

A NESTED EMBEDDING THEOREM FOR HARDY-LORENTZ SPACES WITH APPLICATIONS TO COEFFICIENT MULTIPLIER PROBLEMS

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ABSTRACT. We prove a nested embedding theorem for Hardy-Lorentz spaces and use it to find coefficient multiplier spaces of certain nonlocally convex Hardy-Lorentz spaces into various target spaces such as Lebesgue sequence spaces, other Hardy spaces and analytic mixed norm spaces.

1. Introduction. In this paper we characterize coefficient multipliers between certain types of analytic function spaces on the open unit disk. We are primarily concerned with multipliers having one of the nonlocally convex Hardy-Lorentz spaces $H^{p,q}$, $0 < p < 1$, $0 < q \leq \infty$, for the domain space. For such multipliers we will consider a variety of target spaces including Lebesgue sequence spaces, other Hardy spaces and various analytic function spaces of mixed norm type. Our method depends upon a nested embedding theorem for Hardy-Lorentz spaces (Theorem 4.1) obtained through interpolation from embedding theorems of Hardy and Littlewood and of Flett. Thus, the strategy is to trap $H^{p,q}$ between a pair of mixed norm spaces of Bergman-type and then deduce multiplier results for $H^{p,q}$ from corresponding known multiplier results for the endpoint spaces.

The paper is organized as follows. In Section 2 we define the Hardy-Lorentz spaces and the analytic mixed norm spaces. Also included in this section are some results from interpolation, fractional calculus and H^p -theory needed for the sequel. Our primary references for Lorentz spaces and Hardy spaces are [5, 10], respectively. Section 3 covers preliminary material on coefficient multipliers. In Section 4 we state and prove the embedding theorem for $H^{p,q}$. We then indicate how this theorem may be used to obtain the duality results of [33]. In

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addition, we determine the Abel dual of $H^{p,\infty}$. In Sections 5, 6 and 7, respectively, we find multipliers of $H^{p,q}$ into the Lebesgue sequence spaces l^s , into mixed norm spaces of Bergman-type and into certain Hardy spaces. In Section 8 we discuss the case when the target space is an analytic Lipschitz or Zygmund space, a Bloch space, or BMOA.

Throughout the paper \mathbf{D} will denote the open unit disk in the complex plane and \mathbf{T} will denote its boundary. The symbol $H(\mathbf{D})$ is used to denote the space of analytic functions on \mathbf{D} . If X and Y are topological spaces with $X \subset Y$, we write $X \hookrightarrow Y$ to indicate continuous inclusion. All vector spaces are assumed to be complex. By a Frechet space we mean a locally convex F -space. If E is a topological vector space, then E^* denotes the topological dual space of E consisting of all continuous linear functionals on E . The symbol $A \sim B$ is used to indicate the existence of absolute positive constants C_j , $j = 1, 2$, such that $C_1 \leq A/B \leq C_2$.

2. Hardy-Lorentz spaces and mixed norm spaces. Let m denote normalized Lebesgue measure on \mathbf{T} , and let $L^0(m)$ be the space of complex-valued Lebesgue measurable functions on \mathbf{T} . For $f \in L^0(m)$ and $s \geq 0$, we write $\lambda_f(s) = m(\{z \in \mathbf{T} : |f(z)| > s\})$ for the distribution function and $f^*(s) = \inf(\{t \geq 0 : \lambda_f(t) \leq s\})$ for the decreasing rearrangement of $|f|$, each taken with respect to m . Let $0 < p, q \leq \infty$. For the reader's convenience, we recall the definition of the Lorentz spaces $L^{p,q}(m)$. The Lorentz functional $\|\cdot\|_{p,q}$ is defined at $f \in L^0(m)$ by $\|f\|_{p,q} = (\int_0^1 [f^*(s)s^{1/p}]^q (ds/s))^{1/q}$ for $0 < q < \infty$ and $\|f\|_{p,\infty} = \sup_{s>0} [f^*(s)s^{1/p}]$. The corresponding Lorentz space is $L^{p,q}(m) = \{f \in L^0(m) : \|f\|_{p,q} < \infty\}$. Since $\|f\|_{p,p} = \|f\|_p$, where $\|f\|_p$ denotes the standard L^p -functional on $L^0(m)$, the Lorentz spaces form a 2-parameter array $\{(L^{p,q}(m), \|\cdot\|_{p,q})\}_{0 < p, q \leq \infty}$ of quasi-Banach spaces containing the Lebesgue space scale $\{(L^p(m), \|f\|_p)\}_{0 < p \leq \infty}$ as the main diagonal. Inclusions among the Lorentz spaces are given by

$$(2.1) \quad L^{p,q}(m) \hookrightarrow L^{p,r}(m), \quad 0 < p \leq \infty, \quad 0 < q \leq r \leq \infty,$$

and, since $m(\mathbf{T}) < \infty$,

$$(2.2) \quad L^{r,s}(m) \hookrightarrow L^{p,q}(m), \quad 0 < p < r \leq \infty, \quad 0 < q, s \leq \infty.$$

The space $L^{p,q}(m)$ is separable if and only if $q \neq \infty$. The class of functions $f \in L^{p,\infty}(m)$ satisfying $\lim_{s \rightarrow 0} [f^*(s)s^{1/p}] = 0$ is a separable

closed subspace of $L^{p,\infty}(m)$ which is denoted by $L^{p,\infty}(m)$. We observe here that for $q \neq \infty$, the space $L^{\infty,q}(m) = 0$. In the sequel we will follow the convention that in all discussions concerning the space $L^{p,q}(m)$ it is assumed that $q = \infty$ whenever $p = \infty$.

For $w \in \mathbf{D}$, and $f \in H(\mathbf{D})$, the function f_w is defined on $|z| < 1/|w|$ by $f_w(z) = f(wz)$. The space of continuous complex-valued functions on \mathbf{T} will be denoted by $C(\mathbf{T})$. The function f_w is considered as both an analytic function on the disk $|z| < 1/|w|$, and as a function in $C(\mathbf{T})$. For $0 < r < 1$, the functions f_r are called the dilations of f . Recall that the means $M_p(r, f)$ are defined in the usual way by $M_p(r, f) = (\int_{\mathbf{T}} |f_r(z)|^p dm(z))^{1/p}$, $0 < p < \infty$ and $M_\infty(r, f) = \sup_{z \in \mathbf{T}} |f_r(z)|$. The Hardy space H^p is defined as $H^p = \{f \in H(\mathbf{D}) : \|f\|_{H^p} < \infty\}$, where $\|f\|_{H^p} = \sup_{0 < r < 1} M_p(r, f)$. The Nevanlinna class N is the subclass of functions $f \in H(\mathbf{D})$ for which $\sup_{0 < r < 1} \int_{\mathbf{T}} \log^+ |f_r(z)| dm(z) < \infty$. Functions in N are known to have nontangential limits m -almost everywhere on \mathbf{T} . Consequently, every $f \in H(\mathbf{D})$ determines a boundary value function which we also denote by f . Thus, $f(z) = \lim_{r \rightarrow 1^-} f_r(z)$, m -almost all $z \in \mathbf{T}$. The Smirnov class N^+ is the subclass of N consisting of those functions f for which $\lim_{r \rightarrow 1^-} \int_{\mathbf{T}} \log^+ |f_r(z)| dm(z) = \int_{\mathbf{T}} \log^+ |f(z)| dm(z)$.

It follows from standard H^p -theory that a function $f \in H(\mathbf{D})$ belongs to H^p if and only if $f \in N^+$ with boundary value function in $L^p(m)$, in which case $\|f\|_{H^p} = \|f\|_p$, [10]. Motivated by this characterization of H^p , we define the Hardy-Lorentz space $H^{p,q}$, $0 < p, q \leq \infty$, to be the space of functions $f \in N^+$ with boundary value function in $L^{p,q}(m)$, and we put $\|f\|_{H^{p,q}} = \|f\|_{p,q}$. Then $\{(H^{p,q}, \|\cdot\|_{H^{p,q}})\}_{0 < p, q \leq \infty}$ is an array of quasi-Banach spaces of analytic functions on \mathbf{D} with the standard Hardy space scale as the main diagonal. As with $L^{p,q}(m)$, $H^{p,q}$ is separable if and only if $q \neq \infty$. The functions in $H^{p,\infty}$ with boundary value function in $L_0^{p,\infty}(m)$ form a closed separable subspace of $H^{p,\infty}$ which is denoted by $H_0^{p,\infty}$. Analogs of the inclusion relations (2.1) and (2.2) hold for the Hardy-Lorentz spaces and $H^{p,q} \hookrightarrow H_0^{p,\infty}$ for all $q \neq \infty$. The polynomials are dense in $H_0^{p,\infty}$ and in $H^{p,q}$, $q \neq \infty$. For $f \in H^{p,q}$, $q \neq \infty$, the dilations $f_r \rightarrow f$ in $H^{p,q}$ as $r \rightarrow 1^-$. If $f \in H^{p,\infty}$, then the dilations $f_r \rightarrow f$ in $H^{p,\infty}$ as $r \rightarrow 1^-$ if and only if $f \in H_0^{p,\infty}$. We note that a similar statement can be made for the disk algebra $A(\mathbf{D})$ which is defined as the subspace of H^∞ with boundary

function in $C(\mathbf{T})$. That is, the polynomials are dense in $A(\mathbf{D})$ and the dilations $f_r \rightarrow f$ in H^∞ as $r \rightarrow 1^-$ if and only if $f \in A(\mathbf{D})$, [32].

An important result in the theory of Lorentz spaces is the identification of these spaces with the intermediate spaces arising in the real interpolation theory of the Lebesgue spaces. An analytic analog of this result is given in Theorem 2.1 below. Theorem 2.1 is one of two interpolation theorems needed for the sequel. It was proved in [15] but omitted the endpoint case corresponding to H^∞ . The complete version was proved in [29], see also [44].

Theorem 2.1. *Let $0 < \theta < 1$ and for $j = 0, 1$, let $0 < p_j, q_j \leq \infty$ with $p_0 \neq p_1$.*

(i) *Set $1/p = (1 - \theta)/p_0 + \theta/p_1$. Then for every $0 < q \leq \infty$, we have, with equivalent quasinorms,*

$$(H^{p_0, q_0}, H^{p_1, q_1})_{\theta, q} = H^{p, q}.$$

(ii) *Set $1/q = (1 - \theta)/q_0 + \theta/q_1$. Then for every $0 < p < \infty$, we have, with equivalent quasinorms,*

$$(H^{p, q_0}, H^{p, q_1})_{\theta, q} = H^{p, q}.$$

The second collection of domain spaces $E \hookrightarrow H(\mathbf{D})$ that we define are spaces of mixed “norm” type. Before introducing these spaces, we describe the fractional calculus that we will be using. Let $0 < \beta < \infty$ and suppose $f \in H(\mathbf{D})$ with Taylor series representation

$$(2.3) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbf{D}.$$

The fractional derivative and fractional integral of f of order β are the functions respectively defined at $z \in \mathbf{D}$ by

$$f^{[\beta]}(z) = \sum_{n=0}^{\infty} [(\Gamma(n + \beta + 1))/n!] z^n$$

and

$$f_{[\beta]}(z) = \sum_{n=0}^{\infty} [n! / (\Gamma(n + \beta + 1))] z^n,$$

where Γ is the gamma function. The symbols D^β and D_β stand for the associated operators defined on $H(\mathbf{D})$ by $D^\beta(f) = f^{[\beta]}$ and $D_\beta(f) = f_{[\beta]}$, $f \in H(\mathbf{D})$. We adopt the convention that for $-\infty < \beta < 0$, $f^{[\beta]} = f_{[-\beta]}$, $f_{[\beta]} = f^{[-\beta]}$ and similarly for D^β and D_β . The operators D^0 and D_0 are understood to be the identity on $H(\mathbf{D})$ and $f^{[0]} = f_{[0]} = f$.

