ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 38, Number 3, 2008

δ-SPIRALLIKE FUNCTIONS WITH RESPECT TO A BOUNDARY POINT

ADAM LECKO

ABSTRACT. The aim of this paper is to present a new method of the proof of an analytic characterization of δ-spirallike functions with respect to a boundary point.

1. Introduction. In this paper we examine the class $\mathcal{S}_0^{\supset}(\delta)$ of δ-spirallike functions with respect to a boundary point. This geometric idea arises from the concepts of δ-spirallikeness with respect to an inner point as well as from starlikeness with respect to a boundary point. Spirallikeness with respect to a boundary point is a quite fresh idea introduced and studied by Elin, Reich and Shoikhet [**3**] and Aharonov, Elin and Shoikhet [**1**], who developed the methods based on Robertson's formula for starlike functions with respect to a boundary point [**11**], and on some dynamical system.

An alternative analytic formula for functions in $\mathcal{S}_0^{\supset}(\delta)$ was proposed in [**7**, Theorems 3.5 and 3.8], where the method based on the Julia lemma was explored. This technique of study of the class $\mathcal{S}_0^{\supset}(\delta)$ is a continuation of ideas from [**6, 8**], where an analytic description of starlike functions with respect to a boundary point, other than the characterization found by Robertson [**11**] and completed by Lyzzaik [**9**], was shown.

In this paper we reprove results from [**7**] in a new way. Let us emphasize that the proofs of main results in [**7**] were based on geometrical argument, now are mainly analytical.

2000 AMS *Mathematics subject classification.* Primary 30C45.
Keywords and phrases. δ-spirallike functions with respect to a boundary point, spirallike functions with respect to a boundary point, starlike functions with respect to a boundary point, Julia lemma.
Received by the editors on May 16, 2005.

DOI:10.1216/RMJ-2008-38-3-979

2. Preliminaries.

2.1. For $z_0 \in \mathbf{C}$ and $r > 0$, let $\mathbf{D}(z_0, r) = \{z \in \mathbf{C} : |z - z_0| < r\}$. Let $\mathbf{D} = \mathbf{D}(0,1)$ and $\mathbf{T} = \partial\mathbf{D}$. For $c \in \mathbf{R}$ let $\mathbf{H}_c = \{w \in \mathbf{C} : \operatorname{Re} w > c\}$.

For $A \subset \mathbf{C}$ and $w \in \mathbf{C}$, let $wA = \{wu \in \mathbf{C} : u \in A\}$.

2.2. For each $k > 0$, let

$$\mathbf{O}_k = \left\{ z \in \mathbf{D} : \frac{|1-z|^2}{1-|z|^2} < k \right\}$$

denote the disk in $\mathbf{D}$ called an *oricycle*. The boundary circle $\partial\mathbf{O}_k$ is tangent to $\mathbf{T}$ at 1. Notice that $\mathbf{O}_k = \mathbf{D}(1/(1+k), k/(1+k))$ for every $k > 0$. Let $O_k = \partial\mathbf{O}_k \setminus \{1\}$.

2.3. The set of all analytic functions in $\mathbf{D}$ is denoted by $\mathcal{A}$. Its subset of univalent functions is denoted by $\mathcal{S}$.

The set of all functions $\omega \in \mathcal{A}$ with $\omega(\mathbf{D}) \subset \mathbf{D}$ will be denoted by $\mathcal{B}$.

2.4. By Δ we denote a Stolz angle of $f \in \mathcal{A}$ at 1.

An angular limit, respectively angular derivative, of $f \in \mathcal{A}$ at $\zeta \in \mathbf{T}$ will be denoted by $f_\angle(\zeta) \in \overline{\mathbf{C}}$, respectively $f'_\angle(\zeta) \in \overline{\mathbf{C}}$.

2.5. Let $f \in \mathcal{A}$. Assume that there exists a finite radial limit $v = \lim_{r\to 1^-} f(r)$. Denote by

$$Q_f(z) = \frac{(z-1)f'(z)}{f(z) - v}, \quad z \in \mathbf{D},$$

the *Visser-Ostrowski quotient* of f at 1, see e.g. [**10**, page 251].

