2.4 in [**6**].

Remark 3.9. Let $\mathcal{L}$ again be the system $\mathcal{L}_{2,9}(-6P_1 - 6P_2 - 6P_3)$. As already seen in Example 3.2, we know that each of the lines L_{ij} has the 3-special effect property for $\mathcal{L}$. As we can see, a single line is not a 3-special effect curve for $\mathcal{L}$, since $\nu(\mathcal{L} - 3L_{ij}) < 0$.

Remark 3.9 shows that α-special effect curves are not sufficient to describe all known special systems. However it is clear, now, in which way we proceed. If Y has the α-special effect property for a system $\mathcal{L}$ and $\nu(\mathcal{L} - \alpha Y) < 0$, we substitute system $\mathcal{L}$ with $\mathcal{L} - \alpha Y$ and we investigate this new system.

Definition 3.10. Let $\mathcal{L}$ be a system as above. Fix a sequence of (not necessarily distinct) irreducible curves $Y_1, \dots Y_t$, Suppose further that

(1) Y_j has the α_j-special effect property for $\mathcal{L} - \sum_{i=1}^{j-1} \alpha_i Y_i$, for $j = 1, \dots, t$,

(2) $\nu(\mathcal{L} - \sum_{i=1}^{t} \alpha_i Y_i) \geq 0$.

Then we call both $X := \sum_{i=1}^{t} \alpha_i Y_i$ and $\{Y_1, \dots, Y_t\}$ an $(\alpha_1, \dots, \alpha_t)$-*special effect configuration for* $\mathcal{L}$.

Example 3.11. Consider again the system $\mathcal{L} := \mathcal{L}_{2,9}(-6P_1 - 6P_2 - 6P_3)$. We prove now that $X = 3L_{12} + 3L_{13} + 3L_{23}$ is a $(3,3,3)$-special effect configuration. Recall that $\nu(\mathcal{L}) = -9$. In Example 3.2 we proved that L_{12} has the 3-special effect property for $\mathcal{L}$. We can go ahead and check if L_{13} has the 3-special effect property for $\mathcal{L} - 3L_{12}$. We obtain:

$$\begin{aligned}
\nu(\mathcal{L} - 3L_{12} - L_{13}) &= \nu(|5H - 2P_1 - 3P_2 - 5P_3|) = -4 \\
\nu(\mathcal{L} - 3L_{12} - 2L_{13}) &= \nu(|4H - P_1 - 3P_2 - 4P_3|) = -3 \\
\nu(\mathcal{L} - 3L_{12} - 3L_{13}) &= \nu(|3H - 3P_2 - 3P_3|) = -3
\end{aligned}$$

Finally we check if L_{23} has the 3-special effect property for $\mathcal{L} - 3L_{12} - 3L_{13}$:

$$\begin{aligned}
\nu(\mathcal{L} - 3L_{12} - 3L_{13} - L_{23}) &= \nu(|2H - 2P_2 - 2P_3|) = -1 \\
\nu(\mathcal{L} - 3L_{12} - 3L_{13} - 2L_{23}) &= \nu(|H - P_2 - P_3|) = 0. \\
\nu(\mathcal{L} - 3L_{12} - 3L_{13} - 3L_{23}) &= 0.
\end{aligned}$$

Thus X is a $(3,3,3)$-special effect configuration for $\mathcal{L}_{2,9}(-6P_1 - 6P_2 - 6P_3)$.

As in the case of α-special effect curves also a special effect configuration X forces a system to be special. In fact, one has again

$$\dim(\mathcal{L}) \geq \dim(\mathcal{L} - X) \geq \nu(\mathcal{L} - X) > \nu(\mathcal{L})$$

and, together with condition (2) in Definition 3.10, one has $\dim(\mathcal{L}) > \varepsilon(\mathcal{L})$.

These facts permit us to define a particular kind of speciality.