Suppose then that $0 < p, q \leq \infty$, $0 < \alpha < \infty$, and let $f \in H(\mathbf{D})$. We set $\|f\|_{H(p,q,\alpha)} = (\int_0^1 M_p(r, f)^q (1-r)^{q\alpha-1} dr)^{1/q}$, $q \neq \infty$, and $\|f\|_{H(p,\infty,\alpha)} = \sup_{0 < r < 1} [M_p(r, f)(1-r)^\alpha]$. Then the weighted mixed Bergman space $(H(p, q, \alpha))$ is defined as $H(p, q, \alpha) = \{f \in H(\mathbf{D}) : \|f\|_{H(p,q,\alpha)} < \infty\}$. We also define $H_0(p, \infty, \alpha)$ to be the subspace of functions $f \in H(p, \infty, \alpha)$ satisfying $M_p(r, f)(1-r)^\alpha \rightarrow 0$ as $r \rightarrow 1^-$. In this notation, $H(p, p, 1/p)$ is the standard Bergman space $A^p = \{f \in H(\mathbf{D}) : \int_{\mathbf{D}} |f(z)|^p d\nu(z) < \infty\}$, where ν is Lebesgue measure on \mathbf{D} . For $-\infty < \beta < \infty$, we set $\|f\|_{H(p,q,\alpha,\beta)} = \|f^{[\beta]}\|_{H(p,q,\alpha)}$. The weighted mixed Bergman-Sobolev space $H(p, q, \alpha, \beta)$ is then defined as $H(p, q, \alpha, \beta) = \{f \in H(\mathbf{D}) : \|f\|_{H(p,q,\alpha,\beta)} < \infty\}$. Similarly $H_0(p, \infty, \alpha, \beta)$ is the subspace of $H(p, \infty, \alpha, \beta)$ consisting of those functions f satisfying $f^{[\beta]} \in H_0(p, \infty, \alpha)$. The spaces $(H(p, q, \alpha), \|\cdot\|_{H(p,q,\alpha)})$ and $(H(p, q, \alpha, 0), \|\cdot\|_{H(p,q,\alpha,0)})$ are of course identical and we will continue to use the former notation when $\beta = 0$. As with $H^{p,q}$, $H(p, q, \alpha, \beta)$ is separable if and only if $q \neq \infty$. The space $H_0(p, \infty, \alpha, \beta)$ is a closed separable subspace of $H(p, \infty, \alpha, \beta)$. The polynomials are dense in $H(p, q, \alpha, \beta)$, $q \neq \infty$ and in $H_0(p, \infty, \alpha, \beta)$. If $f \in H(p, q, \alpha, \beta)$, $q \neq \infty$, then $f_r \rightarrow f$ in $H(p, q, \alpha, \beta)$ as $r \rightarrow 1^-$. On the other hand, for $f \in H(p, \infty, \alpha, \beta)$, $f_r \rightarrow f$ in $H(p, \infty, \alpha, \beta)$ as $r \rightarrow 1^-$ if and only if $f \in H_0(p, \infty, \alpha, \beta)$, see [28, 50]. The spaces $H(p, q, \alpha, \beta)$ are often called mixed norm spaces. In the sequel we will simply say $H(p, q, \alpha, \beta)$ is a Bergman-Sobolev space and $H(p, q, \alpha)$ is a Bergman space. Many authors use an equivalent definition of these spaces obtained by replacing the fractional calculus operators D^β and D_β , $0 \leq \beta < \infty$, in the definition of $H(p, q, \alpha, \beta)$ with the multiplier operators J^β and J_β defined at a function $f \in H(\mathbf{D})$ with Taylor series representation (2.3) by $J^\beta(f)(z) = \sum_{n=0}^{\infty} (n+1)^\beta a_n z^n$ and $J_\beta(f)(z) =$

$\sum_{n=0}^{\infty} (n+1)^{-\beta} a_n z^n$, $z \in \mathbf{D}$. Then the mixed norm space obtained using J^β or J_β is identical to the space $H(p, q, \alpha, \beta)$ as previously defined. For the equivalence of the fractional calculus operators with

the multiplier operators in defining the spaces $H(p, q, \alpha, \beta)$ as well as proofs of the following results, the reader is referred to [16, 48, 49].

Lemma 2.1. *Let $0 < p, q \leq \infty$, $0 < \alpha, \beta < \infty$. Then the following mappings are continuous surjective isomorphisms.*

- (i) $D^\beta : H(p, q, \alpha) \rightarrow H(p, q, \alpha + \beta)$,
- (ii) $D^\beta : H_0(p, \infty, \alpha) \rightarrow H_0(p, \infty, \alpha + \beta)$,
- (iii) $D_\beta : H(p, q, \alpha) \rightarrow H(p, q, \alpha - \beta)$ for $\beta < \alpha$,
- (iv) $D_\beta : H_0(p, \infty, \alpha) \rightarrow H_0(p, \infty, \alpha - \beta)$ for $\beta < \alpha$.

Lemma 2.1 and the definition of $H(p, q, \alpha, \beta)$ imply the following.

Lemma 2.2. *Let $0 < p, q \leq \infty$, $0 < \alpha < \infty$, $-\infty < \beta < \infty$. Then for $-\infty < \gamma < \alpha$, the following identifications hold with equivalent quasi-norms.*

- (i) $H(p, q, \alpha, \beta) = H(p, q, \alpha - \gamma, \beta - \gamma)$,
- (ii) $H_0(p, \infty, \alpha, \beta) = H_0(p, \infty, \alpha - \gamma, \beta - \gamma)$.

In particular, Lemma 2.2 implies $H(p, q, \alpha, \beta) = H(p, q, \alpha - \beta)$ and $H_0(p, \infty, \alpha, \beta) = H_0(p, \infty, \alpha - \beta)$ if $-\infty < \beta < \alpha$. Lemma 2.3 which follows represents an extension of Lemma 2.1 to the spaces $H(p, q, \alpha, \beta)$.

Lemma 2.3. *Let $0 < p, q \leq \infty$, $0 < \alpha < \infty$, $-\infty < \beta, \gamma < \infty$. Then the following mappings are continuous surjective isomorphisms.*

- (i) $D^\gamma : H(p, q, \alpha, \beta) \rightarrow H(p, q, \alpha + \gamma, \beta)$ for $\gamma > -\alpha$,
- (ii) $D^\gamma : H_0(p, \infty, \alpha, \beta) \rightarrow H_0(p, \infty, \alpha + \gamma, \beta)$ for $\gamma > -\alpha$,
- (iii) $D^\gamma : H(p, q, \alpha, \beta) \rightarrow H(p, q, \alpha, \beta - \gamma)$,
- (iv) $D^\gamma : H_0(p, \infty, \alpha, \beta) \rightarrow H_0(p, \infty, \alpha, \beta - \gamma)$.

The second interpolation result we need is for the Bergman-Sobolev spaces and is due to Fabrega and Ortega, see [14].

Theorem 2.2. *Let $0 < p < \infty$, $0 < q_j \leq \infty$, $0 < \alpha_j$, $\beta < \infty$, $j = 0, 1$, and suppose that $\alpha_0 \neq \alpha_1$. Let $0 < \theta < 1$, $0 < q \leq \infty$, and set $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$. Then we have, with equivalent quasi-norms,*

$$(H(p, q_0, \alpha_0, \beta), H(p, q_1, \alpha_1, \beta))_{\theta, q} = H(p, q, \alpha, \beta).$$

We also need the following embedding theorems. The first of these is due to Flett [16] and indicates how Bergman-Sobolev spaces embed in the standard Hardy spaces. The second result is a well-known embedding theorem of Hardy and Littlewood, see [10].

Theorem 2.3. *Let $0 < p < s < \infty$, $0 < q \leq s$, $\beta > 1/p - 1/s$. Then*

$$H(p, q, \beta + 1/s - 1/p, \beta) \hookrightarrow H^s.$$

Theorem 2.4. *Let $0 < p < s \leq \infty$, $p \leq t \leq \infty$. Then*

$$H^p \hookrightarrow H(s, t, 1/p - 1/s).$$

3. Multipliers. Let \mathbf{N}_0 denote the set of nonnegative integers, and let W denote the space of complex sequences indexed by \mathbf{N}_0 . We always consider W as being equipped with the topology of pointwise convergence. With this topology, W is a Frechet space. A topological vector space X satisfying $X \hookrightarrow W$ is called a K-space. An FK-space is a K-space which is also an F-space. In particular, spaces which are both K-spaces and Frechet spaces are FK-spaces. W is also a topological algebra under the natural product of coordinate-wise multiplication. Thus, for $w = \{w_n\}$, $\lambda = \{\lambda_n\}$, the product λw is defined by $\lambda w = \{\lambda_n w_n\}$. It will sometimes be convenient to use the symbol B for the product map so that $B(\lambda, w) = \lambda w$. Then $B : W \times W \rightarrow W$ is a continuous bilinear operator. For fixed $\lambda \in W$, we will write B_λ for the continuous linear operator $B_\lambda : W \rightarrow W$ defined

by $B_\lambda(w) = \lambda w$, $w \in W$. Suppose now that E and X are a pair of vector subspaces of W . An element $\lambda \in W$ is said to be a multiplier of E into X if $\lambda w \in X$ for every $w \in E$. The set of multipliers from E into X is denoted by either of the symbols (E, X) or E^X . Thus, $\lambda \in (E, X)$ if and only if the linear operator B_λ maps E into X . Consequently, the bilinearity of B gives $(E, X) = \cap_{w \in E} (B_w^{-1}(X))$. If E and X are FK-spaces, an argument based on the closed graph theorem shows that (E, X) , or more precisely $\{B_\lambda : \lambda \in (E, X)\}$, is a subspace of $\mathcal{L}(E, X)$, the space of continuous X -valued linear operators on E . The space (E, X) is sometimes called the X -dual of E . The second X -dual of E is the space $E^{XX} = (E^X)^X$. If $E^{XX} = E$, then E is said to be X -reflexive or X -perfect. We record some of the basic properties of multiplier spaces in the form of a lemma. We omit the obvious proof.

Lemma 3.1. *Let A, B, C, E be vector subspaces of W with $A \subset B$. Then*

- (i) $B^C \subset A^C$,
- (ii) $C^A \subset C^B$,
- (iii) $(A, C) \subset (C^E, A^E)$.

For quasi-Banach spaces $(E, \|\cdot\|_E)$, $(X, \|\cdot\|_X) \hookrightarrow W$, the operator quasi-norm is defined at an operator $L \in \mathcal{L}(E, X)$ in the standard way by $\|L\|_{\mathcal{L}(E, X)} = \sup\{\|L(w)\|_X : w \in E, \|w\|_E \leq 1\}$. Then $(\mathcal{L}(E, X), \|\cdot\|_{\mathcal{L}(E, X)})$ is a quasi-Banach space containing (E, X) as a closed subspace. In particular, (E, X) is a quasi-Banach space under the quasi-norm $\|\cdot\|_{(E, X)}$ defined at $\lambda \in (E, X)$ by $\|\lambda\|_{(E, X)} = \|B_\lambda\|_{\mathcal{L}(E, X)}$.

We regard $H(\mathbf{D})$ as a subspace of W by identifying functions in $H(\mathbf{D})$ with their Taylor coefficient sequences. Thus, a function $f \in H(\mathbf{D})$ with Taylor series representation (2.3) is identified with the sequence $a = \{a_n\}$. $H(\mathbf{D})$ is a Frechet space when equipped with the topology of uniform convergence on compact subsets of \mathbf{D} . In addition, we note that $H(\mathbf{D})$ is a K-space, and hence any F-space E satisfying $E \hookrightarrow H(\mathbf{D})$ is an FK-space. In [55] it is shown that the product map B on $W \times W$ restricts to a bilinear operator $H(\mathbf{D}) \times H(\mathbf{D}) \rightarrow H(\mathbf{D})$. The symbol c will denote the Cauchy function defined by $c(z) = (1 - z)^{-1}$, $z \in \mathbf{D}$. Since $c \in H(\mathbf{D})$, it follows that $(H(\mathbf{D}), H(\mathbf{D})) = H(\mathbf{D})$. If

$f, g \in H(\mathbf{D})$, then $B(f, g)$ is commonly denoted by $f * g$ and is called the Hadamard product of f and g . Thus, if $f, g \in H(\mathbf{D})$, with Taylor series representations $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $z \in \mathbf{D}$, then $B(f, g) = f * g \in H(\mathbf{D})$ has Taylor series representation $B(f, g)(z) = (f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$, $z \in \mathbf{D}$ and, as a sequence, $B(f, g) = f * g = \{a_n b_n\}$.

