ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 1, 2009

GERM FIELDS FOR HARMONIZABLE SYMMETRIC STABLE PROCESSES WITH RATIONAL SPECTRAL DENSITIES

S. REZAKHAH AND A.R. SOLTANI

ABSTRACT. A Hilbert space technique to treat continuous time complex-valued strongly harmonizable symmetric α stable processes was developed in [**9, 15**]. In this work we apply the technique to prove that such a process $\{X(t), t \in \mathbf{R}\}$ will satisfy $\sum_{n=0}^{d} c_n \partial^n X(t) = \overset{\bullet}{Z}(t)$, if its spectral density is given by $\left|\sum_{n=0}^{d} c_n (i\lambda)^n\right|^{-2}$. A germ field is introduced by using the time domain constructed in the cited works. It is also proved that the germ field for such a process with rational spectral density is of finite dimension, generated by certain derivatives of the process at zero, that will be introduced. The process $Z(t)$ is fully specified as well. This work is analogous to that of [**5**] in the context of Gaussian processes.

1. Introduction. Suppose $X = \{X(t),\ t \in \mathbf{R}\}$, $\mathbf{R}$ the set of real numbers, is a complex-valued continuous time strongly harmonizable symmetric α stable process, SH(SαS)P. In this work we assume that $1 < \alpha \le 2$. It is known that $X(t)$ is the Fourier transform of a complex-valued SαS random measure with independent increments Φ,

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda} \Phi(d\lambda), \quad t \in \mathbf{R}. \tag{1.1}$$

The quantity $f(\lambda) = \|\Phi(d\lambda)\|_\alpha^\alpha / d\lambda$, where $\|\cdot\|_\alpha$ is the Schilder's norm, defines the spectral density of the process, [**3, 6, 13**]. In this work we assume that $f(\cdot)$ is a rational function. More precisely,

$$f(\cdot) = \left|\frac{q(\cdot)}{p(\cdot)}\right|^2, \tag{1.2}$$

2000 AMS *Mathematics subject classification.* Primary 60G52, Secondary 60G25.

Keywords and phrases. Strongly harmonizable stable processes, rational spectral density, germ field.
Received by the editors on September 5, 2005, and in revised form on June 1, 2006.
DOI:10.1216/RMJ-2009-39-1-271

where $q(\cdot)$ and $p(\cdot)$ are polynomials of $i\lambda$ with no common factors, the degree $(q(\cdot)) <$ degree $(p(\cdot))$ and $p(\lambda)$ has no real root.

The customary time domain of a stable process $\{X(t),\ t \in \mathbf{R}\}$, not necessarily harmonizable, is the closed linear span of $\{X(t),\ t \in \mathbf{R}\}$ under $\|\cdot\|_\alpha$, denoted by $(\mathcal{A}, \|\cdot\|_\alpha)$. The space $\mathcal{A}$ is a Banach space and consists of jointly complex-valued SαS random variables generated by the process under the Schilder norm. For a harmonizable process, it is possible to define an inner product and consequently provide an alternative time domain, namely $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$, which is a Hilbert space of jointly complex-valued SαS random variables generated by the process. The inner product is given by the covariation, $\langle X(t), X(s)\rangle_{\mathcal{S}} = [X(t), X(s)]_\alpha$; details are given below in Theorem 1.1. We have learned from a referee that the covariation on the time domain $\mathcal{A}$ is an inner product if and only if the process X is sub-Gaussian, [**13**, Proposition 2.9.3]. Therefore, the inner product given in Theorem 1.1 is not induced by the covariation, as defined in the cited reference. As it is demonstrated in Nikfar and Soltani [**9, 10**] and Soltani and Tarami [**15**], $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$ can be used to apply classical Gaussian techniques for the purpose of filtering and prediction. In this work we extend our study to the class of SH(SαS)P with rational spectral densities. Let

$$\mathcal{S}^{0+} = \bigcap_{T>0} \mathcal{S}^T \quad \text{with} \quad \mathcal{S}^T = \overline{\mathrm{sp}}^{\mathcal{S}}\{X(t) : |t| \le T\},$$

where $\overline{\mathrm{sp}}^{\mathcal{S}}$ denotes the closure of the span in the Hilbert space $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$. We call $\mathcal{S}^{0+}$ the $\mathcal{S}$-germ field of the process. Germ fields for stationary processes and random fields have intensively been studied by several authors, see [**7**] and references therein. To the best of our knowledge, no other work has been produced on germ fields of stable processes. There are, however, very few works on related issues, such as the Markov property for stable processes, see [**1, 8**]. Our aim in this article is to stimulate research work on germ fields of stable processes. As this work indicates, a germ field characterization will lead to useful structural characterizations for the process itself. The following are furnished in this article.