Definition 3.12. A special system arising from the existence of an α-special effect curve (or an $(\alpha_1, \dots, \alpha_r)$-special effect configuration) is called *Numerically Special*.

Finally, we can state the following

Conjecture 3.13. [(NSEC), "Numerical Special Effect" conjecture]. *A linear system of plane curves $\mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ with general multiple base points is special if and only if it is numerically special.*

4. h^1-special effect curves. The second class of curves we introduce are defined via some particular conditions on certain cohomology groups. The original idea for these curves comes from a detailed analysis of the base locus in the special systems listed in Theorem 1.4, that is, linear systems with imposed double points in $\mathbf{P}^n$, $n \geq 2$. In fact, as shown in [**3**], this kind of speciality can be more easily generalized to higher dimensions than the *numerical* one.

Definition 4.1. Let $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ be a linear system of plane curves with general multiple base points. An irreducible curve $Y \subset \mathbf{P}^2$, with $\mathcal{O}_{\mathbf{P}^2}(Y) \ncong \mathcal{L}$, is an h^1-*special effect curve* for system $\mathcal{L}$ if the following conditions are satisfied:

(a) $h^0(\mathcal{L}_{|Y}) = 0$;

(b) $h^0(\mathcal{L} - Y) > 0$;

(c) $h^1(\mathcal{L}_{|Y}) > 0$.

Remark 4.2. Condition (c) is slightly different in the definition in the higher dimension case, in [**3**], where we ask for $h^1(\mathcal{L}_{|Y}) > h^2(\mathcal{L} - Y)$. Instead, in the planar case, we can just ask for $h^1(\mathcal{L}_{|Y}) > 0$ because $h^2(\mathcal{L} - Y) = 0$. In fact, by definition of Y and condition (b), we can suppose $\mathcal{L} - Y = |aH - \sum_{i=1}^{h} s_i P_i|$, with $a, s_1, \dots, s_h$ positive integers. Now define Z as the union of the fat points $s_i P_i$. Then we have the following exact sequence

$$0 \to \mathcal{I}_Z \otimes \mathcal{O}_{\mathbf{P}^2}(a) \to \mathcal{O}_{\mathbf{P}^2}(a) \to \mathcal{O}_Z \to 0.$$

When we consider the cohomology groups, we have

$$\begin{array}{ccc} \cdots \to h^1(\mathcal{O}_Z) \to & h^2(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbf{P}^2}(a)) & \to h^2(\mathcal{O}_{\mathbf{P}^2}(a)) \to \cdots \\ & \| & \\ & h^2(\mathcal{L} - Y) & \end{array}$$

Since Z is a zero-dimensional scheme one has $h^i(\mathcal{O}_Z) = 0$ for $i \geq 1$. Moreover, by the Serre duality, $h^2(\mathcal{O}_{\mathbf{P}^2}(a)) = h^0(\mathcal{O}_{\mathbf{P}^2}(-3-a)) = 0$. Thus $h^2(\mathcal{L} - Y) = 0$.

Example 4.3. Let $\mathcal{L} := \mathcal{L}_{2,2}(-2P_1 - 2P_2)$ be the linear system of conics with two double points. Let Y be a line through P_1 and P_2, i.e., $Y = H - P_1 - P_2$. Since $\mathcal{L} \cdot Y = -2$ the restricted system $\mathcal{L}_{|Y}$ has no effective divisors and $h^0(\mathcal{L}_{|Y})$ is empty. By Riemann-Roch we easily compute $h^1(\mathcal{L}_{|Y}) = g_Y - 1 - \deg(\mathcal{L}_{|Y}) = 1 > 0$. Finally $\mathcal{L} - Y$ is $|H - P_1 - P_2|$, so that $h^0(\mathcal{L} - Y) = 1$. Hence the line Y through P_1 and P_2 is an h^1-special effect curve for $\mathcal{L}$.