We introduce some notation. For $n \in \mathbf{N}_0$ and $z \in \mathbf{D}$, we set $u_n(z) = z^n$. Let $f \in H(\mathbf{D})$ with Taylor series representation given by (2.3). For $N \in \mathbf{N}_0$ we write $S_N(f)$ for the partial sum function

$$S_N(f)(z) = \sum_{n=0}^N a_n z^n, \quad z \in \mathbf{D}.$$

In the sequel we will be interested in the multiplier spaces (E, X) where E is a Hardy-Lorentz space and X is an FK-space, $X \hookrightarrow H(\mathbf{D})$. At this point we would like to consider (E, X) for some specific target spaces X and for a fairly general class of domain spaces E . One of our choices for X is the space $AS(\mathbf{D})$ of Abel summable sequences. Recall that the element $w = \{w_n\} \in W$ is said to be Abel summable if $\lim_{r \rightarrow 1^-} \sum_{n=0}^{\infty} w_n r^n$ exists. The space $AS(\mathbf{D})$ is a Frechet space [45], with respect to the topology induced by the family $\{\rho_n : n \in \mathbf{N}_0 \text{ or } n = \infty\}$ of semi-norms, where $\{r_n\}$ is a fixed sequence in $(0, 1)$ strictly increasing to 1, and for $n \in \mathbf{N}_0$ and $w = \{w_n\} \in AS(\mathbf{D})$,

$$\rho_{\infty}(w) = \sup_{0 < r < 1} \left| \sum_{k=0}^{\infty} w_k r^k \right|$$

and

$$\rho_n(w) = \sum_{k=0}^{\infty} |w_k| r_n^k.$$

Furthermore, we have $AS(\mathbf{D}) \hookrightarrow H(\mathbf{D})$ and hence $AS(\mathbf{D})$ is an FK-space. The $AS(\mathbf{D})$ -dual of a vector subspace E of W is known as the Abel dual of E and will be denoted by E^a . The second Abel dual of E is the space $(E^a)^a$ and will be denoted by E^{aa} . If $E^{aa} = E$, then

E is said to be Abel reflexive. Note that the functional Ψ on $AS(\mathbf{D})$ defined by

$$(3.1) \quad \Psi(a) = \lim_{r \rightarrow 1^-} \sum_{n=0}^{\infty} a_n r^n, \quad a = \{a_n\} \in AS(\mathbf{D})$$

belongs to $AS(\mathbf{D})^*$.

Propositions 3.2 through 3.4 below describe the relationship between the spaces E^* , (E, H^∞) and E^a for a certain general type of FK-space E . First we need the following.

Proposition 3.1. *Suppose that E is an FK-space satisfying*

- (i) $E \hookrightarrow H(\mathbf{D})$,
- (ii) $u_n \in E$ for all $n \in \mathbf{N}_0$,
- (iii) $c_w \in E$ for all $w \in \mathbf{D}$,
- (iv) *For all $w \in \mathbf{D}$, $\{S_N(c_w)\}$ converges to c_w in E as $N \rightarrow \infty$.*

Then for any topological vector space X , operator $T \in \mathcal{L}(E, X)$, and $w \in \mathbf{D}$, the X -valued series

$$(3.2) \quad \sum_{n=0}^{\infty} x_n w^n, \quad x_n = T(u_n), \quad n \in \mathbf{N}_0,$$

converges in X to $T(c_w)$.

Proof. The partial sums of the series (3.2) satisfy

$$(3.3) \quad \sum_{n=0}^N x_n w^n = T(S_N(c_w)).$$

Since T is continuous, the lemma follows from (iv) and (3.3). \square

Suppose then that E is an FK-space satisfying (i)–(iv) of Proposition 3.1, that X is a topological vector space and $T \in \mathcal{L}(E, X)$. Define a function g_T on \mathbf{D} by $g_T(w) = T(c_w)$, $w \in \mathbf{D}$. Then g_T is a well-defined X -valued function on \mathbf{D} with power series representation given by (3.2).

The function g_T is called the analytic or Cauchy transform of T . The linear operator $T \rightarrow g_T$, taking $\mathcal{L}(E, X)$ into the space of X -valued power series on \mathbf{D} will also be referred to as the analytic or Cauchy transform on $\mathcal{L}(E, X)$. Consider now the case when X is an FK-space satisfying $X \hookrightarrow H(\mathbf{D})$ and $T = B_\lambda$ for some multiplier $\lambda \in (E, X)$. For this case, if $\lambda = \{\lambda_n\}$ as a sequence of complex numbers, then $T(u_n) = B_\lambda(u_n) = \lambda u_n$. Since λu_n is identified with the sequence having λ_n in the n th entry and 0 elsewhere, we will write λ_n in place of $T(u_n)$, $n \in \mathbf{N}_0$, and g_λ in place of g_T . The other situation we will be considering is when X is the complex field and $T = \varphi \in E^*$. For this case we may form the sequence $\lambda = \{\lambda_n\}$, where $\lambda_n = \varphi(u_n)$, $n \in \mathbf{N}_0$, and we again write g_λ in place of g_T . Proposition 3.1 ensures that $g_\lambda \in H(\mathbf{D})$. But, as was previously noted, $H(\mathbf{D}) = (H(\mathbf{D}), H(\mathbf{D}))$. Since $(H(\mathbf{D}), H(\mathbf{D})) \subset (E, H(\mathbf{D}))$ by Lemma 3.1 (i), it follows that $\varphi \in E^*$ induces the multiplier $\lambda \in (E, H(\mathbf{D}))$ with analytic transform g_λ . In fact, by convention, we have the identification $\lambda \leftrightarrow g_\lambda$. Let us note here that it is possible to have $\lambda = 0$ for $\varphi \neq 0$, so that in general the analytic transform on $\mathcal{L}(E, X)$ is not one-to-one. However, we do have the following.

Proposition 3.2. *Suppose that in addition to satisfying conditions (i)–(iv) of Proposition 3.1, the FK-space E satisfies*

- (i) $f_w \in E$ for each $w \in \mathbf{D}$ and $f \in E$,
- (ii) for each $f \in E$, the set $\{f_w : w \in \mathbf{D}\}$ is bounded in E ,
- (iii) $\{S_N(f_w)\}$ converges to f_w in E for every $w \in \mathbf{D}$ and $f \in E$.

Let $\varphi \in E^$ with induced multiplier $\lambda = \{\lambda_n\}$, $\lambda_n = \varphi(u_n)$ and analytic transform g_λ . Then, for each $w \in \mathbf{D}$ and $f \in E$,*

$$(3.4) \quad \varphi(f_w) = (f * g_\lambda)(w).$$

Consequently, $\lambda \in (E, H^\infty)$. Furthermore, the analytic transform on E^ is one-to-one whenever E satisfies the additional property*

- (iv) for every $f \in E$, the dilations f_r , $0 < r < 1$, converge to f in E as $r \rightarrow 1^-$.

Proof. Let $\varphi \in E^*$, and put $\lambda = \{\lambda_n\}$, $n \in \mathbf{N}_0$. Then, by the comments made in the paragraph following the proof of Proposition 3.1,

we have $g_\varphi = g_\lambda \in (E, H(\mathbf{D}))$. If $w \in \mathbf{D}$ and $f \in E$ has Taylor series representation (2.3), then the continuity of φ and conditions (i) and (iii) of Proposition 3.2 yield

$$\begin{aligned}\varphi(f_w) &= \varphi\left(\lim_{N \rightarrow \infty} S_N(f_w)\right) \\ &= \lim_{N \rightarrow \infty} \varphi(S_N(f_w)) = \lim_{N \rightarrow \infty} \varphi\left(\sum_{n=0}^N a_n w^n u_n\right) \\ &= \lim_{N \rightarrow \infty} \sum_{n=0}^N a_n \lambda_n w^n = \lim_{N \rightarrow \infty} S_N(f * g_\lambda)(w) = f * g_\lambda(w),\end{aligned}$$

which is (3.4). Then (3.4), the continuity of φ , and condition (ii) imply $\lambda \in (E, H^\infty)$. Finally, suppose E satisfies (iv). It then follows from this property and (3.4) that if λ is the induced multiplier for $\varphi_j \in E^*$, $j = 1, 2$, we have $\varphi_j(f) = \varphi_j(\lim_{r \rightarrow 1^-} f_r) = \lim_{r \rightarrow 1^-} \varphi_j(f_r) = \lim_{r \rightarrow 1^-} (f * g_\lambda)(r)$, so that $\varphi_1 = \varphi_2$. \square

In view of the last two propositions, we may regard $E^* \subset (E, H^\infty)$ for any FK-space satisfying (i)–(iv) of Propositions 3.1 and 3.2. Note that if E is an FK-space with $A(\mathbf{D}) \hookrightarrow E$ then E satisfies conditions (ii)–(iv) of Proposition 3.1 and conditions (i)–(iii) of Proposition 3.2. Also in [42] it is shown that for, the A-spaces, which form a large class of quasi-Banach spaces $E \hookrightarrow H(\mathbf{D})$, condition (iv) of Proposition 3.2 is equivalent to the density of the polynomials in E .

Proposition 3.3. *Suppose E is an FK-space. Then $E^a \subset E^*$.*

Proof. Let $\lambda \in E^a$, and put $\varphi_\lambda = \Psi \circ B_\lambda$, where Ψ is the functional (3.1). The mapping $\lambda \rightarrow \varphi_\lambda$ is a one-to-one linear operator taking E^a into E^* , and we may identify E^a with its image in E^* . \square

Proposition 3.4. *Suppose E is an FK-space satisfying conditions (i)–(iv) of Propositions 3.1 and 3.2. Then $(E, H^\infty) = (E, A(\mathbf{D})) \subset E^a$.*

Proof. Let $\lambda = \{\lambda_n\} \in (E, H^\infty)$. Let $f \in E$ have Taylor series representation (2.3). By Proposition 3.2 (iv) and the continuity of B_λ , we have $B_\lambda(f_r) \rightarrow B_\lambda(f)$ in H^∞ . Hence, $B_\lambda(f) \in A(\mathbf{D})$. Therefore,

$(E, H^\infty) \subset (E, A(\mathbf{D}))$. By Lemma 3.1 (ii) the reverse inclusion holds. Thus, $(E, H^\infty) = (E, A(\mathbf{D}))$. Finally, the continuity of $B_\lambda(f)$ at 1 gives $\lim_{r \rightarrow 1^-} \sum_{n=0}^{\infty} a_n \lambda_n r^n = \lim_{r \rightarrow 1^-} B_\lambda(f)(r) = B_\lambda(f)(1)$. Hence, $\lambda \in E^a$. \square

Corollary 3.1. *Let E be an FK-space satisfying conditions (i)–(iv) of Propositions 3.1 and 3.2. Then $E^* = E^a = (E, H^\infty)$.*

Corollary 3.2. *Let $0 < p, q < \infty$. Then*

- (i) $(H^{p,q})^* = (H^{p,q})^a = (H^{p,q}, H^\infty)$,
- (ii) $(H_0^{p,\infty})^* = (H_0^{p,\infty})^a = (H_0^{p,\infty}, H^\infty)$.