(i) If the spectral density is rational, given by (1.2), then the $\mathcal{S}$-germ field will be of finite dimension, generated by certain derivatives of the process at zero. Indeed, it is proved that, at each t, the process $X(t)$

possesses certain finite derivatives that fall in the time domain $\mathcal{S}$, as well as in $\mathcal{A}$.

(ii) If the spectral density $f(\lambda) = |\sum_{n=0}^{d} c_n(i\lambda)^n|^{-2}$ with real coefficients c_n, then $\sum_{n=0}^{d} c_n\partial^n X(t) = \dot{Z}(t)$, $t \in \mathbf{R}$, where $\dot{Z}$ is a complex-valued SH(SαS)P with orthogonal values in $\mathcal{S}$.

These results are analogous to that of the celebrated work of Hida [**3**] in the context of Gaussian processes. Interestingly, as convergence in $\mathcal{A}$ is implied by convergence in $\mathcal{S}$, the derived representations are also valid in $\mathcal{A}$.

The domain of integration, if not specified, is over the real line $\mathbf{R}$, and L^2 stands for the classical space of absolute square integrable complex-valued functions with respect to Lebesgue measure on $\mathbf{R}$. The Hardy space of absolute integrable complex-valued functions on $\mathbf{R}$ is denoted by H^2. For more on H^2 theory and its applications in second order processes, see [**4**]. For covariation and more on stable random vectors and processes, see [**2**].

The following theorem is brought from Soltani and Tarami [**15**]. It provides a Hilbert space as a time domain for an SH(SαS)P.

Theorem 1.1. *Let $X = \{X(t),\ t \in \mathbf{R}\}$ be a purely nondeterministic complex-valued* SH(SαS)P *given by* (1.1). *Then there is a Hilbert space of jointly* SαS *complex-valued random variables, denoted by $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$, for which the following are satisfied*:

(i) *$\mathcal{S} \subset \mathcal{A}$, as a point inclusion.*

(ii) *$\|Y\|_\alpha \le C\|Y\|_{\mathcal{S}}$, for every $Y \in \mathcal{S}$, where C is a constant independent of Y.*

(iii) *If $Y \in \mathcal{S}$, then $Y = \int g\, dM$ where $g \in L^2$ and $M(A) = \int_A 1/h^*\, d\Phi$, and $h \in H^2$ is the outer function for which $f = |h|^2$.*

(iv) *For $Y_1 = \sum_{l=1}^{n} d_l X(t_l)$ and $Y_2 = \sum_{j=1}^{m} b_j X(s_j)$,*

$$\begin{aligned}\langle Y_1, Y_2\rangle_{\mathcal{S}} &= \sum_{l,j} d_l b_j^*[X(t_l), X(s_j)]_\alpha \\ &= 2\pi \sum_{l,j} d_l b_j^* f^\vee(s_j - t_l),\end{aligned}$$

where $f^{\vee}(t) = 1/2\pi \int_{-\infty}^{\infty} f(u)e^{-itu}\,du$ *is the inverse Fourier transform of* f, *where* $*$ *stands for the complex conjugate and* $[\cdot,\cdot]_{\alpha}$ *stands for the covariation. Furthermore, if* $Y_1 = \int g_1\,dM$ *and* $Y_2 = \int g_2\,dM$, $\langle Y_1, Y_2\rangle_{\mathcal{S}} = \langle g_1, g_2\rangle_{L^2}$.

The following corollary follows immediately from Theorem 1.1 (iii).

Corollary 1.1. *Let* g_n *be a sequence in* L^2. *Then* $g_n \to g$ *in* L^2 *if and only if* $Y_n = \int g_n dM \to Y = \int g\,dM$ *in* $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$, *and only if* $Y_n \to Y$ *in* $\mathcal{A}$.