2.6. Recall Julia's lemma, see [**5**], [**2**, pages 53–56], [**12**, vol. II, pages 68–72].

Lemma 2.1. *Let $\omega \in \mathcal{B}$. Assume that there exists a sequence (z_n) in $\mathbf{D}$ such that*

$$\lim_{n\to\infty} z_n = 1, \qquad \lim_{n\to\infty} \omega(z_n) = 1 \tag{2.1}$$

and

$$\lim_{n\to\infty}\frac{1-|\omega(z_n)|}{1-|z_n|}=\lambda<\infty. \tag{2.2}$$

Then

$$\frac{|1-\omega(z)|^2}{1-|\omega(z)|^2}\le\lambda\frac{|1-z|^2}{1-|z|^2},\quad z\in\mathbf{D}, \tag{2.3}$$

i.e., for every $k>0$,

$$\omega(\mathbf{O}_k)\subset\mathbf{O}_{\lambda k}.$$

Equality in (2.3) *for some* z *can occur only for an automorphism of* $\mathbf{D}$.

Note that the constant λ defined in (2.2) is positive [**2**, page 54].

For $\omega\in\mathcal{B}$ with $\omega_\angle(1)=1$ let

$$\Lambda(\omega)=\sup\left\{\frac{|1-\omega(z)|^2}{1-|\omega(z)|^2}\cdot\frac{1-|z|^2}{|1-z|^2}:\ z\in\mathbf{D}\right\}.$$

The following corollary completes the Julia lemma [**2**, page 57], [**4**, page 44].

Corollary 2.2. *Let* $\omega\in\mathcal{B}$. *Assume that there exists a sequence* (z_n) *in* $\mathbf{D}$ *such that* (2.1) *and* (2.2) *holds. Then*

(1) $\Lambda(\omega)\le\lambda$

(2) *the following limits exist*:

$$\lim_{\mathbf{R}\ni x\to 1^-}\frac{1-|\omega(x)|}{1-x}=\lim_{\mathbf{R}\ni x\to 1^-}\frac{|1-\omega(x)|}{1-x}=\lim_{\mathbf{R}\ni x\to 1^-}\frac{1-\omega(x)}{1-x}=\Lambda(\omega)$$

and, for every Stolz angle Δ,

$$\lim_{\Delta\ni z\to 1}\frac{1-\omega(z)}{1-z}=\omega'_\angle(1)=\Lambda(\omega). \tag{2.4}$$

2.7. For further considerations it is convenient to use the class of functions denoted as $\mathcal{B}(\lambda)$ introduced first in [**6**]. It plays a crucial role in analytic characterization of starlike functions with respect to a boundary point [**6**] and spirallike functions with respect to a boundary point [**7**].

Definition 2.3. Let $\lambda \in (0,\infty]$. By $\mathcal{B}(\lambda)$ we denote the class of all $\omega \in \mathcal{B}$ such that $\omega_\angle(1) = 1$ and $\omega'_\angle(1) = \lambda$.

Remark 2.4. 1. From Corollary 2.2 we see that (2.4) holds under the assumption that there exists a sequence (z_n) of points in $\mathbf{D}$ satisfying (2.1) and (2.2) with some positive constant λ. Thus (2.1) and (2.2) are sufficient conditions for ω to be in $\mathcal{B}(\lambda)$ with $\lambda = \Lambda(\omega) < \infty$. Vice versa, let $\omega \in \mathcal{B}(\lambda)$ with a finite λ. Then, by the converse of the Julia lemma ([**12**, volume II, page 72], [**4**, pages 42–44]), from (2.3) it follows that there exists a sequence (z_n) of points in $\mathbf{D}$ satisfying (2.1) and (2.2) with $\lambda = \Lambda(\omega)$. Thus the existence of a sequence (z_n) of points in $\mathbf{D}$ satisfying (2.1) and (2.2) with some $\lambda \in (0,\infty)$ is a necessary and sufficient condition for ω to be in $\mathcal{B}(\lambda)$ with $\lambda = \Lambda(\omega)$.

3. An analytic characterization of δ-spirallike functions with respect to a boundary point.

3.1. For $\delta \in (-\pi/2, \pi/2)$ let $L(\delta) = \{\exp(e^{-i\delta}t) : t \le 0\}$. For $\delta \neq 0$, $L(\delta)$ is the logarithmic spiral joining 0 and 1. Clearly, $L(0) = (0,1]$.