4. A nested embedding theorem for Hardy-Lorentz spaces.

Our main tool for identifying certain multiplier spaces $(H^{p,q}, X)$ is the following nested embedding theorem for $H^{p,q}$. Its proof consists of using Theorems 2.1 and 2.2 to interpolate Theorems 2.3 and 2.4.

Theorem 4.1. *Let $0 < p_0 < p < s \leq \infty$, $0 < q \leq t \leq \infty$, and $\beta > 1/p_0 - 1/p$. Then*

- (i) $H(p_0, q, \beta + 1/p - 1/p_0, \beta) \hookrightarrow H^{p,q} \hookrightarrow H(s, t, 1/p - 1/s)$.
- (ii) $H_0(p_0, \infty, \beta + 1/p - 1/p_0, \beta) \hookrightarrow H_0^{p,\infty} \hookrightarrow H_0(s, \infty, 1/p - 1/s)$.

Proof. (i) Choose s_j , $j = 0, 1$, such that $0 < p_0 < s_0 < p < s_1 < \infty$. Let us further stipulate that for the case $\beta \leq 1/p_0$ we require that $0 < s_1 < p_0/(1 - \beta p_0)$. This ensures that $\beta > 1/p_0 - 1/s_1$. We can then apply Theorem 2.4 to obtain embeddings

$$(4.1) \quad H(p_0, s_j, \beta + 1/s_j - 1/p_0, \beta) \hookrightarrow H^{s_j}, \quad j = 0, 1.$$

Interpolation of (4.1) results in the embeddings

$$(4.2) \quad \left(H(p_0, s_0, \beta + 1/s_0 - 1/p_0, \beta), H(p_0, s_1, \beta + 1/s_1 - 1/p_0) \right)_{\theta, q} \\ \hookrightarrow (H^{s_0}, H^{s_1})_{\theta, q},$$

for all $0 < \theta < 1$, $0 < q \leq \infty$. Then the first embedding in (i) follows by choosing θ to satisfy $1/p = (1 - \theta)/s_0 + \theta/s_1$ and applying Theorems

2.1 and 2.2 to (4.2). The proof of the second embedding in (i) is similar and is in [33]. We omit the details.

(ii) Again, we prove only the first embedding in (ii) since the proof of the second embedding in (ii) is similar and is also in [33]. Thus, let $f \in H_0(p_0, \infty, \beta + 1/p - 1/p_0, \beta)$. Then $f \in H^{p, \infty}$ by Theorem 4.1 (i). In order to show $f \in H_0^{p, \infty}$, it is enough to show the dilations f_r converge to f in $H^{p, \infty}$ as $r \rightarrow 1^-$. But the functions f_r converge to f in $H(p_0, \infty, \beta + 1/p - 1/p_0, \beta)$, and this fact combined with (i) implies $f_r \rightarrow f$ in $H^{p, \infty}$. Hence, $f \in H_0^{p, \infty}$. \square

Recall that if E is a quasi-Banach space with separating dual E^* , then there exists a unique Banach space Y in which E embeds as a dense subspace and for which $Y^* = E^*$. The space Y is called the Banach envelope of E and is denoted by $[E]_1$. In [33] we identified the Banach envelopes and dual spaces of the spaces $H^{p, q}$ and $H_0^{p, \infty}$ for indices in the range $0 < p < 1$, $0 < q < \infty$. The specific result was the following.

Theorem 4.2. *Let $0 < p < 1$, $0 < q < \infty$. Set $q_* = \max(1, q)$, and let q' be the Hölder conjugate of q_* , $1/q_* + 1/q' = 1$. Then*

- (i) $[H^{p, q}]_1 = H(1, q_*, 1/p - 1)$ and $(H^{p, q})^* = H(\infty, q', 1, 1/p)$,
- (ii) $[H_0^{p, \infty}]_1 = H_0(1, \infty, 1, 1/p - 1)$ and $(H_0^{p, \infty})^* = H(\infty, 1, 1, 1/p)$.

Note that for $p = q$, (i) becomes

$$(4.3) \quad [H^p]_1 = H(1, 1, 1/p - 1) \quad \text{and} \quad (H^p)^* = H(\infty, \infty, 1, 1/p).$$

This is the well-known Duren-Romberg-Shields theorem [11]. The proof of Theorem 4.2 (i) given in [33] essentially consisted of two steps. The first step was the establishment of the second embedding in Theorem 4.1 (i). The second step was a constructive proof of the embedding

$$(H^{p, q})^* \hookrightarrow H(\infty, q', 1, 1/p).$$

The proof of Theorem 4.2 (ii) in [33] was carried out in an analogous fashion. A short proof of Theorem 4.2 can be based on Theorems 4.1 and 4.3 below. Theorem 4.3 is a general Banach envelope-duality theorem for separable Bergman-Sobolev spaces and is due to the efforts

of several authors. For statements and proofs of Theorem 4.3 for Bergman spaces in some specific cases, the reader is referred to [1–3, 6, 7, 9, 11, 17–19, 23–25, 35, 37, 39, 46, 47, 50, 52, 54, 56]. Pavlovic's paper [42] contains a very general and complete version of Theorem 4.3 for the case $\beta = 0$. To obtain the result for Bergman-Sobolev spaces, one uses the validity of Theorem 4.3 for Bergman spaces with Lemmas 2.1–2.3.

Theorem 4.3. *Let $0 < p \leq \infty$, $0 < q$, $\alpha < \infty$, $-\infty < \beta < \infty$. Let $p_0 = \min(1, p)$, $p_1 = \max(1, p)$ and $q_1 = \max(1, q)$. Let $1/p_1 + 1/p'_1 = 1/q_1 + 1/q'_1 = 1$. Then*

- (i) $([H(p, q, \alpha, \beta)]_1)_1 = H(p_1, q_1, \alpha + 1/p_0 - 1, \beta)$,
- (ii) $(H(p, q, \alpha, \beta))^* = H(p'_1, q'_1, 1, \alpha - \beta + 1/p_0)$,
- (iii) $[H_0(p, \infty, \alpha, \beta)]_1 = H_0(p_1, \infty, \alpha + 1/p_0 - 1, \beta)$,
- (iv) $(H_0(p, \infty, \alpha, \beta))^* = H(p'_1, 1, 1, \alpha - \beta + 1/p_0)$.

For $0 < p \leq \infty$, $0 < q$, $\alpha < \infty$, $-\infty < \beta < \infty$, the spaces $H(p, q, \alpha, \beta)$ and $H_0(p, \infty, \alpha, \beta)$ satisfy conditions (i)–(iv) of Propositions 3.1 and 3.2. So, by Corollary 3.1, $H(p, q, \alpha, \beta)^* = H(p, q, \alpha, \beta)^a = (H(p, q, \alpha, \beta), H^\infty)$ and similarly for $H_0(p, \infty, \alpha, \beta)$. Thus, duality and Abel duality coincide for the separable Bergman-Sobolev spaces. More explicitly, say in the case of Theorem 4.3 (ii), if $\Lambda \in H(p, q, \alpha, \beta)^*$ and $f \in H(p, q, \alpha, \beta)$ has Taylor series representation (2.3), then the proof of Theorem 4.3 (ii) shows that

$$(4.4) \quad \Lambda(f) = \lim_{r \rightarrow 1^-} \sum_{n=0}^{\infty} a_n \lambda_n r^n, \quad \lambda_n = \Lambda(u_n), \quad n \in \mathbf{N}_0.$$

Furthermore, the analytic transform g_λ of the sequence $\lambda = \{\lambda_n\}$ in (4.4) satisfies $g_\lambda \in H(p'_1, q'_1, 1, \alpha - \beta + 1/p_0)$, $\|g_\lambda\|_{H(p'_1, q'_1, 1, \alpha - \beta + 1/p_0)} \sim \|\Lambda\|_{H(p, q, \alpha, \beta)^*}$. Conversely, if $g \in H(p_1, 1, 1, \alpha - \beta + 1/p_0)$ has Taylor coefficient sequence $\lambda = \{\lambda_n\}$, then we may define Λ_g as in (4.4). The resulting functional Λ_g belongs to $H(p, q, \alpha, \beta)^*$, has analytic transform g , and satisfies

$$\|\Lambda_g\|_{H(p, q, \alpha, \beta)^*} \sim \|g\|_{H(p_1, 1, 1, \alpha - \beta + 1/p_0)}.$$

Theorem 4.3 also implies that the spaces $H(p, q, \alpha, \beta)$, $1 \leq p \leq \infty$, $1 < q < \infty$, are reflexive with the properties of reflexivity and Abel reflexivity being the same for these spaces.

To see how Theorem 4.2 follows from Theorems 4.1 and 4.3, let $0 < p < 1$, choose $0 < p_0 < p$, and take $s = 1$, $t = \max(1, q) = q_*$ in Theorem 4.1 (i) to obtain the nested embedding

$$(4.5) \quad H(p_0, q, \beta + 1/p - 1/p_0, \beta) \hookrightarrow H^{p,q} \hookrightarrow H(1, q_*, 1/p - 1).$$

Applying the functor $[\cdot]_1$ to (4.5) we find

$$[H(p_0, q, \beta + 1/p - 1/p_0, \beta)]_1 \hookrightarrow [H^{p,q}]_1 \hookrightarrow [H(1, q_*, 1/p - 1)]_1.$$

Since $H(1, q_*, 1/p - 1)$ is a Banach space,

$$(4.6) \quad [H(1, q_*, 1/p - 1)]_1 = H(1, q_*, 1/p - 1).$$

Using Theorem 4.3, we also find

$$(4.7) \quad [H(p, q, \beta + 1/p - 1/p_0, \beta)]_1 = H(1, q_*, \beta + 1/p - 1, \beta).$$

But the spaces on the righthand sides of (4.6) and (4.7) are identical by Lemma 2.2 (i). This establishes the first equality in Theorem 4.2 (i) and hence the second inequality as well via Theorem 4.3 (ii). The proof of Theorem 4.2 (ii) is similar.

Combining Theorems 4.2 and 4.3 one sees that the Banach envelopes of the spaces $H^{p,q}$ are reflexive and Abel reflexive for $0 < p < 1 < q < \infty$. Similarly, from Corollary 3.2 and Theorem 4.2, we deduce that, for $0 < p < 1$,

$$(4.8) \quad (H_0^{p,\infty})^a = (H_0^{p,\infty})^* = H(\infty, 1, 1, 1/p).$$

It then follows from (4.8), Theorem 4.3 (ii), and Lemma 2.2 (i), that

$$(4.9) \quad (H_0^{p,\infty})^{aa} = H(1, \infty, 1/p - 1).$$

That we also have $(H^{p,\infty})^a = H(\infty, 1, 1, 1/p)$ is a consequence of the following result of Shi [51].

Lemma 4.1. *Let $0 < \alpha < \infty$. Then*

- (i) $H(1, \infty, \alpha)^a = H_0(1, \infty, \alpha)^a$,
- (ii) $H(1, \infty, \alpha)^{aa} = H(1, \infty, \alpha)$.

Corollary 4.1. *Let $0 < p < 1$. Then*

- (i) $(H^{p,\infty})^a = H(\infty, 1, 1, 1/p)$,
- (ii) $(H^{p,\infty})^{aa} = H(1, \infty, 1/p - 1)$.

Proof. (i) Using Lemma 3.1 (i) twice, (4.8), Theorems 4.2, 4.3 and Lemma 4.1 (i), we obtain

$$(4.10) \quad H(1, \infty, 1/p - 1)^a \subset (H^{p,\infty})^a \subset (H_0^{p,\infty})^a = (H_0^{p,\infty})^* \\ = H_0(1, \infty, 1/p - 1)^* = H_0(1, \infty, 1/p - 1)^a = H(1, \infty, 1/p - 1)^a.$$

Since the endpoint spaces in (4.10) are the same, (i) follows from (4.8) and (4.10).