Let $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^h m_i P_i)$ and consider, on the blow-up of $\mathbf{P}^2$ at the points P_i's, the exact sequence

$$0 \to \mathcal{L} - Y \to \mathcal{L} \to \mathcal{L}_{|Y} \to 0,$$

which gives the following long exact sequence in cohomology:

$$\begin{aligned} 0 \to H^0(\mathcal{L} - Y) \to H^0(\mathcal{L}) \to H^0(\mathcal{L}_{|Y}) \to H^1(\mathcal{L} - Y) \to H^1(\mathcal{L}) \\ \to H^1(\mathcal{L}_{|Y}) \to 0. \end{aligned}$$

Conditions (a) and (b) assure us that $H^0(\mathcal{L}) \neq 0$, while condition (c) implies $H^1(\mathcal{L}) \neq 0$. Thus the existence of such Y forces the system $\mathcal{L}$ to have $h^0(\mathcal{L}) \cdot h^1(\mathcal{L}) \neq 0$ so that, by (2.3), $\mathcal{L}$ is special. Again, we can give a particular name to this kind of system:

Definition 4.4. A special system arising from the existence of an h^1-special effect curve is called *Cohomologically Special*.

And again we can state a conjecture:

Conjecture 4.5 [(CSEC), "Cohomological Special Effect" conjecture]. *A linear system of plane curves $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ with general multiple base points is special if and only if it is cohomologically special.*

Lemma 4.6. *Suppose that the Cohomological Special Effect conjecture holds. Let $C \subset \mathbf{P}^2$ be an irreducible curve passing through the general points $P_1, \ldots, P_h$ with multiplicity at least $m_1, \ldots, m_h$. Then $\widetilde{C}^2 \geq g_{\widetilde{C}} - 1$.*

Proof. Suppose $\nu(|C|) < 0$. Then the system $|C|$ is special. Thus there is an h^1-special effect curve Y for $|C|$ and Y is a fixed part of C. This is a contradiction since C is irreducible. Hence, $\nu(|C|) \geq 0$ and, by formula (2.2), one has $\widetilde{C}^2 \geq g_{\widetilde{C}} - 1$. □

5. The four conjectures. In the previous sections we introduced two new conjectures for the characterization of special linear systems in the planar case. At this point it is natural to ask if these conjectures are equivalent to the Segre and Harbourne-Hirschowitz ones. The answer is given in the following

Theorem 5.1. *Conjectures* (SC), (HHC), (NSEC) *and* (CSEC) *are equivalent.*

Proof. First of all, we recall that the equivalence between (SC) and (HHC) is proved in [**7**]. Then we just need to prove the following implications:

$$\begin{gathered}\text{(HHC)} \Longrightarrow \text{(NSEC)} \Longrightarrow \text{(SC)} \\ \text{(HHC)} \Longrightarrow \text{(CSEC)} \Longrightarrow \text{(SC)}.\end{gathered}$$

[**(HHC) $\Rightarrow$ (NSEC)**]. Suppose that the Harbourne-Hirschowitz conjecture holds. Let $\mathcal{L}$ be a special system. Then it splits as $\mathcal{L} = \sum_{i=1}^{t} N_i C_i + \mathcal{M}$, where $\nu(\mathcal{M}) \geq 0$ and there is at least one index j such that $N_j > 1$. After a permutation in the indexes we can suppose that $N_i > 1$ for $i = 1, \ldots, s$, $s \leq t$. Thus, we can write the

$(-1)-$configuration $C = \sum_{i=1}^{t} N_i C_i$ appearing in $\mathcal{L}$ as

$$C = \sum_{i=1}^{s} N_i C_i + \sum_{i=s+1}^{t} C_i.$$

At this point it is enough to show that $\sum_{i=1}^{s} N_i C_i$ is an $(N_1, \dots, N_s)$-special effect configuration for $\mathcal{L}$. By formula (3.4) each (-1)-curve C_j with $N_j > 1$ increases the virtual dimension of the residual system by

$$\nu\Big(\mathcal{L} - \sum_{i=1}^{j} N_i C_i\Big) = \nu\Big(\mathcal{L} - \sum_{i=1}^{j-1} N_i C_i\Big) + \binom{N_i}{2}.$$

