ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 38, Number 3, 2008

# GERBES, 2-GERBES AND SYMPLECTIC FIBRATIONS

TSEMO ARISTIDE

ABSTRACT. Let $p : P \to N$ be a symplectic bundle whose typical fiber is the symplectic manifold $(F, \omega)$. McDuff has defined a subgroup $\mathrm{Ham}^s(F, \omega)$ of the group of symplectic automorphisms of $(F, \omega)$ and has shown that the cohomology class $[\omega]$ extends to $P$ if and only if $p$ has a $\mathrm{Ham}^s(F, \omega)$ reduction. The purpose of this paper is to interpret the result of McDuff using gerbe theory. We define fundamental gerbes in symplectic geometry which allows us to define a 2-gerbe which represents the geometric obstruction to lift $\omega$ to $P$. Using these gerbes, we define a geometric quantization of symplectic manifolds.

**1. Introduction.** A *symplectic fibration* $P \to N$ is a differentiable fibration whose typical fiber is the closed connected symplectic manifold $(F, \omega)$, and such that there exists a trivialization $(U_i, g_{ij})$, such that $g_{ij}(u)$ is a symplectic automorphism of the fiber over $u$, endowed with a symplectic structure $\omega_u$, symplectomorphic to $(F, \omega)$. We suppose that the cohomology class $[\omega_u]$ of $\omega_u$ is fixed. The theory of symplectic bundles has been studied by different authors, see [**8, 9, 12, 16**]. One purpose of the paper [**16**] is to determine whether the structural group of the symplectic bundle can be reduced to the Hamiltonian group of $(F, \omega)$, that is, whether there exists a symplectic bundle $P' \to N$ isomorphic to $P$, whose coordinate changes $g'_{ij}(u)$ are Hamiltonian automorphisms of the fiber above $u$; such a reduction will be called a *Hamiltonian structure*, or a Ham-reduction. In [**16**], it is shown that the existence of such Hamiltonian reductions on a finite cover of $N$ is equivalent to the following two conditions:

(i) There exists a closed 2-form $\Omega$ defined on $P$ whose cohomology class $[\Omega]$ extends $[\omega]$. This means that the restriction to the fiber above $u$ of the cohomology class $[\Omega]$ is the cohomology class $[\omega]$. Following McDuff, we will call the form $\Omega$ a *closed connection form.*

(ii) Let $\mathrm{Symp}\,(F, \omega)_0$ be the connected component of the group of symplectomorphisms $\mathrm{Symp}\,(F, \omega)$, of $(F, \omega)$. The symplectic bundle is

Received by the editors on September 8, 2005, and in revised form on October 5, 2005.

DOI:10.1216/RMJ-2008-38-3-727 

isomorphic to a symplectic bundle whose coordinate changes take their values in $\mathrm{Symp}\,(F,\omega)_0$.

In [**16**] it was necessary to impose condition (ii) because the Hamiltonian subgroup is connected. In [**16**], McDuff has defined a disconnected subgroup $\mathrm{Ham}^s$ of the group $\mathrm{Symp}\,(F,\omega)$, and has shown that the existence of a $\mathrm{Ham}^s$-reduction of a symplectic bundle is equivalent to the existence of a closed connection form.

One purpose of this paper is to study the problem of the existence of Hamiltonian and $\mathrm{Ham}^s$-reductions of a symplectic bundle using gerbes, and 2-gerbes. The theory of gerbes has been defined by Giraud [**6**] with the purpose of giving geometric interpretations of cohomology classes. These classes represent the obstruction to globally extending locally defined bundles, as is the case for Hamiltonian bundles. Breen [**2**] has also defined a theory of 2-gerbes. A 2-gerbe represents geometrically the obstruction for a 2-geometric type structure to be defined globally. This theory will be also involved here. For $n \geq 2$, such a geometric obstruction theory has been defined by Tsemo [**20**].