(ii) This follows from (4.9) and (4.10). \square

5. Multipliers of $H^{p,q}$ and $H_0^{p,\infty}$ into l^s , $0 < p < 1$, $0 < q < \infty$, $0 < s \leq \infty$. In this section we determine the multiplier spaces (E, X) where E is either $H^{p,q}$ or $H_0^{p,\infty}$, $0 < p < 1$, $0 < q \leq \infty$, and X is l^s , $0 < s \leq \infty$. Here l^s is the usual Lebesgue sequence space consisting of s -summable sequences in W when $s \neq \infty$ and bounded sequences in W when $s = \infty$. Actually we do a little more. If E is either $H^{p,q}$ or $H_0^{p,\infty}$, $0 < p < 1$, $0 < q < \infty$, we find (E, X) whenever X is l^s -reflexive for some $0 < s \leq \infty$. We also find the multiplier spaces $(H^{p,\infty}, X)$ for solid target spaces X . Recall that a vector subspace X of W is said to be solid if for every $x = \{x_n\}$, $y = \{y_n\} \in W$, we have $y \in X$ whenever $x \in X$ and $|y_n| \leq |x_n|$, $n \in \mathbf{N}_0$. Equivalently, X is solid if either of the conditions

$$(5.1) \quad l^\infty \subset (X, X) \quad \text{or} \quad (l^\infty, X) = X$$

are satisfied. The notation $s(X) = (l^\infty, X)$ is commonly used. In general, $s(X)$ is the largest solid subspace of X . We note here that for arbitrary spaces E and X , the multiplier space (E, X) is solid whenever the target space X is solid. Consequently, l^s -reflexive spaces are solid. Thus the result for $H^{p,\infty}$ is more general than the corresponding result for $H_0^{p,\infty}$.

The determination of (E, X) in these cases and others frequently requires using the analytic transform to identify (E, X) with a weighted

sequence space. If X is a vector subspace of W and the element $w \in W$, we define the weighted space $X_w = B_w^{-1}(X) = \{y \in W : wy \in X\}$. If X is a quasi-Banach space with quasi-norm $\|\cdot\|_X$, then $(X_w, \|\cdot\|_{X_w})$ is a quasi-Banach space where $\|y\|_{X_w} = \|yw\|_X$, $y \in X_w$. For $-\infty < \alpha < \infty$, let $w_\alpha = \{w_\alpha(n) : n \in \mathbf{N}_0\}$ be the power sequence defined by $w_\alpha(0) = 1$ and $w_\alpha = n^\alpha$ for $n \neq 0$. In this case we will write $(X_\alpha, \|\cdot\|_{X_\alpha})$ in place of $(X_w, \|\cdot\|_{X_w})$. The following lemma gives the relationship between (E, X) and (E_α, X_β) . The proof is purely algebraic, [48].

Lemma 5.1. *Let E and X be vector subspaces of W , and let $-\infty < \alpha, \beta < \infty$. Then*

$$(E_\alpha, X_\beta) = (E_{\alpha-\beta}, X) = (E, X_{\beta-\alpha}) = (E, X)_{\beta-\alpha}.$$

Of special interest to us are the dyadically blocked sequence spaces $l(p, q)$ and their weighted analogs. These spaces are defined as follows. Let $0 < p, q \leq \infty$, set $I_0 = \{0\}$, and for $n \in \mathbf{N}_0$, $n > 0$, set $I_n = \mathbf{N}_0 \cap [2^{n-1}, 2^n)$. Then $l(p, q)$ is the subspace of W consisting of elements $x = \{x_n\}$ such that $\|x\|_{l(p, q)} < \infty$, where $\|x\|_{l(p, q)} = \| \{ \|x_k\|_{l^p} \}_{k \in I_n} \|_{l^q}$. For $-\infty < \alpha < \infty$, we write $(l(p, q, \alpha), \|\cdot\|_{l(p, q, \alpha)})$ for the weighted space $(l(p, q)_\alpha, \|\cdot\|_{l(p, q)_\alpha})$. The spaces $(l(p, q, \alpha), \|\cdot\|_{l(p, q, \alpha)})$ are quasi-Banach spaces and $(l(p, p, 0), \|\cdot\|_{l(p, p, 0)}) = (l^p, \|\cdot\|_{l^p})$, where $\|\cdot\|_{l^p}$ is the standard quasi-norm on l^p . Furthermore, $l(p, q, \alpha) \hookrightarrow l_\alpha^\infty \hookrightarrow H(\mathbf{D})$, and hence $l(p, q, \alpha)$ is also an FK-space. The multipliers between these spaces are well known. The following result is due mainly to Kellogg [31], see also [26–28, 51]. Before stating the theorem we introduce notation. Let $0 < q, s \leq \infty$. Then $q * s$ is the extended real number defined by $q * s = s$ if $q = \infty$, $q * s = qs/(q - s)$ if $0 < s < q < \infty$, and $q * s = \infty$ if $0 < q \leq s \leq \infty$.

Theorem 5.1. *Let $0 < p, q, r, s \leq \infty$, $-\infty < \alpha, \beta < \infty$. Then*

$$(l(p, q, \alpha), l(r, s, \beta)) = l(p * r, q * s, \beta - \alpha).$$

We remark that a consequence of Theorem 5.1 is that $l(p, q, \alpha)$ is l^s -reflexive for $-\infty < \alpha < \infty$ and $0 < s \leq p, q \leq \infty$. The space

c_0 , of null sequences, is an example of a space which fails to be l^s -reflexive for every $0 < s \leq \infty$. In addition to Theorem 5.1 we need a lemma. Lemma 5.2 is due to Aleksandrov [2] but may also be seen to follow from Theorem 4.1. In the lemma the spaces $H^{p,\infty}$ and $H_0^{p,\infty}$ are considered as sequence spaces of Taylor coefficients.

Lemma 5.2. *Let $0 < p < 1$. Then*

- (i) $H^{p,\infty} \hookrightarrow l_{1-1/p}^\infty$,
- (ii) $H_0^{p,\infty} \hookrightarrow (c_0)_{1-1/p}$.

Theorem 5.2. *Let $0 < p < 1$, and let X be a solid FK-space satisfying $X \hookrightarrow H(\mathbf{D})$. Then*

$$(H^{p,\infty}, X) = X_{1/p-1}.$$

Proof. Since X is solid, it follows that $X_{1/p-1}$ is solid. Therefore, using Lemma 5.2, we find

$$(5.2) \quad X_{1/p-1} = (l^\infty, X_{1/p-1}) = (l_{1-1/p}^\infty, X).$$

Then the inclusion

$$(5.3) \quad X_{1-1/p} \subset (H^{p,\infty}, X)$$

results from (5.2) and Lemma 5.1.

We obtain the reverse inclusion of (5.3) as follows. Let

$$(5.4) \quad g(z) = (1 - z)^{-1/p}, \quad z \in \mathbf{D}.$$

Then, for $0 < t < 1$,

$$(5.5) \quad g^*(t) \sim t^{-1/p} \quad \text{and} \quad g \in H^{p,\infty}.$$

The Taylor coefficient sequence of g is

$$(5.6) \quad \left\{ \frac{\Gamma(n + 1/p)}{\Gamma(1/p)n!} \right\}.$$

A well-known consequence of Stirling's formula is that

$$(5.7) \quad \left\{ \frac{n^{1/p-1} n!}{\Gamma(n+1/p)} \right\} \in l^\infty.$$

Therefore, we deduce the reverse inclusion of (5.3) from (5.1) and (5.5) through (5.7). \square

Corollary 5.1. *Let $0 < p < 1$, $0 < s \leq \infty$. Then $(H^{p,\infty}, l^s) = l_{1/p-1}^s$.*

Theorem 5.3. *Let $0 < p < 1$, $0 < s \leq \infty$. Then $(H_0^{p,\infty}, l^s) = l_{1/p-1}^s$.*

Proof. We prove only the case $s \neq \infty$, the other case being similar. Using Corollary 5.1 and Lemma 3.1 (i) we obtain

$$(5.8) \quad l_{1/p-1}^s = (H^{p,\infty}, l^s) \subset (H_0^{p,\infty}, l^s).$$

Next we show the reverse inclusion of (5.8) holds. Fix $\lambda = \{\lambda_n\} \in (H_0^{p,\infty}, l^s)$. Since l^s is solid there is no loss of generality in assuming $\lambda_n \geq 0$, $n \in \mathbf{N}_0$. Furthermore, the solidity of $l_{1/p-1}^s$ and (5.7), show that $\lambda \in l_{1/p-1}^s$ if and only if $\{\Gamma(n+1/p)\lambda_n/(\Gamma(1/p)n!)\} \in l^s$. Consider the operator $B_\lambda \in \mathcal{L}(H_0^{p,\infty}, l^s)$ corresponding to λ . That is, for $f \in H_0^{p,\infty}$, with Taylor series representation (2.3) we have

$$(5.9) \quad B_\lambda(f) = a\lambda = \{a_n\lambda_n\},$$

and

$$(5.10) \quad \|B_\lambda\|_{\mathcal{L}(H_0^{p,\infty}, l^s)} = \|\lambda\|_{(H_0^{p,\infty}, l^s)} < \infty.$$

Let g be the Cauchy-type function in (5.4). Then for $0 < r, t < 1$,

$$(5.11) \quad (g_r)^*(t) \sim C_1(1-r)^{-1/p}\chi_{[0,1-r)}(t) + C_2t^{-1/p}\chi_{(1-r,1)}(t)$$

where the constants C_j , $j = 1, 2$, are independent of r and t , and χ_A denotes the characteristic function of a set A . Thus, (5.5) and (5.11) imply

$$(5.12) \quad \sup_{0 \leq r < 1} \|g_r\|_{H^{p,\infty}} \leq C\|g\|_{H^{p,\infty}} < \infty.$$

For $0 \leq r < 1$, put $\Phi(r) = \|B_\lambda(g_{r^{1/s}})\|_{l^s}^s$. It follows from (5.10) and (5.12) that Φ is bounded on $[0, 1]$. From (5.6) and (5.9) we see that $g_{r^{1/s}} = \{(\Gamma(n+1/p)\lambda_n r^{n/s})/(\Gamma(1/p)n!)\}$ and $\Phi(r) = \|\{(\Gamma(n+1/p)\lambda_n r^{n/s})/(n!\Gamma(1/p))\}\|_{l^s}^s$. It therefore follows that Φ is an increasing function of r . From these observations we deduce that $\lim_{r \rightarrow 1^-} \Phi(r)$ exists; hence, the positive sequence $\{((\Gamma(n+1/p)\lambda_n)/(n!\Gamma(1/p)))^s\}$ is Abel summable and consequently belongs to l^1 . But then $\{(\Gamma(n+1/p)\lambda_n)/(n!\Gamma(1/p))\} \in l^s$ which is what we needed to show. \square

The space l^s is a rearrangement invariant quasi-Banach function space with \mathbf{N}_0 equipped with counting measure as the underlying measure space. Quasi-Banach function spaces are always solid. A rearrangement invariant quasi-Banach function space X is called maximal if every quasi-norm bounded increasing sequence in X is bounded above in X , see [30]. It is not hard to see that if X is a maximal rearrangement invariant quasi-Banach function space, then $(H_0^{p,\infty}, X) = X_{1/p-1}$. Another situation where we have $(H_0^{p,\infty}, X) = X_{1/p-1}$ is when X is l^s -reflexive. For $0 < s \leq \infty$ and an arbitrary space X , we use the notation $X^{K(s)}$ for (X, l^s) and $X^{K(s)K(s)}$ for $(X^{K(s)})^{K(s)}$. Thus, the space X is l^s -reflexive if and only if $X^{K(s)K(s)} = X$. We have the following generalization of Theorem 5.3.