Thus, if $N_j > 1$, then C_i has the N_j-special effect property for $\mathcal{L} - \sum_{i=1}^{j-1} N_i C_i$. Finally, we can observe that $\mathcal{M} = \mathcal{L} - \sum_{i=1}^{t} N_i C_i$. By hypothesis on the (-1)-special system, we know that $\nu(\mathcal{M}) \geq 0$. Moreover the C_i's are fixed for $i = s+1, \dots, t$; hence, one has

$$\nu\Big(\mathcal{L} - \sum_{i=1}^{s} N_i C_i\Big) = \nu\Big(\mathcal{L} - \sum_{i=1}^{t} N_i C_i\Big) = \nu(\mathcal{M}) \geq 0,$$

and we can conclude that $C = \sum_{i=1}^{s} N_i C_i$ is an $(N_1, \dots, N_s)$-special effect configuration for $\mathcal{L}$. Then $\mathcal{L}$ is numerically special.

[**(HHC)** $\Rightarrow$ **(CSEC)**]**.** Suppose that the Harbourne-Hirschowitz conjecture holds. As in the previous case, we prove that a (-1)-curve appearing in a (-1)-special system and splitting off with at least multiplicity two is an h^1-special effect curve. Let $\mathcal{L}$ be a special system. Then there is at least a (-1)-curve C such that $\mathcal{L} \cdot C < -N$, $N > 1$. Then $h^0(\mathcal{L}_{|C}) = 0$ and, by Riemann-Roch, $h^1(\mathcal{L}_{|C}) = N - 1 > 0$ so that conditions (a) and (c) of Definition 4.1 are satisfied. At this point it is important to observe that $\mathcal{L} - C$ could be special. However the speciality of $\mathcal{L} - C$ has no effect on $h^0(\mathcal{L} - C)$. In fact if $\mathcal{L} - C$ is nonspecial, then $\mathcal{L} - C$ contains the residual system $\mathcal{M}$ and, by definition of the (-1)-special system, $\nu(\mathcal{M}) \geq 0$ so that $h^0(\mathcal{L} - C) \neq 0$. If $\mathcal{L} - C$ is special, then, by (2.3) we surely have $h^0(\mathcal{L} - C) \neq 0$. Hence condition (b) is satisfied.

[**(NSEC)** $\Rightarrow$ **(SC)**]**.** Suppose that the Numerical Special Effect conjecture holds. Let $\mathcal{L}$ be a special system. Then there is an $(\alpha_1, \dots, \alpha_t)$-special effect configuration or an α-special effect curve for $\mathcal{L}$. We prove only the case in which there is a special effect configuration for $\mathcal{L}$, since the other one is similar.

Let $X = \sum_{i=1}^{t} \alpha_i Y_i$ be the special effect configuration. It is enough to fix our attention on Y_1. Since, by hypothesis, $\nu(\mathcal{L} - \sum_{i=1}^{t} \alpha_i Y_i) \geq 0$, one has $h^0(\mathcal{L} - \alpha_1 Y_1) \geq 1$ and we can apply Lemma 3.5. Thus $Y_1^2 \leq -1$ and, by Lemma 3.4, we have

$$\mathcal{L} \cdot Y_1 < \frac{(\alpha_1 + 1)}{2} \widetilde{Y_1}^2 < -1.$$

Thus Y_1 is a fixed multiple component of $\mathcal{L}$ and Segre's conjecture holds.