Definition 3.1. Let $\delta \in (-\pi/2, \pi/2)$. By $\mathcal{Z}_0^{\partial}(\delta)$ we denote the class of all simply connected domains $\Omega \subset \mathbf{C}$, $\Omega \neq \mathbf{C}$, such that $0 \in \partial\Omega$ and $wL(\delta) \subset \Omega$ for every $w \in \Omega$. Such domains will be called *δ-spiralshaped with respect to a boundary point.* Let

$$\mathcal{S}_0^{\partial}(\delta) = \{f \in \mathcal{S} : f(\mathbf{D}) \in \mathcal{Z}_0^{\partial}(\delta)\}$$

be the class of univalent functions called *δ-spirallike with respect to a boundary point.*

Domains in $\mathcal{Z}_0^* = \mathcal{Z}_0^{\partial}(0)$ and functions in $\mathcal{S}_0^* = \mathcal{S}_0^{\partial}(0)$ are called *starlike with respect to a boundary point.*

3.2. Let us start with the following theorem.

Theorem 3.2. *Let $\delta \in (-\pi/2, \pi/2)$. For every $f \in \mathcal{S}_0^{\supset}(\delta)$ there exists a unique $\zeta_0 \in \mathbf{T}$ such that $f_\angle(\zeta_0) = 0$.*

Proof. Let $\delta \in (-\pi/2, \pi/2)$ and $f \in \mathcal{S}_0^{\supset}(\delta)$.

1. Fix $w_1 \in f(\mathbf{D})$. Then $w_1 L(\delta) \subset f(\mathbf{D})$ and $w_1 L(\delta)$ is a curve ending at 0. Thus, 0 is an asymptotic value of f at some point $\zeta_0 \in \mathbf{T}$ along the curve $f^{-1}(w_1 L(\delta))$. By the Lehto-Virtanen theorem [**10**, page 71], 0 is an angular limit of f at ζ_0, i.e., $f_\angle(\zeta_0) = 0$.

2. Now we prove that ζ_0 is unique. Suppose that ζ_1 and ζ_2 are two distinct points of $\mathbf{T}$ such that

$$f_\angle(\zeta_1) = f_\angle(\zeta_2) = 0. \tag{3.1}$$

Without loss of generality, we can assume that $\zeta_1 = -1$ and $\zeta_2 = 1$. Let $\mathbf{D}^+ = \{z \in \mathbf{D} : \operatorname{Im} z > 0\}$, $\mathbf{D}^- = \{z \in \mathbf{D} : \operatorname{Im} z < 0\}$, $\mathbf{T}^+ = \{z \in \mathbf{T} : \operatorname{Im} z > 0\}$ and $\mathbf{T}^- = \{z \in \mathbf{T} : \operatorname{Im} z < 0\}$. Let $\Gamma_1 = f((-1, 0])$ and $\Gamma_2 = f([0, 1))$. From (3.1) it follows that Γ_1 and Γ_2 are two curves ending at 0. Since the curve $\Gamma = \Gamma_1 \cup \Gamma_2$ is a crosscut of $f(\mathbf{D})$, we see by [**10**, Proposition 2.12] that $f(\mathbf{D}) \setminus (\Gamma_1 \cup \Gamma_2)$ has exactly two components, say G_1 and G_2. We can assume that $G_1 = f(\mathbf{D}^+)$ and $G_2 = f(\mathbf{D}^-)$. Since $f(\mathbf{D}) \in \mathcal{Z}_0^{\supset}(\delta)$, from (3.1) we can write that $\partial G_1 = \Gamma \cup \{0\}$ and $\partial G_2 = \partial f(\mathbf{D}) \cup \Gamma$. Let

$$g(z) = \frac{\sqrt{i(1+z)} + \sqrt{1-z}}{\sqrt{i(1+z)} - \sqrt{1-z}}, \quad z \in \mathbf{D},$$

with $\sqrt{i} = -e^{i\pi/4}$. The function g maps univalently $\overline{\mathbf{D}}$ onto $\overline{\mathbf{D}^+}$. Moreover, $g(\mathbf{T}^-) = (-1, 1)$. Thus, $h = f_{|\mathbf{D}^+} \circ g$ is a conformal map of $\mathbf{D}$ onto G_1 having a continuous extension to $\overline{\mathbf{D}}$. We have

$$h(\mathbf{T}^-) = f_{|\mathbf{D}^+} \circ g(\mathbf{T}^-) = f_{|\mathbf{D}^+}((-1, 1)) = \Gamma. \tag{3.2}$$