Theorem 5.4. *Let $0 < p < 1$, and let X be an FK-space which is l^s -reflexive for some $0 < s \leq \infty$. Then*

$$(H_0^{p,\infty}, X) = X_{1/p-1}.$$

Proof. Since X is l^s -reflexive, $X_{1/p-1}$ is solid. Hence, Theorem 5.2 and Lemma 3.1 (i) produce the inclusion

$$(5.13) \quad X_{1/p-1} = (H^{p,\infty}, X) \subset (H_0^{p,\infty}, X).$$

To get the reverse inclusion of (5.13), we use Lemma 3.1 (ii), the l^s -reflexivity of X , and the identity

$$(5.14) \quad (l_{1/p-1}^s)^{K(s)} = l_{1-1/p}^\infty.$$

Then, using (5.2), (5.13), Theorem 5.3, Lemma 3.1 (ii), Lemma 5.1 and (5.14), we get

$$\begin{aligned} (H_0^{p,\infty}, X) &\subset (X^{K(s)}, (H_0^{p,\infty})^{K(s)}) = (X^{K(s)}, l_{1/p-1}^s) \\ &\subset (l_{1-1/p}^\infty, X^{K(s)K(s)}) = (l_{1-1/p}^\infty, X) \\ &= X_{1/p-1} \subset (H_0^{p,\infty}, X), \end{aligned}$$

and the proof is complete. \square

In Theorem 5.4, the hypothesis that X be l^s -reflexive for some $0 < s \leq \infty$ cannot be omitted.

Corollary 5.2. *Let $0 < p < 1$. Then $(H_0^{p,\infty}, c_0) = l_{1/p-1}^\infty$.*

Proof. Use Lemma 3.1 (i) and (ii), Lemma 5.2 (ii), Theorem 5.4 and the identity $l_{1/p-1}^\infty = ((c_0)_{1-1/p}, c_0)$. \square

We turn now to the study of the multiplier space $(H^{p,q}, X)$ where X is an l^s -reflexive FK-space. We need two results from the theory of mixed norm spaces. The first of these is part of the folklore. The case $q = t$ may be found in [8].

Lemma 5.3. *Let $0 < p \leq 2$, $0 < q \leq t \leq \infty$, $0 < \alpha < \infty$, $-\infty < \beta < \infty$. Set $p_0 = \min(1, p)$ and $p_1 = \max(1, p)$. Let p'_1 be the Hölder conjugate of p_1 , $1/p_1 + 1/p'_1 = 1$. Then there is the embedding*

$$H(p, q, \alpha, \beta) \hookrightarrow l(p'_1, t, 1 - 1/p_0 + \beta - \alpha).$$

The second result we need is a theorem of Pavlovic characterizing the multiplier spaces $(H(p, q, \alpha), l^s)$ for $0 < p \leq 1$, $0 < q$, $s \leq \infty$, $0 < \alpha < \infty$, [43]. See also [28]. For some special cases of the theorem, see [1, 39].

Theorem 5.5. *Let $0 < p \leq 1$, $0 < q$, $s \leq \infty$, $0 < \alpha < \infty$. Then*

$$(H(p, q, \alpha), l^s) = l(s, q * s, \alpha + 1/p - 1).$$

Lemma 2.3 may be used to obtain the following extension of Theorem 5.5.

Corollary 5.3. *Let $0 < p \leq 1$, $0 < q$, $s \leq \infty$, $0 < \alpha < \infty$, $-\infty < \beta < \infty$. Then*

$$(H(p, q, \alpha, \beta), l^s) = l(s, q * s, \alpha - \beta + 1/p - 1).$$

Duren and Shields showed that $(H^p, l^s) = l(s, \infty, 1/p - 1)$ for $0 < p < 1$, $p \leq s \leq \infty$, [10, 12, 13]. In [27], Jevtic and Pavlovic showed that

$$(H^p, l^s) = l(s, p * s, 1/p - 1)$$

for the case $0 < s < p < 1$. Theorem 5.6 below extends these results to the Hardy-Lorentz space setting.

Theorem 5.6. *Let $0 < p < 1$, $0 < q < \infty$, $0 < s \leq \infty$. Then*

$$(5.15) \quad (H^{p,q}, l^s) = l(s, q * s, 1/p - 1).$$

Proof. From Theorem 4.1 we have the embeddings

$$(5.16) \quad H(p_0, q, \beta + 1/p - 1/p_0, \beta) \hookrightarrow H^{p,q} \hookrightarrow H(1, q, 1/p - 1)$$

where $\beta > 1/p_0 - 1/p > 0$. Applying Lemma 3.1 (i) to (5.16) and using Corollary 5.3, we have

$$\begin{aligned} l(s, q * s, 1/p - 1) &= (H(1, q, 1/p - 1), l^s) \subset (H^{p,q}, l^s) \\ &\subset (H(p_0, q, \beta + 1/p - 1/p_0, \beta), l^s) \\ &= l(s, q * s, 1/p - 1), \end{aligned}$$

which establishes (5.15). \square

Theorem 5.6 generalizes to

Theorem 5.7. *Let $0 < p < 1$, $0 < q < \infty$. Let X be an FK-space which is l^s -reflexive for some $0 < s \leq \infty$, and set $t = \max(q, s)$. Then*

$$(H^{p,q}, X) = (l(\infty, t, 1 - 1/p), X).$$

Proof. Observe that, for $0 < q, s \leq \infty$,

$$(5.17) \quad (q * s) * s = t.$$

Now apply Lemma 3.1 (iii) followed by Theorem 5.6 to obtain

$$(5.18) \quad \begin{aligned} (H^{p,q}, X) &\subset (X^{K(s)}, (H^{p,q})^{K(s)}) \\ &= (X^{K(s)}, l(s, q * s, 1/p - 1)). \end{aligned}$$

Since X is l^s -reflexive, a second application of Lemma 3.1 (iii) to the last space in (5.18) together with Theorem 5.1 and (5.17) yields

$$(5.19) \quad \begin{aligned} (X^{K(s)}, l(s, q * s, 1/p - 1)) &\subset (l(s, q * s, 1/p - 1)^{K(s)}, X^{K(s)K(s)}) \\ &= (l(\infty, (q * s) * s, 1 - 1/p), X) \\ &= (l(\infty, t, 1 - 1/p), X). \end{aligned}$$

Then, starting with the last space in (5.19), use Lemma 3.1 (iii) three times, first with Lemma 5.3, then with Theorem 4.1, and finally with the Hardy-Lorentz analog of inclusion (2.1). As a result, we get

$$(5.20) \quad \begin{aligned} l(\infty, t, 1 - 1/p), X) &\subset (H(1, t, 1/p - 1), X) \subset (H^{p,t}, X) \\ &\subset (H^{p,q}, X). \end{aligned}$$

Combining (5.18)–(5.20) completes the proof. \square

Theorems 5.4 and 5.7 may be used to compute the multiplier spaces $(E, ces(s))$, where E is one of the Hardy-Lorentz spaces $H_0^{p,\infty}$ or $H^{p,q}$, $0 < p < 1$, $0 < q \leq \infty$ and, for $1 < s < \infty$, $ces(s)$ is the Cesaro sequence space consisting of sequences $\{x_k\} \in W$ satisfying $\sum_{n=1}^{\infty} ((1/n) \sum_{k=1}^n |x_k|)^s < \infty$. Since it is known that $ces(s) = l(1, s, 1/s - 1)$, [20], then we have the following result.

Corollary 5.4. *Let $0 < p < 1$, $0 < q \leq \infty$, $1 < s < \infty$. Then*

- (i) $(H^{p,q}, ces(s)) = l(1, q * s, 1/p + 1/s - 2)$,
- (ii) $H_0^{p,\infty}, ces(s) = l(1, s, 1/p + 1/s - 2)$.

Corollary 5.5. *Let $0 < p < 1$, and suppose X is an FK-space which is l^s -reflexive for some $0 < s \leq \infty$. Then for every $0 < q \leq s$,*

$$(H^{p,q}, X) = (H^{p,s}, X) = (l(\infty, s, 1 - 1/p), X).$$

Corollary 5.5 asserts that given a FK-space X which is l^s -reflexive for some $0 < s \leq \infty$ and a number $0 < p < 1$, the X -valued multiplier spaces for the Hardy-Lorentz space scale $\{H^{p,q}\}_{0 < q \leq s}$ will coincide. This is really due to the fact that the Bergman-Sobolev spaces appearing in the proof of Theorem 5.5 enjoy this property. We conclude this section with a Hardy-Lorentz analog of a well-known result of Hardy and Littlewood for H^p spaces.

Corollary 5.6. *Let $0 < p < 1$, $0 < q < \infty$. Suppose $f \in H^{p,q}$ has Taylor series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbf{D}$. Then*

$$(5.21) \quad \left(\sum_{n=1}^{\infty} n^{q(1-1/p)-1} |a_n|^q \right)^{1/q} \leq C \|f\|_{H^{p,q}}.$$

The diagonal case $p = q$ is a result of Hardy and Littlewood and is actually valid for $0 < p \leq 2$. We also mention that, for $f \in H^p$, $0 < p \leq 1$, $p < q < \infty$, the series (5.21) is known to converge, [12]. Since $\{n^{1-1/p-1/q}\}$ belongs to $l(q, \infty, 1/p - 1)$, Corollary 5.5 follows from the identification $(H^{p,q}, l^q) = l(q, \infty, 1/p - 1)$. Finally, we note that (5.21) remains valid if $H^{p,q}$ is replaced by $H(1, q, 1/p - 1)$.

6. Multipliers of $H^{p,q}$ and $H_0^{p,\infty}$, $0 < p < 1$, $0 < q < \infty$, into Bergman-Sobolev spaces. Our main tools in this section are Theorem 4.1 and the following result of Pavlovic [43], see also [28].

Theorem 6.1. *Let $0 < q, s, t \leq \infty$, $0 < p \leq \min(1, s)$, $0 < \alpha$, $\beta < \infty$. Then*

$$(6.1) \quad (H(p, q, \alpha), H(s, t, \beta)) = H(s, q * t, \beta, \alpha + 1/p - 1).$$

Note that by Lemma 2.2 the space on the righthand side of (6.1) coincides with $H(s, q * t, 1, 1/p + \alpha - \beta)$.

Corollary 6.1. *Let $0 < q, s, t \leq \infty$, $0 < p \leq \min(1, s)$, $0 < \alpha, \beta < \infty$, $-\infty < \delta, \gamma < \infty$. Then*

- (i) $(H(p, q, \alpha, \delta), H(s, t, \beta, \gamma)) = H(s, q * t, 1, 1/p + \alpha - \beta + \gamma - \delta),$
- (ii) $(H_0(p, \infty, \alpha, \delta), H(s, t, \beta, \gamma)) = H(s, t, 1, 1/p + \alpha - \beta + \gamma - \delta),$
- (iii) $(H_0(p, \infty, \alpha, \delta), H_0(s, \infty, \beta, \gamma)) = H(s, \infty, 1, 1/p + \alpha - \beta + \gamma - \delta),$
- (iv) $H(p, \infty, \alpha, \delta), H_0(s, \infty, \beta, \gamma)) = H_0(s, \infty, 1, 1/p + \alpha - \beta + \gamma - \delta),$
- (v) $(H(p, q, \alpha, \delta), H_0(s, \infty, \beta, \gamma)) = H(s, \infty, 1, 1/p + \alpha - \beta + \gamma - \delta)$ if $q \neq \infty$.