Let $\omega$ be a 2-closed form defined on the manifold $F$, and let $T^1$ be the circle. It has been shown by Kostant and Weil that the cohomology class $[\omega]$ of $\omega$ is integral, if and only if $[\omega]$ is the Chern class of a $T^1$-bundle. When the class is not integral, we define a flat gerbe $C'(\omega)$ bounded by the sheaf of locally constant $\mathbf{R}$-functions defined on $F$ which represents the obstruction of $[\omega]$ to zero. We can construct from this gerbe, another gerbe $C(\omega)$ bounded by the sheaf of locally constant $T^1$-functions defined on $F$, which represents the obstruction of $[\omega]$ to integral (see 2.4). These gerbes are used to study the extension of $[\omega]$. We have:

**Theorem 2.5.4.** *Let $p : P \to N$ be a symplectic bundle whose typical fiber is $(F,\omega)$. There exists a gerbe $C^1_F(\omega)$ whose classifying cocycle $c^1_F(\omega)$ represents the obstruction of the symplectic bundle $p$ to a Hamiltonian reduction.*

To show an analogous theorem for $\mathrm{Ham}^s$-reductions, one has to show first, as in [**12**], that the automorphisms group of a $\mathrm{Ham}^s$-reduction of a symplectic bundle is independent of the chosen $\mathrm{Ham}^s$-reduction, in order to define the band of the classifying gerbe. We prove also the following result:

**Theorem 8.2, 8.2.2.** *There exists a 2-gerbe $C^2_F(\omega)$ whose classifying cocycle $c^2_F(\omega)$ represents the obstruction of the class $[\omega]$ to be extended to $P$. The class $[c^2_F(\omega)]$ can be deduced from $[c^1_F(\omega)]$ as follows: Let $L_1$ and $L_0$ be the respective bands of $C^1_F(\omega)$ and $C^2_F(\omega)$. There exists an exact sequence of sheaves $1 \to L_0 \to L'_1 \to L_1 \to 1$, such that the class $[c^2_F(\omega)]$ is the image of the class $[c^1_F(\omega)]$ by the connecting morphism $H^2(N, L_1) \to H^3(N, L_0)$ of the last exact sequence. This shows that the existence of a Hamiltonian reduction implies that the form $\omega$ can be extended to $P$.*

In [**16**], McDuff defines a discrete subgroup $H^1(F, P_\omega)$ of $H^1(F, \mathbf{R})$ and a class in $H^2(N, H^1(F, P_\omega))$ which is the obstruction to have a $\mathrm{Ham}^s$-reduction, that is, to obtain a closed connection form. We show that this last class and $[c^2_F(\omega)]$ are the image of the Chern class of an $H^1(F, \mathbf{R})/H^1(F, P_\omega)$-principal bundle by connecting morphisms related to exact sequences of sheaves, see subsection 8.3.

The holonomy of a connective structure defined on a gerbe is an analogue of the holonomy of a connection. It is used to represent the action in string theory. We relate the holonomy of the gerbe $C(\omega)$ to the flux, see Section 4.

We generalize the methods applied here to solve other geometric problems, as for example to find an $H$-reduction of a $G$-bundle such that $G/H$ is a $K(\pi, 1)$ space. For this problem, we define also a gerbe $C_H$ which represents the geometric obstruction to solve it: More precisely we have:

**Theorem 2.6.3.** *Let $f : P \to N$ be a $G$-bundle defined on $N$, and let $H$ be a subgroup of $G$ such that the right quotient of $G$ by $H$, $G/H$ is a $K(\pi, 1)$ space. Suppose that:*

(i) *either the coordinate changes take their values in* $\mathrm{Nor}\,(H)$*, the normalizer of $H$ in $G$. This condition is satisfied for symplectic bundles whose coordinate changes take their values in the connected component* $\mathrm{Symp}\,(F, \omega)_0$ *of the group of symplectic automorphisms* $\mathrm{Symp}\,(F, \omega)$.

*We consider $G$ to be* $\mathrm{Symp}\,(F,\omega)_0$*, and $H$ to be* $\mathrm{Ham}\,(F,\omega)$ *the group of Hamiltonian diffeomorphisms, or*

(ii) *$H$ intersects every connected component of $G$, and there exists a commutative group $L$, a continuous and surjective cocycle $F: G \to L$, for a representation $\rho: G \to L$, such that $\rho(G_0)$ the image of the connected component $G_0$ of $G$ is the identity of $L$, and the kernel of $F$ is $H$. Here $L$ is a quotient of a vector space by a discrete subgroup. This condition is satisfied if $H$ is the subgroup* $\mathrm{Ham}^s$*, and $G$ is* $\mathrm{Symp}\,(F,\omega)$.

*Then there exists a gerbe $C_H$, whose classifying cocycle represents the obstruction to reduce $G$ to $H$.*

When the gerbe $C_H$ is defined by a cocycle $F$, (ii) the classifying cocycle of this gerbe is the Chern class of a $G/H = L$-bundle.

Analogues of the gerbe which appear in the last theorem can be constructed in more abstract situations: we generalize this construction to the case of topoi (elementary topoi). This will perhaps suggest applications to algebraic geometry and arithmetic.