[**(CSEC)** $\Rightarrow$ **(SC)**]. Suppose that the Cohomological Special Effect conjecture holds. Let $\mathcal{L}$ be a special system. Then there exists an h^1-special effect curve Y for $\mathcal{L}$. By condition (b) of Definition 4.1 we know that Y splits from $\mathcal{L}$, and it is enough to show that Y splits off at least with multiplicity 2. Since Y is irreducible, we have $Y^2 \geq g - 1$ where g is the genus of Y. By Riemann-Roch and $h^0(\mathcal{L}_{|Y}) = 0$ we have $\mathcal{L} \cdot Y = g - 1 - h^1(\mathcal{L}_{|Y})$. Then we compute

$$(\mathcal{L} - Y) \cdot Y = \mathcal{L} \cdot Y - Y^2 \leq g - 1 - h^1(\mathcal{L}_{|Y}) - (g - 1) = -h^1(\mathcal{L}_{|Y}) < 0,$$

and the claim follows. $\square$

6. First examples of special effect varieties in higher dimension. Since a curve in $\mathbf{P}^2$ is also a divisor, when we pass to analyze the case of special linear systems in $\mathbf{P}^n$, $n \geq 3$, we can pose the question if it is natural to consider special effect varieties of every codimension (i.e., not only curves or not only divisors). This more general situation is justified in [**3**] where we prove, for example, that $\mathbf{P}^s$, $1 \leq s \leq n-1$ can be a special effect varieties for a given system $\mathcal{L}$.

The definition of a special effect variety Y such that $\operatorname{codim}(Y, \mathbf{P}^n) \leq n - 2$ is more difficult than the codimension one case. Thus, here, we consider only when Y is a divisor. Obviously, in this situation, Definitions 3.1, 3.6, 3.10 and 4.1 remain the same.

Example 6.1. Let $\mathcal{L}$ be the system $\mathcal{L}_{3,4}(2^9)$ in Theorem 1.4. Consider a quadric $Q \subset \mathbf{P}^3$ through the nine points of $\mathcal{L}$. Obviously

$\nu(|Q_3|) = 0$. Moreover, one has

$$\nu(\mathcal{L} - Q) = \nu(\mathcal{L} - 2Q) = 0$$

while

$$\nu(\mathcal{L}) = -2.$$

Thus Q is a 2-special effect variety (hypersurface) for $\mathcal{L}_{3,4}(2^9)$.

Example 6.2. In the same way we can prove that the quadric $Q \subset \mathbf{P}^4$ is a 2-special effect variety for $\mathcal{L}_{4,4}(2^{14})$.

Example 6.3. Consider again the situation of Example 6.1. We prove that Q is an h^1-special effect variety for $\mathcal{L}$. Since $\mathcal{L}(-Q) \cong \mathcal{O}_{\mathbf{P}^3}(Q)$ one has

$$H^0(\mathcal{L} - Q) = 1 \quad \text{and} \quad H^i(\mathcal{L} - Q) = 0, \quad i \geq 1$$

and condition (b) is satisfied. Since we know that

$$H^0(\mathcal{L}) = 1, \quad H^1(\mathcal{L}) = 2 \quad \text{and} \quad H^i(\mathcal{L}) = 0, \quad i \geq 2,$$

we can conclude that

$$H^0(\mathcal{L}_{|Q}) = 0 \quad \text{and} \quad H^1(\mathcal{L}_{|Q}) = 2.$$

Thus, conditions (a) and (c) hold and the claim follows.

Example 6.4. In the same way we can prove that the quadric $Q \subset \mathbf{P}^4$ is an h^1-special effect variety for $\mathcal{L}_{4,4}(2^{14})$.

Remark 6.5. In the previous examples we shows that the quadrics are both α-and h^1-special effect varieties for the same system. This is not true in general. In fact, in [**2**] we show that a plane $\pi \subset \mathbf{P}^3$ is a 1-special effect variety for $\mathcal{L} := \mathcal{L}_{3,6}(4^3)$, but it is not an h^1-special effect variety for the same system.

7. Special effect curves on surfaces. It could be interesting to extend the concept of special effect curves to surfaces different from $\mathbf{P}^2$.