Proof. (i) Let $g \in (H(p, q, \alpha, \delta), H(s, t, \beta, \gamma))$. By Lemma 2.3 the maps D_δ and D^γ are continuous surjective isomorphisms

$$\begin{aligned} D_\delta &: H(p, q, \alpha) \longrightarrow H(p, q, \alpha, \delta) \\ D^\gamma &: H(s, t, \beta, \gamma) \longrightarrow H(s, t, \beta) \end{aligned}$$

and hence $D^\gamma \circ g \circ D_\delta \in (H(p, q, \alpha), H(s, t, \beta))$. But $D^\gamma \circ g \circ D_\delta$ is the same multiplier as $(g^{[\gamma]})_{[\delta]}$. Therefore, from (6.1), Lemmas 2.2 and 2.3, we deduce

$$\begin{aligned} (6.2) \quad & (H(p, q, \alpha, \delta), H(s, t, \beta, \gamma)) \\ & \subset D_\gamma(D^\delta((H(p, q, \alpha), H(s, t, \beta)))) \\ & = D_\gamma(D^\delta(H(s, q * t, \beta, \alpha + 1/p - 1))) \\ & = D_\gamma(H(s, q * t, \beta, \alpha - \delta + 1/p - 1)) \\ & = H(s, q * t, \beta, \alpha + \gamma - \delta + 1/p - 1). \end{aligned}$$

Similarly,

$$(6.3) \quad (H(p, q, \alpha), H(s, t, \beta)) \subset D_\delta(D^\gamma(H(p, q, \alpha, \delta), H(s, t, \beta, \gamma))).$$

Therefore, (6.3) implies

$$\begin{aligned} (6.4) \quad & H(s, q * t, \alpha + \gamma - \delta + 1/p - 1) \\ & = D_\gamma(D^\delta(H(s, q * t, \beta, \alpha + 1/p - 1))) \\ & = D_\gamma(D^\delta((H(p, q, \alpha), H(s, t, \beta)))) \\ & \subset (H(p, q, \alpha, \delta), H(s, t, \beta, \gamma)). \end{aligned}$$

Then (6.2) and (6.4) together give us (i).

(ii) This follows from (i) and the monotonicity of the means $M_k(\cdot, f)$ for $f \in H(\mathbf{D})$, $0 < k \leq \infty$.

(iii) This follows from (ii) and the fact that for every $0 < k \leq \infty$, $0 < \eta < \infty$, $-\infty < \nu < \infty$, the function $F \in H(k, \infty, \eta, \nu)$ belongs to $H_0(k, \infty, \eta, \nu)$ if and only if $F_r \rightarrow F$ in $H(k, \infty, \eta, \nu)$ as $r \rightarrow 1^-$.

(iv) First note that the inclusion

$$(6.5) \quad (H_0(s, \infty, 1, 1/p + \alpha - \beta + \gamma - \delta) \subset (H_0(p, \infty, \alpha, \gamma), H(s, \infty, \beta, \gamma)))$$

follows from (i) and the fact that for every $0 < k \leq \infty$, $0 < \eta < \infty$, $-\infty < \nu < \infty$, the function $F \in H(k, \infty, \eta, \nu)$ belongs to $H_0(k, \infty, \eta, \nu)$ if and only if $F_r \rightarrow F$ in $H(k, \infty, \eta, \nu)$ as $r \rightarrow 1^-$.

To prove the reverse inclusion of (6.5), we consider the case $\delta = \gamma = 0$ first. Let $g \in (H(p, \infty, \alpha), H_0(s, \infty, \beta))$. The Cauchy-type function $F(z) = (1 - z)^{-\alpha-1/p}$, $z \in \mathbf{D}$, belongs to $H(p, \infty, \alpha)$. Therefore $g * F \in H_0(s, \infty, \beta)$. But for any $w \in \mathbf{D}$, $g * F(w) = \Gamma(\alpha + 1/p)^{-1} g^{[\alpha+1/p-1]}(w)$. Hence, $g \in H_0(s, \infty, \beta, \alpha + 1/p - 1)$ which is equivalent to $g \in H_0(s, \infty, 1, 1/p + \alpha - \beta)$ by Lemma 2.2. For the general case we argue as in the proof of (i), using Lemma 2.3 to write

$$(H(p, \infty, \alpha, \delta), H_0(s, \infty, \beta, \gamma)) = D^\delta(D_\gamma((H(p, \infty, \alpha), H_0(s, \infty, \beta)))).$$

Then use the validity of (iv) for the case $\delta = \gamma = 0$.

(v) The proof is similar to the proof of (iv). \square

Theorem 6.2. Let $0 < q, s, t \leq \infty$, $0 < \beta < \infty$, $0 < p < \min(1, s)$, $-\infty < \gamma < \infty$. Then

- (i) $(H^{p,q}, H(s, t, \beta, \gamma)) = H(s, q * t, 1, 1/p + \gamma - \beta)$,
- (ii) $(H^{p,q}, H_0(s, \infty, \beta, \gamma)) = H(s, \infty, 1, 1/p + \gamma - \beta)$, $q \neq \infty$,
- (iii) $(H^{p,\infty}, H_0(s, \infty, \beta, \gamma)) = H_0(s, \infty, 1, 1/p + \gamma - \beta)$,
- (iv) $(H_0^{p,\infty}, H(s, t, \beta, \gamma)) = H(s, t, 1, 1/p + \gamma - \beta)$,
- (v) $(H_0^{p,\infty}, H_0(s, \infty, \beta, \gamma)) = H(s, \infty, 1, 1/p + \gamma - \beta)$.

Proof. We prove (i) only, the proofs of (ii) through (v) being similar. Let $0 < p_0 < p$ and $\delta > 1/p_0 - 1/p$. By Theorem 4.1 we have embeddings

$$(6.6) \quad H(p_0, q, \delta + 1/p - 1/p_0, \delta) \hookrightarrow H^{p,q} \hookrightarrow H(s, q, 1/p - 1/s).$$

By Corollary 6.1, the multiplier spaces

$$(H(p_0, q, \delta + 1/p - 1/p_0, \delta), H(s, t, \beta, \gamma))$$

and

$$(H(s, q, 1/p - 1/s), H(s, t, \beta, \gamma))$$

are both equal to $H(s, q * t, 1, 1/p + \gamma - \beta)$. Then (i) follows from this fact and (6.6). \square

7. Multipliers of $H^{p,q}$ and $H_0^{p,\infty}$, $0 < p < 1$, $0 < q \leq \infty$, into Hardy spaces. In this section we consider the multiplier spaces $(H^{p,q}, H^s)$ and $(H_0^{p,\infty}, H^s)$ for $0 < p < 1$, $0 < q \leq \infty$, $0 < s < \infty$. We are able to determine these spaces for the cases $0 < q \leq \min(2, s)$ and $0 < q \leq \infty$, $s = 2$. Since $H^2 = l^2$, the second case was addressed in Section 5. We restate that result in terms of Bergman-Sobolev spaces below. To do this, we need the following lemma from [8], see also [40, 43, 53].

Lemma 7.1. *Let $0 < q \leq \infty$, $0 < \alpha < \infty$, $-\infty < \beta < \infty$. Then*

$$H(2, q, \alpha, \beta) = l(2, q, \beta - \alpha).$$

Using Lemma 7.1, the identification $H^2 = l^2$, and either Corollary 5.1 or Theorem 5.6, we have the following result.

Theorem 7.1. *Let $0 < p < 1$, $0 < q \leq \infty$. Then*

$$(H^{p,q}, H^2) = H(2, q * 2, 1, 1/p).$$

We turn now to the case $0 < q \leq \min(2, s)$. First we record the following.

Theorem 7.2. *Let $0 < s \leq \infty$, $0 < p < \min(1, s)$. Then*

$$(H^p, H^s) = H(s, \infty, 1, 1/p).$$

Theorem 7.2 dates back to Hardy and Littlewood who observed that $H(s, \infty, 1, p) \subset (H^p, H^s)$ for $0 < p < 1 \leq s \leq \infty$, [20, 21]. The case $s = \infty$ corresponding to the Duren-Romberg-Shields theorem of [11] reduces to (4.3). Duren and Shields proved Theorem 7.2 for the case $0 < p < 1 \leq s < \infty$, [13]. The proof for the case $0 < p < s \leq 1$ is due to Mateljevic and Pavlovic [38]. Theorem 7.3 below represents an extension of Theorem 7.2 to $H^{p,q}$ for $0 < q \leq \min(2, s)$. We will need the following lemma.

Before stating this result we introduce some notation. For $0 < s < \infty$, the Dirichlet-type space \mathcal{D}^s is defined to be the space $H(s, s, 1, 1)$.

Lemma 7.2. *Let $0 < s \leq 2 \leq t < \infty$. Then*

- (i) $\mathcal{D}^s \hookrightarrow H^s \hookrightarrow H(s, 2, 1, 1)$,
- (ii) $H(t, 2, 1, 1) \hookrightarrow H^t \hookrightarrow \mathcal{D}^t$.

For statements and proofs of Lemma 7.2, the reader may consult [8, 17, 28, 34, 36, 41]. Recently, Baernstein, Girela, and Pelaez have shown that for all $0 < s < \infty$, $H^s \cap \mathcal{U} = \mathcal{D}^s \cap \mathcal{U}$, where \mathcal{U} is the class of univalent functions on \mathbf{D} , [4].

Theorem 7.3. *Let $0 < s < \infty$, $0 < p < \min(1, s)$ and $0 < q \leq \min(2, s)$. Then*

$$(H^{p,q}, H^s) = H(s, \infty, 1, 1/p).$$

Proof. Assume first that $0 < s \leq 2$. Then using Lemma 7.2 and Lemma 3.1 (ii), we have

$$(7.1) \quad (H^{p,q}, \mathcal{D}^s) \subset (H^{p,q}, H^s) \subset (H^{p,q}, H(s, 2, 1, 1)).$$

By Theorem 6.2 (i), both of the endpoint spaces in (7.1) are equal to $H(s, \infty, 1, 1/p)$. For the case $2 \leq s < \infty$, the reverse inclusion of (7.1) holds and the rest of the proof is exactly the same as in the first case. \square

Theorem 7.3 has the following corollary.

Corollary 7.1. *Let $0 < q \leq s \leq 2$, $0 < p < \min(1, s)$. Then*

$$(7.2) \quad (H^{p,q}, H^{s,q}) = \bigcap_{q \leq t \leq s} (H^{p,t}, H^{s,t}).$$

Proof. We prove the inclusion $(H^{p,q}, H^{s,q}) \subset \bigcap_{q \leq t \leq s} (H^{p,t}, H^{s,t})$ with the reverse conclusion being obvious. Let $g \in (H^{p,q}, H^{s,q})$. Since $q \leq s \leq 2$, we have $(H^{p,q}, H^{s,q}) \subset (H^{p,q}, H^s) = (H^{p,s}, H^s)$ by the Hardy-Lorentz analog of (2.1), Lemma 3.1 (i) and Theorem 7.3. Thus g is a bounded multiplier for

$$g : H^{p,s} \rightarrow H^s \quad \text{and} \quad g : H^{p,q} \rightarrow H^{s,q}.$$

Therefore, by interpolation, we find g is also bounded as a multiplier

$$g : (H^{p,q}, H^{p,s})_{\theta,t} \rightarrow (H^{s,q}, H^s)_{\theta,t},$$

for $0 < \theta < 1$ and $1/t = (1 - \theta)/q + \theta/s$. Then an application of Theorem 2.1 (ii) implies g is bounded as a multiplier

$$g : H^{p,t} \rightarrow H^{s,t} \text{ for all } q \leq t \leq s. \quad \square$$

8. Multipliers of $H^{p,q}$ and $H_0^{p,\infty}$, $0 < p < 1$, $0 < q \leq \infty$, into analytic Lipschitz spaces, analytic Zygmund spaces, Bloch spaces and $BMOA$. In this section we apply the results of the previous two sections to some specific target spaces belonging to the class of Bergman-Sobolev spaces. The target spaces we have in mind are the analytic Lipschitz and Zygmund spaces and the Bloch spaces. We also have some results for the case when the target space is $BMOA$. For the discussion that follows we assume $f \in H(\mathbf{D})$ and that f has nontangential limits m almost everywhere on \mathbf{T} . We denote the resulting boundary value function by the same symbol f . For $1 \leq s \leq \infty$, the moduli of continuity $\omega_s(f)(t)$ and $\Omega_s(f)(t)$ of f are defined for $t > 0$ by $\omega_s(f)(t) = \sup_{0 < |h| \leq t} \|T_h(f) - f\|_s$ and $\Omega_s(f)(t) = \sup_{0 < |h| \leq t} \|T_h(f) - 2f + T_{-h}(f)\|_s$, where T_h is the translation operator given by $T_h(f)(e^{i\theta}) = f(e^{i(\theta+h)})$. Let $0 < \alpha \leq 1$. Then f is said to belong to the analytic Lipschitz space $\Lambda_\alpha^s(\mathbf{D})$,

respectively $\lambda_\alpha^s(\mathbf{D})$, if $\omega_s(f)(t) = O(t^\alpha)$, respectively $o(t^\alpha)$, as $t \rightarrow 0^+$. If the boundary value function $f \in C(\mathbf{T})$ and $\Omega_\infty(f)(t) = O(t)$, respectively $o(t)$, as $t \rightarrow 0^+$, then f is said to belong to the analytic Zygmund space $\Lambda_\infty^s(\mathbf{D})$, respectively $\lambda_\infty^s(\mathbf{D})$. For $1 \leq s < \infty$, f is said to belong to the analytic Zygmund space $\Lambda_\infty^s(\mathbf{D})$, respectively $\lambda_\infty^s(\mathbf{D})$, if $\Omega_s(f)(t) = O(t^\alpha)$, respectively $o(t^\alpha)$, as $t \rightarrow 0^+$. Theorem 8.1 below is a collection of well-known results of Hardy and Littlewood identifying various analytic Lipschitz and Zygmund spaces as Bergman-Sobolev spaces. See [10, 57]. In order to cover the case $\alpha = 1$, we recall that for $0 < s \leq \infty$, $0 < \beta < \infty$, the Hardy-Sobolev space $H_\beta^s = \{f \in H(\mathbf{D}) : f^{[\beta]} \in H^s\}$.