By [**10**, Theorem 1.7] the angular limits $h_\angle(\zeta)$ exist for almost all $\zeta \in \mathbf{T}$. Since $\partial h(\mathbf{D}) = \partial G_1 = \Gamma \cup \{0\}$, we see in view of (3.2) that $h_\angle(\zeta) = 0$ for almost all $\zeta \in \mathbf{T}^+$. By the Privalov uniqueness theorem [**10**, page 126] the function h vanishes identically in $\mathbf{D}$. Hence, $f \equiv 0$ in $\mathbf{D}^+$

and, consequently, $f \equiv 0$ in $\mathbf{D}$. This yields a contradiction. Thus, ζ_0 is unique. □

3.3. The theorem below was proved in [**7**, Theorem 3.5].

Theorem 3.3. *Let* $\delta \in (-\pi/2, \pi/2)$ *and* $f \in \mathcal{S}$. *Then* $f \in \mathcal{S}_0^{\supset}(\delta)$ *and* $f_{\angle}(1) = 0$ *if and only if* $f(\mathbf{O}_k) \in \mathcal{Z}_0^{\supset}(\delta)$ *for every* $k > 0$.

Now we reprove a necessary condition for functions in $\mathcal{S}_0^{\supset}(\delta)$ first shown in [**7**, Theorem 3.6]. The main goal of this paper is showing a new technique of its proof.

Theorem 3.4. *Let* $\delta \in (-\pi/2, \pi/2)$. *If* $f \in \mathcal{S}_0^{\supset}(\delta)$ *and* $f_{\angle}(1) = 0$, *then there exist* $\lambda \in (0, 1]$ *and* $\omega \in \mathcal{B}(\lambda)$ *such that*

$$-e^{i\delta}(1-z)^2 \frac{f'(z)}{f(z)} = 4\frac{1-\omega(z)}{1+\omega(z)}, \quad z \in \mathbf{D}. \tag{3.3}$$

Proof. 1. First, we show that

$$\operatorname{Re}\left\{e^{i\delta}(1-z)^2 \frac{f'(z)}{f(z)}\right\} < 0, \quad z \in \mathbf{D}. \tag{3.4}$$

Let $h(z) = (1+z)/(1-z)$, $z \in \mathbf{D}$. Fix $z \in \mathbf{D}$. Then $z \in O_k$ for some $k > 0$. By Theorem 3.3, $f(\mathbf{O}_k) \in \mathcal{Z}_0^{\supset}(\delta)$.

(a) We show first that

$$f(z)L(\delta) \subset f(\overline{\mathbf{O}_k} \setminus \{1\}). \tag{3.5}$$

Indeed, let $w_0 \in f(z)L(\delta)$, i.e., $w_0 = f(z)u_0$ for some $u_0 \in L(\delta)$. The case $u_0 = 1$ is evident since $f(z) \in f(O_k)$. Assume that $u_0 \neq 1$. Since $f(\mathbf{D}) \in \mathcal{Z}_0^{\supset}(\delta)$, we have $f(z)u_0 \in f(\mathbf{D})$ and, consequently, $f^{-1}(f(z)u_0) \in \mathbf{D}$. Thus,

$$f(z)u_0 = f(f^{-1}(f(z)u_0)).$$

We will prove that $f^{-1}(f(z)u_0) \in \overline{\mathbf{O}_k} \setminus \{1\}$. Let (z_n) be any sequence in $\mathbf{O}_k$ convergent to z. Clearly, $f(z_n) \in f(\mathbf{O}_k)$ for every $n \in \mathbf{N}$. The inclusion $f(z_n)L(\delta) \subset f(\mathbf{O}_k)$ shows that $f(z_n)u_0 \in f(\mathbf{O}_k)$ for every $n \in \mathbf{N}$. Therefore, for every $n \in \mathbf{N}$ there exists $\xi_n \in \mathbf{O}_k$ such that $f(z_n)u_0 = f(\xi_n)$. Namely, by the univalence of f, we have $\xi_n = f^{-1}(f(z_n)u_0)$, $n \in \mathbf{N}$. Since $(f(z_n))$ is a sequence in $f(\mathbf{O}_k)$ convergent to $f(z)$, we obtain

$$\lim_{n\to\infty} f(\xi_n) = \lim_{n\to\infty}(f(z_n)u_0) = f(z)u_0.$$

Hence and from the fact that $f(z)u_0 \in f(\mathbf{D})$ it follows that the sequence (ξ_n) is convergent to $f^{-1}(f(z)u_0)$. Thus $f^{-1}(f(z)u_0) \in \overline{\mathbf{O}_k}$. In consequence,

$$f^{-1}(f(z)u_0) \in \mathbf{D} \cap \overline{\mathbf{O}_k} = \overline{\mathbf{O}_k} \setminus \{1\}.$$

This proves our claim.