We just give here some examples which show some important evidence.

Example 7.1. Hirzebruch surfaces. Let $\mathbf{F}_e$, $e \geq 0$, be the Hirzebruch surface with invariant e, i.e., such that $-e$ is the minimal self-intersection of a section of the ruling of $\mathbf{F}_e$. We have $\mathrm{Pic}(\mathbf{F}_e) \cong \mathbf{Z} \oplus \mathbf{Z}$ and we take, as a basis of $\mathrm{Pic}(\mathbf{F}_e)$, a section h of the ruling $f : \mathbf{F}_e \to \mathbf{P}^1$ with $h^2 = -e$ and a class, F, of f. Thus $h \cdot F = 1$ and $F^2 = 0$. The dimension of $H^0(\mathbf{F_e}, \mathcal{O}_{\mathbf{F_e}}(\mathbf{ah} + \mathbf{bF}))$ is given by

$$\begin{cases} 0 & \text{if } a \geq 0 \text{ and } b < 0, \\ \sum_{i=0}^{t-1}(b - ie + 1) & \text{if } 0 \leq b < te \text{ for some } t \in \mathbf{Z}, \text{ with } 0 \leq t \leq a \\ \frac{(2b+2-ae)(a+1)}{2} & \text{if } a \geq 0 \text{ and } b \geq ae - 1 \end{cases}$$

and $h^1(\mathbf{F}_e, \mathcal{O}_{\mathbf{F}_e}(ah + bF)) = 0$ if $a \geq 0$ and $b \geq ae - 1$.

We denote a system on $\mathbf{F}_e$ by $\mathcal{L}(a, b) := |ah + bF|$.

Laface, in [**11**], gives a different definition of the (-1)-special system. For that, we need the following procedure.

Given a linear system $\mathcal{L} := |ah + bF - \sum_{i=1}^{h} m_i P_i|$ on $\mathbf{F}_e$

1) if a (-1)-curve E does exist such that $-t := \mathcal{L} \cdot E < 0$, then substitute $\mathcal{L}$ with $\mathcal{L} - tE$ and go to step 1), else go to step 2).

2) if $\mathcal{L} \cdot h < 0$, then substitute $\mathcal{L}$ with $\mathcal{L} - h$ and go to step 1), else finish.

After a finite number of steps, we have a new linear system $\mathcal{M}$, i.e., the residual linear system.

Definition 7.2. Let $\mathcal{L} := |ah + bF - \sum_{i=1}^{h} m_i P_i|$ and $\mathcal{M}$ on $\mathbf{F}_e$ be as above. Then $\mathcal{L}$ is (-1)-*special* if $v(\mathcal{M}) > v(\mathcal{L})$.

Then we can state again a modified Harbourne-Hirschowitz conjecture:

Conjecture 7.3 [**11**]. *A system $\mathcal{L}(-\sum_{i=1}^{h} m_i P_i)$ on an $\mathbf{F}_e$ is special if and only if is (-1)-special.*

This time, for the speciality of a linear system $\mathcal{L}$ such that $\mathcal{L} = \sum_{i=1}^{t} N_i C_i + \mathcal{M}$, it is not enough to have $v(\mathcal{M}) \geq 0$ and $N_i \geq 2$ for at least one index i. Following the argument of the main theorem in [**11**], it is easy to construct several examples of special system in $\mathbf{F}_e$, $e \geq 4$, such that the Harbourne-Hirschowitz does not hold (see [**2**, Example 3.4.4]).

The interested reader can look at Laface's article for a deep understanding. We just recall the main results contained in it.