The outer function $h \in H^2$ that satisfies $f = |h|^2$ is called the outer factor of the density f in H^2. The outer factor is unique up to a multiplicative constant of absolute value one. Rational densities (rational integrable functions on $\mathbf{R}$) are intensively studied in [**4, 12**]. The construction of the outer factor of a rational density is given in [**12**, page 44]. A deep insight to the rational spectral densities is given in [**4**], from where the following facts, Lemmas 1.1–1.4, are recalled. In this work we take the outer factor to be either

$$h = \frac{p_0 p_1}{p_2}, \tag{1.3}$$

where p_0, p_1 and p_2 are polynomials of $i\lambda$ with real coefficients, with no common factors, of degrees n_0, n_1 and d, $n_0+n_1 < n_2 = d$, respectively, and the roots of p_0 lie on the line and the roots of p_1 and p_2 lie in the lower half plane; or

$$h = \frac{1}{p} \quad \text{with} \quad p(\lambda) = \sum_{n=0}^{d} c_n(-i\lambda)^n, \quad [p_0 = p_1 = 1,\ p_2 = p]. \tag{1.4}$$

Lemma 1.1. *Let* $\mathcal{L}^T = \operatorname{span} L^2 - \text{closure}\,\{e^{i\lambda t}h(\lambda) : |t| \le T\}$ *and* $\mathcal{L}^{0+} = \cap_{T>0}\mathcal{L}^T$. *Assume the function* h *is an outer rational function given by* (1.3). *Then* $\mathcal{L}^{0+}$ *will be the class of polynomials of degree less than* $d - n_0 - n_1$. *Also,* $\partial^k h^{\vee}$, $0 \le k \le d - n_0 - n_1 - 1$, *is a basis for* $\mathcal{L}^{0+}$.

Lemma 1.2. *Let $D = (2\pi)^{-1/2} p(i\partial)$ be a real differentiable operator of degree d, where the polynomial $p(\cdot)$ is given in* (1.4). *Also let $\{e_n,\ 0 \le n < d\}$ be a basis for the solutions of $D[l] = 0$ that satisfies*

$$\partial^k e_n(0+) = \begin{cases} 1 & \text{if } n = k < d \\ 0 & \text{if } n \ne k < d. \end{cases}$$

Then the solution of $D[l] = g$, with initial data $\partial^n l(0+)$, $0 \le n < d$, is

$$l(t) = \sum_{n=0}^{d-1} \partial^n l(0+) e_n(t) + \frac{1}{c_d} \int_0^t e_{d-1}(t-s) g(s)\, ds.$$

Lemma 1.3. *Suppose that D and p are as given in Lemma* 1.2 *and the outer factor h is given by* (1.4). *Then $D[h^\vee(t)] = 0$ for $t > 0$, and $\partial^n h^\vee(0+) = 0$, $0 \le n \le d-2$, and $\partial^{d-1} h^\vee(0+) = 1/c_d$.*

Lemma 1.4. *Suppose the function h is given by* (1.4). *Then $h^\vee = e_{d-1}/c_d$, where $\{e_n,\ 0 \le n < d\}$ is the basis given in Lemma* 1.2.

Proof. Apply Lemmas 1.2 and 1.3 and note that $g \equiv 0$.

We also need the following results from Soltani and Tarami [**15**]. For $f \in \mathrm{L}^2$, let $\hat{f}(u) = \int_{-\infty}^{\infty} f(t) e^{itu}\, dt$ denote the Fourier transform of f.

Theorem 1.2. *Let $X = \{X(t), t \in \mathbf{R}\}$ be a purely nondeterministic* SH(SαS)P. *Then*

$$X(t) = \int_{-\infty}^{t} h^\vee(t-s)\, dZ(s), \quad t \in \mathbf{R}, \tag{1.5}$$

in $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$, and consequently in $(\mathcal{A}, \|\cdot\|_\alpha)$, where for bounded Borel sets $\mathbf{A} \subset \mathbf{R}$, $Z(\mathbf{A}) = \int \hat{I}_{\mathbf{A}}(\lambda)\, dM(\lambda)$. Furthermore, $\langle Z(\mathbf{A}), Z(\mathbf{B})\rangle_{\mathcal{S}} = 0$, $\mathbf{A} \cap \mathbf{B} = \varnothing$, and $\mathcal{S}_t(X) = \mathcal{S}_t(\Delta Z)$, $t \in \mathbf{R}$, where $\mathcal{S}_t(X) = \overline{\mathrm{sp}}^{\mathcal{S}}\{X(s),\ s \le t\}$ and $\mathcal{S}_t(\Delta Z) = \overline{\mathrm{sp}}^{\mathcal{S}}\{Z(\mathbf{A}) : \mathbf{A} \subset (-\infty, t] \text{ and bounded}\}$.