(b) We see from (3.5) that $\exp(e^{-i\delta}t)f(z) \in f(\overline{\mathbf{O}_k} \setminus \{1\})$ for $t \le 0$. Hence,

$$\omega_t(\delta; z) = f^{-1}(\exp(e^{-i\delta}t)f(z)) \in \overline{\mathbf{O}_k} \setminus \{1\}$$

and, consequently,

$$h \circ \omega_t(\delta; z) \in h(\overline{\mathbf{O}_k} \setminus \{1\}) = \overline{\mathbf{H}_{1/k}} \setminus \{\infty\}, \quad t \le 0.$$

Thus,

$$\text{Re}\,\{h \circ \omega_t(\delta; z)\} \ge \text{Re}\, h(z), \quad t \le 0. \tag{3.6}$$

For $t \le 0$, let

$$\psi_z(\delta; t) = h \circ \omega_t(\delta; z).$$

Since $\psi_z(\delta; 0) = h(z)$, from (3.6) we have

$$\begin{aligned} 0 &\ge \lim_{t\to 0^-} \frac{\text{Re}\,\psi_z(\delta;t) - \text{Re}\,\psi_z(\delta;0)}{t} = \text{Re}\left\{\frac{\partial}{\partial t}\psi_z(\delta;t)\right\}_{|t=0} \\ &= \text{Re}\left\{\frac{h'(f^{-1}(\exp(e^{-i\delta}t)f(z)))}{f'(f^{-1}(\exp(e^{-i\delta}t)f(z)))} f(z)\exp(e^{-i\delta}t)e^{-i\delta}\right\}_{|t=0} \\ &= \text{Re}\left\{e^{-i\delta}\frac{h'(z)}{f'(z)}f(z)\right\} = 2\text{Re}\left\{e^{-i\delta}\frac{f(z)}{(1-z)^2 f'(z)}\right\} \\ &= \frac{2}{|1-z|^4|f'(z)|^2}\text{Re}\left\{e^{i\delta}(1-z)^2\frac{f'(z)}{f(z)}\right\}, \quad z \in \mathbf{D}. \end{aligned}$$

Thus, we proved that

$$\operatorname{Re}\left\{e^{i\delta}(1-z)^2\frac{f'(z)}{f(z)}\right\} \le 0. \tag{3.7}$$

(c) Suppose now that the equality holds in (3.7) for some $z_0 \in \mathbf{D}$. By the maximum principle for harmonic functions it holds in the whole disk $\mathbf{D}$, which yields

$$e^{i\delta}(1-z)^2\frac{f'(z)}{f(z)} \equiv ai, \quad z \in \mathbf{D}.$$

for some $a \in \mathbf{R}\setminus\{0\}$. But the solution

$$f(z) = f_0(z) = f(0)\exp\left(\frac{e^{-i\delta}aiz}{1-z}\right), \quad z \in \mathbf{D},$$

of the last equation is not univalent in $\mathbf{D}$. So $f_0 \notin \mathcal{S}_0^{\supset}(\delta)$ and hence strict inequality in (3.7), i.e., (3.4) holds.

2. Let $p(z) = -e^{i\delta}(1-z)^2 f'(z)/f(z)$, and

$$\omega(z) = \frac{4-p(z)}{4+p(z)}, \quad z \in \mathbf{D}. \tag{3.8}$$

Then $\omega \in \mathcal{B}$. We now prove that $\omega \in \mathcal{B}(\lambda)$ for some $\lambda \in (0,1]$. Since $f_\angle(1) = 0$, we can write

$$p(z) = e^{i\delta}(1-z)Q_f(z), \quad z \in \mathbf{D}.$$

As $0 \in \partial f(\mathbf{D})$, we see from [**10**, page 92] that

$$\left|\frac{f'(z)}{f(z)}\right| \le \frac{4}{1-r^2}, \quad |z| = r < 1.$$

Hence,

$$|Q_f(r)| = (1-r)\left|\frac{f'(r)}{f(r)}\right| \le \frac{4}{1+r}, \quad r \in (0,1). \tag{3.9}$$

Consequently,

$$\lim_{r\to 1^-}\left\{e^{-i\delta}p(r)\right\} = \lim_{r\to 1^-}\{(1-r)Q_f(r)\} = 0.$$

Thus, $\lim_{r\to 1^-} p(r) = 0$ and, in view of (3.8), $\lim_{r\to 1^-}\omega(r) = 1$ so (2.1) holds.