Proposition 7.4. *Denote by* $\mathcal{L}_e(a,b,m^h)$ *the system* $\mathcal{L}_e(a,b)(-\sum_{i=1}^h mP_i)$. *All homogeneous* (-1)*-special systems with multiplicity* $m \leq 3$ *on* $\mathbf{F}_e$ *are listed in the following table*:

system	*virt dim* $(\mathcal{L})$	*dim* $(\mathcal{L})$
$\mathcal{L}_1(4,4,2^5)$	-1	0
$\mathcal{L}_1(6,6,3^5)$	-3	0
$\mathcal{L}_5(4,21,3^{10})$	-1	0
$\mathcal{L}_6(4,24,3^{11})$	-1	0
$\mathcal{L}_e(2,2d+2e,2^{2d+e+1})$	-1	0
$\mathcal{L}_e(0,d,2^r)$	$d-3r$	$d-2r$
$\mathcal{L}_e(2,4d+3e+1,3^{2d+e+1})$	-1	0
$\mathcal{L}_e(3,3d+3e+1,3^{2d+e+1})$	1	2
$\mathcal{L}_e(3,3d+3e,3^{2d+e+1})$	-3	0
$\mathcal{L}_e(1,d+e,3^r)$	$2d+e-6r+1$	$2d+e-5r+1$
$\mathcal{L}_e(0,d,3^r)$	$d-6r$	$d-3r$

Theorem 7.5. *Every special homogeneous system of multiplicity* ≤ 3 *on an* $\mathbf{F}_e$ *surface is a* (-1)*-special system.*

After we modify the condition for α by respect to the degree of $\mathcal{L}$ and Y, we can give again the definition of α-special effect property and arrive again to state the Numerical Special Effect conjecture. One has the following

Theorem 7.6. *The Numerical Special Effect conjecture on the Hirzebruch surface holds for all special systems listed in Proposition* 7.4.

Proof. It is enough to check by hand every single case on the previous table. As an example we prove the case $\mathcal{L} := \mathcal{L}_e(0, d, 2^r)$, $d \geq 2r$. Consider the curve Y_1 of bidegree $(0,1)$ passing through one of the r points in $\mathcal{L}$, i.e., Y_1 corresponds to the system $\mathcal{L}_e(0,1,1)$ Thue one has $\nu(\mathcal{L}) = d - 3r$ and $\nu(\mathcal{L} - Y_1) = \nu(\mathcal{L} - 2Y_1) = d - 3r + 1$. If $d - 3r + 1 \geq 0$ we conclude that Y_1 is a 2-special effect curve for $\mathcal{L}$. In the other case we pass to study the system $\mathcal{L}' := \mathcal{L}_e(0, d-2, 2^{r-1})$ and we consider a new curve Y_2 passing through one of the $r-1$ points of $\mathcal{L}'$. As in the case of Y_1 we conclude that Y_2 is a 2-special effect curve for $\mathcal{L}' = \mathcal{L} - 2Y_1$. Going further we will obtain a $(2, \ldots, 2)$-special effect configuration $X = \sum_{i=1}^{r} 2Y_i$ for $\mathcal{L}$. $\square$

Consider now the Cohomological Special Effect conjecture. Unluckily it does not hold for all special systems listed in Proposition 7.4.

In fact, let $\mathcal{L}$ be the special system $\mathcal{L}_6(4, 24, 3^{11})$. We know, by [**11**], that $\mathcal{L}$ splits as $3E + h$, where E is the $(-1)-$curve corresponding to the system $\mathcal{L}_6(1, 8, 1^{11})$. By condition $h^0(\mathcal{L}_{|Y}) = 0$, we know that an h^1-special effect variety must split from $\mathcal{L}$. Thus only E and h are the candidates to be h^1-special for $\mathcal{L}$. Since $\mathcal{L} \cdot E = -1$ (in fact h "hides" the effective multiplicity of E, see [**2**] or [**11**]) one has $h^1(\mathcal{L}_{|E}) = 0$. Similarly, since $\mathcal{L} \cdot h = 0$, we have again $h^1(\mathcal{L}_{|h}) = 0$. Thus condition (c) is never satisfied and both E and h are not h^1-special effect curves for $\mathcal{L}$. $\square$