Theorem 8.1. *Let $0 < \alpha < 1 \leq s \leq \infty$. Then*

- (i) $\Lambda_\alpha^s(\mathbf{D}) = H(s, \infty, 1 - \alpha, 1)$ and $\lambda_\alpha^s(\mathbf{D}) = H_0(s, \infty, 1 - \alpha, 1)$,
- (ii) $\Lambda_\infty^s(\mathbf{D}) = H(s, \infty, 1 - \alpha, 2)$ and $\lambda_\infty^s(\mathbf{D}) = H_0(s, \infty, 1 - \alpha, 2)$,
- (iii) $\Lambda_1^s(\mathbf{D}) = H_1^s$.

We combine Theorem 8.1 with the Duren-Romberg-Shields theorem to find the multipliers from $H^{p,q}$ into the analytic Lipschitz spaces $\Lambda_\alpha^\infty(\mathbf{D})$, $\lambda_\alpha^\infty(\mathbf{D})$ and analytic Zygmund spaces $\Lambda_\infty^\infty(\mathbf{D})$, $\lambda_\infty^\infty(\mathbf{D})$, $0 < \alpha$, $p < 1$, $0 < q \leq \infty$.

Corollary 8.1. *Let $0 < \alpha$, $p < 1$, $0 < q \leq \infty$. Then*

- (i) $(H^{p,q}, \Lambda_\alpha^\infty(\mathbf{D})) = (H_0^{p,\infty}, \Lambda_\alpha^\infty(\mathbf{D})) = H(\infty, \infty, 1, 1/p + \alpha) = (H^{p/(1+\alpha p)})^*$,
- (ii) For $q \neq \infty$,

$$\begin{aligned} (H^{p,q}, \lambda_\alpha^\infty(\mathbf{D})) &= (H_0^{p,\infty}, \lambda_\alpha^\infty(\mathbf{D})) \\ &= H(\infty, \infty, 1, 1/p + \alpha) = (H^{p/(1+\alpha p)})^*, \end{aligned}$$

- (iii) $(H^{p,\infty}, \lambda_\alpha^\infty(\mathbf{D})) = H_0(\infty, \infty, 1, 1/p + \alpha)$,
- (iv) $(H^{p,q}, \Lambda_\infty^\infty(\mathbf{D})) = (H_0^{p,\infty}, \Lambda_\infty^\infty(\mathbf{D})) = H(\infty, \infty, 1, 1/p + 1) = (H^{p/(1+p)})^*$,

(v) For $q \neq \infty$,

$$\begin{aligned} (H^{p,q}, \lambda_*^\infty(\mathbf{D})) &= (H_0^{p,\infty}, \lambda_*^\infty(\mathbf{D})) = H(\infty, \infty, 1, 1/p + 1) \\ &= (H^{p/(1+p)})^*, \end{aligned}$$

(vi) $(H^{p,\infty}, \lambda_*^\infty(\mathbf{D})) = H_0(\infty, \infty, 1, 1/p + 1)$.

Corollary 8.1 shows that for $0 < \alpha < 1$, the secondary index q is irrelevant with respect to the multiplier spaces $(H^{p,q}, E)$ for the target spaces $E = \Lambda_\alpha^\infty(\mathbf{D})$ or $E = \Lambda_*^\infty(\mathbf{D})$. The same is true for the target spaces $E = \lambda_\alpha^\infty(\mathbf{D})$ or $E = \lambda_*^\infty(\mathbf{D})$ provided $q \neq \infty$. A similar phenomenon occurs when the target spaces are the Bloch spaces. Let us recall that the Bloch space \mathcal{B} and the little Bloch space \mathcal{B}_0 are realized as Bergman-Sobolev spaces using the identifications $\mathcal{B} = H(\infty, \infty, 1, 1)$ and $\mathcal{B}_0 = H_0(\infty, \infty, 1, 1)$. The Bloch space analog of Corollary 8.1 is the following.

Corollary 8.2. *Let $0 < p < 1$, $0 < q \leq \infty$. Then*

- (i) $(H^{p,q}, \mathcal{B}) = (H_0^{p,\infty}, \mathcal{B}) = H(\infty, \infty, 1, 1/p) = (H^p)^*$,
- (ii) $(H^{p,q}, \mathcal{B}_0) = (H_0^{p,\infty}, \mathcal{B}_0) = H(\infty, \infty, 1, 1/p) = (H^p)^*$, $q \neq \infty$,
- (iii) $(H^{p,\infty}, \mathcal{B}_0) = H_0(\infty, \infty, 1, 1/p)$.

We observe here that the fractional derivative operator $D = D^1$ is a continuous isomorphism of the analytic Zygmund space $\Lambda_*^\infty(\mathbf{D})$, respectively $\lambda_*^\infty(\mathbf{D})$, onto \mathcal{B} , respectively \mathcal{B}_0 . Thus, Corollary 8.2 may be viewed as an isomorphic version of the Zygmund space portion of Corollary 8.1.

In contrast, the Lipschitz spaces $\Lambda_1^\infty(\mathbf{D})$ will, in general, determine different multiplier spaces $(H^{p,q}, \Lambda_1^\infty(\mathbf{D}))$ for different values of q . This is demonstrated in the next result which follows from Theorem 8.1 and Theorem 4.2.

Corollary 8.3. *Let $0 < p < 1$, $0 < q < \infty$. Then*

- (i) $(H^{p,q}, \Lambda_1^\infty(\mathbf{D})) = (H^{p/(1+p),q})^*$,
- (ii) $(H^{p,\infty}, \Lambda_1^\infty(\mathbf{D})) = (H_0^{p/(1+p),\infty})^*$.

For $1 \leq s < \infty$, we have the following analogs of Corollaries 8.1 and 8.3.

Corollary 8.4. *Let $0 < \alpha$, $p < 1 \leq s < \infty$, $0 < q \leq \infty$. Then*

$$(i) \quad (H^{p,q}, \Lambda_\alpha^s(\mathbf{D})) = (H_0^{p,\infty}, \Lambda_\alpha^s(\mathbf{D})) = H(s, \infty, 1, 1/p + \alpha) = (H^{p/(1+\alpha p)}, H^s),$$

(ii) *For $q \neq \infty$,*

$$(H^{p,q}, \lambda_\alpha^s(\mathbf{D})) = (H_0^{p,\infty}, \lambda_\alpha^s(\mathbf{D})) = H(s, \infty, 1, 1/p + \alpha) = (H^{p/(1+\alpha p)}, H^s),$$

$$(iii) \quad (H^{p,\infty}, \lambda_\alpha^s(\mathbf{D})) = H_0(s, \infty, 1, 1/p + \alpha),$$

$$(iv) \quad (H^{p,q}, \Lambda_*^s(\mathbf{D})) = (H_0^{p,\infty}, \Lambda_*^s(\mathbf{D})) = H(s, \infty, 1, 1/p + 1) = (H^{p/(1+p)}, H^s),$$

(v) *For $q \neq \infty$,*

$$(H^{p,q}, \lambda_*^s(\mathbf{D})) = (H_0^{p,\infty}, \lambda_*^s(\mathbf{D})) = H(s, \infty, 1, 1/p + 1) = (H^{p/(1+p)}, H^s),$$

$$(vi) \quad (H^{p,\infty}, \lambda_*^s(\mathbf{D})) = H_0(s, \infty, 1, 1/p + 1).$$

Corollary 8.5. *Let $0 < p < 1 \leq s < \infty$, $0 < q \leq \infty$. Then*

$$(i) \quad (H^{p,q}, \Lambda_1^s(\mathbf{D})) = (H^{p/(1+p),q}, H^s),$$

$$(ii) \quad (H_0^{p,\infty}, \Lambda_1^s(\mathbf{D})) = (H_0^{p/(1+p),\infty}, H^s).$$

The space $BMOA$ is the space of functions $f \in H(\mathbf{D})$ having nontangential limits m almost everywhere on \mathbf{T} for which the resulting boundary value function f is of bounded mean oscillation on \mathbf{T} . That is, for which

$$(8.1) \quad \sup_{I \subset \mathbf{T}} [m(I)^{-1} \| (f - f_I) \chi_I \|_1] < \infty$$

where the supremum in (8.1) is taken over all subintervals $I \subset \mathbf{T}$ and $f_I = m(I)^{-1} \int_I f(z) dm(z)$. The space $BMOA$ is not a Bergman-Sobolev space. However, we do have the following embedding, see [38].

Lemma 8.1. $H(\infty, 2, 1, 1) \hookrightarrow BMOA$.

Theorem 8.2. Let $0 < p < 1$, $0 < q \leq 2$. Then

$$(H^{p,q}, BMOA) = (H^p)^* = (H^{p,\infty}, \mathcal{B}).$$

Proof. Using the Duren-Romberg-Shields theorem, Theorem 6.2, Lemma 8.1, Lemma 3.1 (ii) and Corollary 8.2 (i), we find $(H^p)^* = H(\infty, \infty, 1, 1/p) = (H^{p,q}, H(\infty, 2, 1, 1)) \subset (H^{p,q}, BMOA) \subset (H^{p,q}, \mathcal{B}) = (H^p)^*$ and $(H^p)^* = (H^{p,\infty}, \mathcal{B})$ by Corollary 8.2 (i). \square

We have not been able to find $(H^{p,q}, BMOA)$ for $2 < q \leq \infty$. However, we can show

$$(H^{1/2,\infty}, BMOA) \neq (H^{1/2,\infty}, \mathcal{B}).$$

To show (8.2), let G be a function in \mathcal{B} which is not in $BMOA$. The fractional integral operator $D = D_1$ is a continuous isomorphism of \mathcal{B} onto $\Lambda_*^\infty(\mathbf{D})$ and hence $G_{[1]} \in \Lambda_*^\infty(\mathbf{D})$, [10]. But $\Lambda_*^\infty(\mathbf{D}) = (H^{1/2,\infty}, \mathcal{B})$ by Theorem 8.1 and Corollary 8.2 (i) and so $G_{[1]} \in (H^{1/2,\infty}, \mathcal{B})$. Since the function $f(z) = (1 - z)^{-2} \in H^{1/2,\infty}$ and since $f * G_{[1]} = G$, then $G_{[1]}$ fails to multiply $H^{1/2,\infty}$ into $BMOA$. Thus, $G_{[1]}$ belongs to $(H^{1/2,\infty}, \mathcal{B})$ but not to $(H^{1/2,\infty}, BMOA)$ which establishes (8.2).

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