The process $\{Z(t),\ t \in \mathbf{R}\}$ given in Theorem 1.2 is an $\mathcal{S}$-orthogonal SαS process for which $\|Z(t) - Z(s)\|_{\mathcal{S}} = t - s$, and it is the Fourier transform of the SαS process with independent increments $M(\lambda)$. Although the process $\{Z(t),\ t \in \mathbf{R}\}$ does not necessarily have independent values, but, in studying harmonizable stable processes, it is a good substitute for the noise process related to a stationary second order process. This process will play a crucial role in the subsequent section.

2. The structure of the Germ field. In this section we will prove that the Germ field $\mathcal{S}^{0+}$ is the finite-dimensional subspace generated by the SαS random variables

$$\partial^k X(0) : 0 \le k \le d - n_0 - n_1 - 1,$$

whenever the outer factor is given by (1.3). Let us first prove that these are well-defined stable random variables.

Lemma 2.1. *Let* $\{X(t),\ t \in \mathbf{R}\}$ *be a complex-valued* SH(SαS)P *with a rational spectral density whose outer factor* h *is given by* (1.3). *Then* $\partial^k X(0) : 0 \le k \le d - n_0 - n_1 - 1$ *are well-defined jointly complex-valued SαS random variables that generate the Germ field* $\mathcal{S}^{0+}$.

Proof. Since the functions $\lambda^k h(\lambda)$, $0 \le k \le d - n_0 - n_1 - 1$ are in L^2, by using the inequality

$$\left| \int \left[e^{i\lambda t} d\lambda - \sum_{k=0}^{n} \frac{(it)^k}{k!} \lambda^k \right] h(\lambda)\, d\lambda \right| \le \int \min\left\{ \frac{|t\lambda|^{n+1}}{(n+1)!}, \frac{2|t\lambda|^n}{n} \right\} h(\lambda)\, d\lambda,$$

$t \in \mathbf{R}$, we deduce from the dominated convergence theorem that

$$\partial^k h^{\vee}(t) = i^k \int \lambda^k e^{i\lambda t} h(\lambda)\, d\lambda, \quad t > 0,$$

and

$$\partial^k h^{\vee}(0+) = i^k \int \lambda^k h(\lambda)\, d\lambda$$

in L^2. Therefore, by Corollary 1.1,

$$\partial^k X(t) = i^k \int \lambda^k e^{i\lambda t} h(\lambda)\, dM, \qquad 0 \le k \le d - n_0 - n_1 - 1, \tag{2.1}$$

and

$$\partial^k X(0+) = i^k \int \lambda^k h(\lambda)\, dM,$$
$$0 \le k \le d - n_0 - n_1 - 1.$$

To characterize $\mathcal{S}^{0+}$, first note that by Theorem 1.1 (iii), $\mathcal{S}^{0+}$ is isomorphic to $\mathcal{L}^{0+}$. Then apply Lemma 1.1. The proof is complete. □

The following theorem is an SH(Sα)P analogue of the fundamental result of Hida [**5**], produced for real stationary Gaussian processes.

Theorem 2.1. *Suppose* $\{X(t),\ t \in \mathbf{R}\}$ *is an* SH(SαS)P *possessing an even density function* f *for which* $f(\lambda) = |p(\lambda)|^{-2}$, $p(\lambda)$ *is given in* (1.4). *Then*

$$\sum_{n=0}^{d} c_n \partial^n X(t) = \overset{\bullet}{Z}(t), \tag{2.2}$$

where $\overset{\bullet}{Z}$ *is the so-called generalized derivative of the SαS process* Z *given in Theorem* 1.2.