Example 7.7 (K3 surfaces). Let X be a K3 surface with $n = H^2 \in 2\mathbf{Z}$. Let $\mathcal{L} := \mathcal{L}^n(d, m_1, \ldots, m_h)$ be the system of curves $|dH|$ passing through points $P_1, \ldots, P_h$ in general position on X with multiplicities at least $m_1, \ldots, m_h$. The virtual dimension of $\mathcal{L}$ is given by

$$\nu(\mathcal{L}) = d^2 \frac{H^2}{2} - \sum_{i=1}^{h} \frac{m_i(m_i+1)}{2} + 1.$$

In [**8**], De Volder and Laface state a conjecture for linear systems on a K3 surface and, moreover, they proved it is equivalent to the Segre Conjecture, i.e., if $\mathcal{L}$ is special on X then $\mathcal{L}$ has a multiple fixed component.

Conjecture 7.8 ([**8**]). *Let $\mathcal{L}$ and X be as above.*

(i) *$\mathcal{L}$ is special if and only if $\mathcal{L} = \mathcal{L}^4(d, 2d)$ or $\mathcal{L} = \mathcal{L}^2(d, d^2)$ with $d \geq 2$;*

(ii) *if $\mathcal{L}$ is nonempty then its general divisor has exactly the imposed multiplicities at the points P_i;*

(iii) *if $\mathcal{L}$ is nonspecial and has a fixed irreducible component C then*

a) $\mathcal{L} := \mathcal{L}^2(m+1, m+1, m) = mC + \mathcal{L}^2(1,1)$ *with* $C = \mathcal{L}^2(1, 1^2)$ *or*

b) $\mathcal{L} = 2C$, $C \in \{\mathcal{L}^4(1, 1^3), \mathcal{L}^6(1,1,2), \mathcal{L}^{10}(1,3)\}$ *or*

c) $\mathcal{L} = C$.

(iv) *if $\mathcal{L}$ has no fixed component then either its general element is irreducible or $\mathcal{L} = \mathcal{L}^2(2,2)$.*

Consider the system $\mathcal{L} = \mathcal{L}^2(d, d^2)$. Its virtual dimensions are

$$\nu(\mathcal{L}) = d^2 - d(d+1) + 1 = 1 - d.$$

Let C_1 be the curve $\mathcal{L}^2(1, 1^2)$; then C_1 is a d-special effect curve for $\mathcal{L}$ since $\nu(\mathcal{L} - dC_1) = 0$. In a similar way we can prove that $C_2 := \mathcal{L}^4(1,2)$ is a d-special effect curve $\mathcal{L}^4(d, 2d)$. Moreover, we can see that $\nu(\mathcal{L} - C) = \nu(\mathcal{L})$ when C is one of the curve in cases (iii) a)–c) of the conjecture and $\mathcal{L}$ is the relative system to C.

Passing to the h^1-special effect curves, we can observe that $C_1 := \mathcal{L}^2(1, 1^2)$ and $C_2 := \mathcal{L}^4(1,2)$ are genus two curves with self-intersection equal to zero. Applying Riemann-Roch we discover that $h^0(\mathcal{L}_{|C_t}) = 0$ and $h^1(\mathcal{L}_{|C_t}) = 1$, where $\mathcal{L}$ is the relative system to C_t in case (i) in Conjecture 7.8 ($t = 1, 2$). Since $h^0(\mathcal{L} - C_i) > 0$, we conclude that systems in (i) are cohomologically special. Finally we can see that no curve C in cases (iii) a)–c) is an h^1-special effect curve. In fact, in all cases in (iii) a)–b) one has $h^i(\mathcal{L}_{|C}) = 0$, $i = 0, 1$, while the curve in (iii) c) does not fit the hypothesis in Definition 4.1, since $\mathcal{L} \cong \mathcal{O}_X(C)$.

Thus we can state the following

Theorem 7.9. *Conjecture* 7.8 *implies both Numerical and Cohomological conjectures.*

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