From (3.9) it follows that there exists a sequence (r_n) in $(0,1)$ convergent to 1 such that

$$\lim_{n\to\infty}|Q_f(r_n)| = 2\lambda_0$$

for some $\lambda_0 \in [0,1]$. Hence, and from (3.8)–(3.9), we have

$$\begin{aligned}\lim_{n\to\infty}\frac{|1-\omega(r_n)|}{1-r_n} &= \lim_{n\to\infty}\left\{\frac{2}{|4+p(r_n)|}\left|\frac{p(r_n)}{1-r_n}\right|\right\}\\ &= \lim_{n\to\infty}\left\{\frac{2}{|4+p(r_n)|}|Q_f(r_n)|\right\} = \lambda_0 \in [0,1].\end{aligned}$$

But

$$\frac{1-|\omega(r_n)|}{1-r_n} \le \frac{|1-\omega(r_n)|}{1-r_n}$$

so we can find a subsequence (r_{n_k}) of (r_n) such that

$$\lim_{k\to\infty}\frac{1-|\omega(r_{n_k})|}{1-r_{n_k}} = \lambda_1 \le \lambda_0.$$

Thus (2.2) holds with the sequence (r_{n_k}). Hence ω satisfies the assumptions of the Julia lemma. Moreover, $\lambda_1 \in (0,1]$. In view of Remark 2.4, $\omega \in \mathcal{B}(\lambda)$ with $\lambda \in (0,\lambda_1]$.

This ends the proof of the theorem. □

3.4. Now we reprove in a new way a sufficient condition for functions in $\mathcal{S}_0^{\circlearrowright}(\delta)$ shown in [**7**, Theorem 3.8].

Theorem 3.5. *Let $\delta \in (-\pi/2,\pi/2)$ and $f \in \mathcal{A}$ with $f_\angle(1) = 0$. If there exist $\lambda \in (0,\cos\delta]$ and $\omega \in \mathcal{B}(\lambda)$ such that* (3.3) *holds, then $f \in \mathcal{S}_0^{\circlearrowright}(\delta)$.*

Proof. 1. First we show that f is univalent in $\mathbf{D}$. It is immediate from (3.3) that f' never vanishes in $\mathbf{D}$. Let g be the solution of the differential equation

$$-(1-z)^2\frac{g'(z)}{g(z)} = 4\frac{1-\omega(z)}{1+\omega(z)}, \quad z \in \mathbf{D}, \tag{3.10}$$

with $g_{\angle}(1) = 0$. Then, by [**8**, Theorem 3], $g \in \mathcal{S}_0^*$, i.e., $g \in \mathcal{S}$ and $g(\mathbf{D})$ is a starlike domain with respect to a boundary point (at zero). Moreover, $g(\mathbf{D})$ lies in a wedge of angle $2\lambda\pi$. Hence, a single-valued analytic branch of $\log g$ in $\mathbf{D}$ exists and $\log g(\mathbf{D})$ lies in a horizontal strip of width not exceeding $2\lambda\pi$. Thus, the function

$$h = g^{e^{-i\delta}} = \exp\{e^{-i\delta}\log g\}$$

is well defined in $\mathbf{D}$ and, since $\lambda \in (0, \cos\delta]$, h is univalent in $\mathbf{D}$. But, in view of (3.3) and (3.10), we have

$$\frac{g'}{g} = e^{i\delta}\frac{f'}{f},$$

so $f = h$. This yields the univalence of f.

2. Now we prove that $f(\mathbf{D}) \in \mathcal{Z}_0^{\supset}(\delta)$.

(a) Let $h(z) = (1+z)/(1-z)$, $z \in \mathbf{D}$. Fix $z \in \mathbf{D}$. Then $z \in O_k$ for some $k > 0$. Set

$$t_z(\delta) = \inf\left\{t \in (-\infty, 0] : \{f(z)\exp(e^{-i\delta}s) : s \in (t, 0]\} \subset f(\mathbf{D})\right\}.$$