Proof. It follows from Theorem 1.2 that

$$\begin{aligned} X(t) &= \int_{-\infty}^{t} h^{\vee}(t-s)\, dZ(s) \\ &= \int_{-\infty}^{0} h^{\vee}(t-s)\, dZ(s) + \int_{0}^{t} h^{\vee}(t-s)\, dZ(s) \\ &= \int_{-\infty}^{0} h^{\vee}(t-s)\overset{\bullet}{Z}(s)\, ds + \int_{0}^{t} h^{\vee}(t-s)\overset{\bullet}{Z}(s)\, ds, \end{aligned}$$

where $\overset{\bullet}{Z}(t)$ is an $\langle\cdot,\cdot\rangle_{\mathcal{S}}$-orthogonal S$\alpha$S process in $\mathcal{S}$ and we consider it as the generalized derivative of the process $Z(t)$. Now apply Lemma 1.4 to observe that

$$\begin{aligned} X(t) = &\int_{-\infty}^{0} h^{\vee}(t-s)\overset{\bullet}{Z}(s)\, ds \\ &+ \frac{1}{c_d}\int_{0}^{t} e_{d-1}(t-s)\overset{\bullet}{Z}(s)\, ds, \quad t \in \mathbf{R}, \end{aligned}$$

where $\{e_n,\ 0 \le n < d\}$ is given in Lemma 1.2. Since by Lemma 1.3 $D[h^\vee] = 0$, it will follow from Lemma 1.2 that

$$\int_{-\infty}^{0} h^\vee(t-s)\overset{\bullet}{Z}(s)\,ds = \sum_{n=0}^{d-1} \partial^n X(0) e_n(t), \quad t \in \mathbf{R}.$$

The existence of $\partial^n X(0)$ was established in the proof of Lemma 2.1. Therefore,

$$X(t) = \sum_{n=0}^{d-1} \partial^n X(0) e_n(t) + \frac{1}{c_d}\int_0^t e_{d-1}(t-s)\overset{\bullet}{Z}(s)\,ds, \quad t \in \mathbf{R},$$

giving the result.

Corollary 2.1. *Suppose $X(t)$ is an* SH(SαS)P *possessing an even density function f for which $f(\lambda) = |(p_0(\lambda)p_1(\lambda))/(p_2(\lambda))|^2$, where $p_0(\lambda)$, $p_1(\lambda)$, $p_2(\lambda)$ are given in* (1.3). *Then*

$$X(t) = \sum_{n=0}^{n_0+n_1} b_n \partial^n Y(t), \quad t \in \mathbf{R}, \tag{2.3}$$

where $\sum_{n=0}^{n_0+n_1} b_n (i\lambda)^n = p_0(\lambda)p_1(\lambda)$ and $Y(t) = \int e^{it\lambda} 1/(p_2(\lambda)) M(d\lambda)$.

Proof. Observe that

$$X(t) = \int e^{it\lambda}\frac{p_0(\lambda)p_1(\lambda)}{p_2(\lambda)} M(d\lambda) = \sum_{n=0}^{n_0+n_1} b_n \int (i\lambda)^n e^{it\lambda}\frac{1}{p_2(\lambda)} M(d\lambda),$$
$$t \in \mathbf{R}.$$

Apply (2.1), with $n_0 = n_1 = 0$ and $d = n_2$, and (2.2) to arrive at (2.3).

3. Discussion and concluding remarks. The work of Hida [**5**] was produced for real stationary Gaussian processes. For this reason the spectral density was taken to be even, forcing $\overline{h}(\lambda) = h(-\lambda)$ and $h^\vee$ to be real, $\overline{h}$ the complex conjugate of h. This assumption is not restrictive, and the results of the previous section also will be true, if f

is not even. This will be discussed in the next paragraph. In connection with this work, the following distinguishes the harmonizable stable processes from the stationary Gaussian processes. A real harmonizable stable process is very naive. According to Rosinski [**11**, Proposition 5.2], if $\{X_t,\ t\in\mathbf{R}\}$ is a real harmonizable SαS process, then $X_t = X_0$, $t\in\mathbf{R}$. Thus, in contrast to the Gaussian processes, a harmonizable SαS process with an even spectral density necessarily is complex. Indeed, in (1.5), $h^{\vee}$ is real, but the process $Z(t)$ is complex-valued, resulting in $X(t)$ being complex-valued.