Since f is an open mapping, $t_z(\delta) < 0$. For $t \in (t_z(\delta), 0]$ define the functions

$$\omega_t(\delta; z) = f^{-1}(f(z)\exp(e^{-i\delta}t))$$

and

$$\psi_z(\delta; t) = h \circ \omega_t(\delta; z).$$

From (3.3) for $t \in (t_z(\delta), 0]$ we have

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Re}\ \psi_z(\delta;t) &= \frac{\partial}{\partial t}\mathrm{Re}\ h\circ\omega_t(\delta;z)\\ &= \mathrm{Re}\left\{\frac{h'(f^{-1}(f(z)\exp(e^{-i\delta}t)))}{f'(f^{-1}(f(z)\exp(e^{-i\delta}t)))}f(z)\exp(e^{-i\delta}t)e^{-i\delta}\right\}\\ &= \mathrm{Re}\left\{e^{-i\delta}h'(\omega_t(\delta;z))\frac{f(\omega_t(\delta;z))}{f'(\omega_t(\delta;z))}\right\}\\ &= 2\mathrm{Re}\left\{e^{-i\delta}\frac{f(\omega_t(\delta;z))}{(1-\omega_t(\delta;z))^2f'(\omega_t(\delta;z))}\right\}\\ &= \frac{2}{|1-u|^4|f'(u)|^2}\mathrm{Re}\left\{e^{i\delta}(1-u)^2\frac{f'(u)}{f(u)}\right\} > 0,\end{aligned}$$

where $u = \omega_t(\delta;z)$. Thus, the function

$$(t_z(\delta), 0] \ni t \longmapsto \mathrm{Re}\ \psi_z(\delta;t)$$

is strictly increasing. Therefore,

$$\mathrm{Re}\ \psi_z(\delta;t) > \mathrm{Re}\ \psi_z(\delta;0) = \mathrm{Re}\ h(z) \tag{3.11}$$

and, consequently,

$$h^{-1}(\psi_z(\delta;t)) = f^{-1}(f(z)\exp(e^{-i\delta}t)) \in \mathbf{O}_k$$

for $t \in (t_z(\delta), 0)$. Hence

$$f(h^{-1}(\psi_z(\delta;t))) = f(z)\exp(e^{-i\delta}t) \in f(\mathbf{O}_k), \quad t \in (t_z(\delta), 0), \tag{3.12}$$

i.e., for every $z \in O_k$,

$$\{f(z)\exp(e^{-i\delta}t) : t \in (t_z(\delta), 0)\} \subset f(\mathbf{O}_k).$$

Moreover, $f(z) \in f(O_k)$ and $w_z(\delta) \in \partial f(\mathbf{O}_k)$, where

$$w_z(\delta) = \begin{cases} f(z)\exp(e^{-i\delta}t_z(\delta)) & t_z(\delta) \neq -\infty,\\ 0 & t_z(\delta) = -\infty.\end{cases} \tag{3.13}$$

Since $f(z)$ and $w_z(\delta)$ are distinct endpoints of the curve

$$\{f(z)\exp(e^{-i\delta}t) : t \in (t_z(\delta), 0)\}$$

on $\partial f(\mathbf{O}_k)$, from (3.12) and [**10**, Proposition 2.14] it follows that

$$h^{-1}(\psi_z(\delta;t)) = f^{-1}(f(z)\exp(e^{-i\delta}t)) \longrightarrow z(\delta) \tag{3.14}$$

as $t \to t_z(\delta)$ for some $z(\delta) \in O_k \cup \{1\}$ and $z(\delta) \neq z$.

Suppose that $z(\delta) \in O_k$. Then $z(\delta) \in \mathbf{D}$ and from (3.14) it follows that $w_z(\delta) = f(z(\delta)) \in f(O_k)$. Hence $\mathbf{D}(w_z(\delta), \varepsilon) \subset f(\mathbf{D})$ for some $\varepsilon > 0$. Thus,

$$\big\{\{f(z)\exp(e^{-i\delta}t) : t \in (t(\varepsilon), t_z(\delta)]\} \subset f(\mathbf{D})\big\}$$

for some $t(\varepsilon) < t_z(\delta)$, so

$$\big\{\{f(z)\exp(e^{-i\delta}t) : t \in (t(\varepsilon), 0]\} \subset f(\mathbf{D})\big\}\,.$$

This contradicts the definition of $t_z(\delta)$ and shows that $z(\delta) = 1$. Hence, and from (3.14), we conclude that for every $z \in O_k$ the following holds

$$h^{-1}(\psi_z(\delta;t)) \longrightarrow 1 \tag{3.15}$$

as $t \to t_z(\delta)$.