By using the same technique as in Section 2, Theorem 1.2 can be applied to do similar work as in Rozanov [**12**, pages 49–50] for a harmonizable SαS process with a spectral density given by (1.2), giving that

$$\int_{-\infty}^{\infty}\left[p\left(\frac{d}{dt}\right)\phi(t-s)\right]X(s)\,ds = \int_{-\infty}^{\infty}\left[q\left(\frac{d}{dt}\right)\phi(t-s)\right]\dot{Z}(s)\,ds,\quad t\in\mathbf{R}, \tag{3.1}$$

for any differentiable function of compact support ϕ. Let us apply (3.1) to derive (2.2), alternatively. Let $q=1$; then, choose a sequence of infinitely differentiable functions with compact supports ϕ_n that converges to the unit mass at 0. We know that $\phi_n^{\vee}\to 1$. Thus, since f is bounded, it will follow from the dominated convergence theorem that $\int_{-\infty}^{\infty}|\phi_n^{\vee}(\lambda)-1|^2 f(\lambda)d\lambda\to 0$, $n\to\infty$. This will allow us to pass the limit through the integral signs in (3.1), arriving at (2.2).

The method presented in this article is based on the structure of the outer factor of the spectral density and $\mathbf{L}^2$ convergence. This will make the extension of the work to the cases of nonrational spectral density, treated for stationary Gaussian processes, promising. The case that the outer factor is the reciprocal of an entire function, discussed in [**4**, page 128], can be worked out for harmonizable SαS processes.

Similar to the $\mathcal{S}$-germ field, the $\mathcal{A}$-germ field can be defined. Our approach is only suitable for the time domain $\mathcal{S}$. The germ field under the Schilder norm is larger than the germ field under the $\mathbf{L}^2$ norm. We believe an approach different from the one presented here is needed to characterize the germ field under the Schilder norm.

Acknowledgments. The authors are grateful to a referee for valuable comments and encouragement.

REFERENCES

1. R. Adler, S. Cambanis and G. Samorodnitsky, *Stable Markov processes*, Stochastic Proc. Appl. **34** (1990), 1–17.

2. S. Cambanis, *Complex symmetric stable variables and processes*, in *Contributions to statistics: Essays in honor of Norman L. Johnson*, P.K. Sen, ed., North Holland, New York, 1982.

3. S. Cambanis and A.R. Soltani, *Prediction of stable processes: Spectral and moving average representations*, Z. Wahr. Verw. Geb. **66** (1984), 593–612.

4. H. Dym and H.P. McKean, *Gaussian processes, function theory, and the inverse spectral problem*, Academic Press Inc., New York, 1976.

5. T. Hida, *Canonical representations of Gaussian processes and their applications*, Mem. Coll. Sci. Univ. Kyoto **33** (1960), 109–155.

6. A. Janicki and A. Weron, *Simulation and chaotic behavior of α-stable stochastic processes*, Marcel Dekker Inc., New York, 1994.

7. V.S. Mandrekar, *Germ-field Markov property for multiparameter processes*, Seminar de Probability X, Lecture Notes Math. **511**, Springer-Verlag, Berlin, 1976.

8. V.S. Mandrekar and B. Thelen, *On multiple Markov* SαS *processes*, in *Stable processes and related topics,* S. Cambanis, et al., eds., Birkhauser, Boston, 1991.

9. M. Nikfar and A.R. Soltani, *A characterization and moving average representation for stable harmonizable processes*, J. Appl. Math. Stochastic Anal. **9** (1996), 263–270.

10. ———, *On regularity of certain stable processes*, Bull. Iranian Math. Soc. **23** (1997), 13–22.

11. J. Rosinski, *On the structure of stationary stable processes*, Ann. Probab. **23** (1995), 1163–1187.

12. Yu.A. Rozanov, *Stationary random processes*, Holden-Day, San Francisco, 1967.

13. G. Samorodnitsky and M. Taqqu, *Stable non-Gaussain random processes*, Chapman & Hall, New York, 1994.

14. M. Schilder, *Some structure theorems for the symmetric stable laws*, Ann. Math. Statist. **41** (1970), 412–421.

15. A.R. Soltani and B. Tarami, *Hilbert spaces formed by strongly harmonizable stable processes*, Georgian Math. J. **8** (2001), 181–188.

Faculty of Mathematics and Computer Sciences, Amirkabir University of Technology, 424 Hafez Avenue, Tehran, Iran
Email address: rezakhah@aut.ac.ir, rezakhah@cic.aku.ac.ir

Department of Statistics and OR, Faculty of Sciences, Kuwait University, P.O. Box 5605, Safat–13060, State of Kuwait and Department of Statistics, College of Science, Shiraz University, Shiraz 71454, Iran
Email address: soltani@kuc01.kuniv.edu.kw