Moreover, from (3.12) and (3.13), for every $z \in O_k$ we have

$$f(h^{-1}(\psi_z(\delta;t))) = f(z)\exp(e^{-i\delta}t) \longrightarrow w_z(\delta) \tag{3.16}$$

as $t \to t_z(\delta)$. Since f, as a normal function, by the Lehto-Virtanen theorem [**10**, page 71] has at most one asymptotic value at 1 which is unique, from (3.15) and (3.16) we deduce that

$$w_{z_1}(\delta) = w_{z_2}(\delta) \tag{3.17}$$

for every z_1, $z_2 \in O_k$. Suppose that $w_{z_1}(\delta) \neq 0$ for some $z_1 \in O_k$. Let $z_2 \in O_k$, $z_2 \neq z_1$, be arbitrary. By (3.17) we have

$$w_{z_1}(\delta) = w_{z_2}(\delta) \neq 0.$$

Thus, in view of (3.13) we see that $t_{z_1}(\delta) \neq -\infty$, $t_{z_2}(\delta) \neq -\infty$ and

$$f(z_1)\exp(e^{-i\delta}t_{z_1}(\delta)) = f(z_2)\exp(e^{-i\delta}t_{z_2}(\delta)),$$

i.e.,

$$f(z_2) = f(z_1)\exp(e^{-i\delta}c), \tag{3.18}$$

where $c = t_{z_1}(\delta) - t_{z_2}(\delta)$. Since $f(z_1) \neq f(z_2)$ by the univalence of f, we can assume that $c < 0$. Observe that from (3.18) and (3.13) it follows that $c \in (t_{z_1}(\delta), 0)$. Therefore, using once again (3.18), we have

$$h(z_2) = h(f^{-1}(f(z_1)\exp(e^{-i\delta}c)) = \psi_{z_1}(\delta; c).$$

Taking into account that $z_1,\, z_2 \in O_k$, the above yields

$$\mathrm{Re}\ h(z_1) = \mathrm{Re}\ h(z_2) = \mathrm{Re}\ \psi_{z_1}(\delta; c)$$

which contradicts (3.11). Therefore, $w_z(\delta) = 0$, so by (3.12) we have $t_z(\delta) = -\infty$ for every $z \in \mathbf{D}$. Thus

$$\big\{f(z)\exp(e^{-i\delta}t) : t \leq 0\big\} \subset f(\mathbf{D})\big\}$$

for every $z \in \mathbf{D}$. This means that $f(\mathbf{D}) \in \mathcal{Z}_0^{\supset}(\delta)$.

This and Part 1 of this proof show that $f \in \mathcal{S}_0^{\supset}(\delta)$. □

REFERENCES

1. D. Aharonov, M. Elin and D. Shoikhet, *Spirallike functions with respect to a boundary point*, J. Math. Anal. Appl. **280** (2003), 17–29.

2. C. Carathéodory, *Conformal representation*, Cambridge University Press, Cambridge, 1963.

3. M. Elin, S. Reich and D. Shoikhet, *Holomorphically accretives mappings and spiral-shaped functions of proper contractions*, Nonlinear Anal. Forum **5** (2000), 149–161.

4. J.M. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.

5. G. Julia, *Extension nouvelle d'un lemme de Schwarz*, Acta Math. **42** (1920), 349–355.

6. A. Lecko, *On the class of functions starlike with respect to a boundary point*, J. Math. Anal. Appl. **261** (2001), 649–664.

7. ———, *The class of functions spirallike with respect to a boundary point*, Inter. J. Math. Math. Sci. **2004** (2004), 2133–2143.

8. A. Lecko and A. Lyzzaik, *A note on univalent functions starlike with respect to a boundary point*, J. Math. Anal. Appl. **282** (2003), 846–851.

9. A. Lyzzaik, *On a conjecture of M.S. Robertson*, Proc. Amer. Math. Soc. **91** (1984), 108–110.

10. Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer-Verlag, Berlin, 1992.

11. M. Robertson, *Univalent functions starlike with respect to a boundary point*, J. Math. Anal. Appl. **81** (1981), 327–345.

12. G. Sansone and J. Gerretsen, *Lectures on the theory of functions of a complex variable*, Wolters-Noordhoff, Groningen, 1969.

Department of Mathematics, Technical University of Rzeszów, ul. W. Pola 2, 35-959 Rzeszów, Poland
Email address: alecko@ewa.prz.rzeszow.pl