The fact that the pull-back of an $\mathrm{Symp}\,(F,\omega)_0$-bundle endowed with a closed connection form to a finite cover of the base space has Hamiltonian reductions, suggests that the natural category for the study of Hamiltonian reductions is the etale topos of the base, see Section 3.

The last part of the paper is devoted to geometric quantization. We give an extension of the Kostant-Souriau quantization whenever the class $[\omega]$ is not supposed to be integral, using the gerbe $C(\omega)$. In particular we obtain the following:

**Theorem.** *Let $(M,\omega)$ be a symplectic manifold, and let $(C^\infty(M), \{\ ,\ \})$ be the Poisson algebra of $(M,\omega)$. There exists a pre-Hilbert space $H$, and a representation $(C^\infty(M), \{\ ,\ \}) \to (\mathrm{Aut}\,(H), [.])$ where $(\mathrm{Aut}\,(H), [\ ,\ ])$ is the algebra of operators of $H$ endowed with the commutator bracket.*

The contents are as follows: Section 2 is on Gerbes theory, Section 3 is on the group $\mathrm{Ham}^s$ and the etale topos of a manifold, Section 4 is on flux and holonomy of gerbes, the classifying cocycle, Section 5 is a geometric interpretation of a section $H_1(F,\mathbf{R}) \to SH_1(F,\mathbf{R})$, Section 6 is on existence of symplectic bundles and gerbes, Section 7 discusses 2-gerbes and 2-gerbed towers, Section 8 is the general case and Section 9 is on quantization of the symplectic gerbe.

**2. Gerbes theory.** The notion of gerbe has been defined by Giraud [**6**] to give a geometric interpretation of 2-Cech cohomology classes and to find obstructions to solve gluing problems. The basic example of a gerbe is defined as follows: consider a $G$-principal bundle defined on the manifold $N$, and $1 \to H \to G' \to G \to 1$ a central extension. The geometric obstruction to the existence of a $G'$-principal bundle over $N$, whose quotient by $H$ is the original $G$-bundle is defined by the classifying cocycle of a gerbe. Gerbe theory also has a lot of applications in algebraic geometry. In theoretical physics, a notion of holonomy of gerbe allows us to represent geometrically the action in string theory. In this part, we summarize the results of gerbe theory used here. We prefer the point of view of sheaf of categories rather to the one of descent.

**Definition 2.1.** Let $N$ be a category. A *sieve* $T$ is a subclass of objects of $N$, such that if $u$ is an element of $T$, and $v \to u$ an arrow of $N$, then $v$ is an element of $T$.

Recall that the category $N_u$ is the category whose objects are objects $v$ of $N$ such that there exists an arrow $h_v : v \to u$, a morphism between two objects $v$ and $v'$ of $N_u$ is an arrow $h : v \to v'$ such that $h_{v'} \circ h = h_v$.

A *topology on the category* $N$ is defined as follows: for each object $u$ of $N$, there is a family of sieves $J(u)$ of $N_u$, such that:

(i) If $h : v \to u$ is an arrow, and $T$ an element of $J(u)$, then $T^h = \{v' \in \mathrm{Ob}\,(N) : v' \in T$, there exists $h' : v' \to v\}$ is an element of $J(v)$.

(ii) Suppose that $T$ is a sieve of the sub-category $N_u$ above $u$, if for each map $h : v \to u$, $T^h$ is an element of $J(v)$, then $T$ is an element of $J(u)$.

For example, one can define a topology $J$ on the category $\mathrm{Top}\,(N)$, whose objects are open sets of a topological manifold $N$, and morphisms are canonical inclusions as follows: For each open set $U$ of $N$, an element of $J(U)$ is a sieve of the category above $U$, which contains a family of open subsets of $(U_i)_{i\in I}$ of $U$ whose union is $U$.

We will suppose in the sequel that our category is a topos; readers unfamiliar with this notion can make the stronger assumptions that the category is stable by finite sums, and products, and that final and initial objects exist, and the limits exist and are universal.

We will also suppose that the topology is generated by a covering family $(u_i)_{i\in I}$, where $u_i$ is an object of $N$. This means that: for each object $u$, there exists a subset $I_u$ contained in $I$, such that for each $i \in I_u$, there exists a map $u_i \to u$ of $N$, the subcategory $u_{(u_i)_{i\in I_u}}$ whose objects are objects $v$ of $N$ such that there exists a map $v \to u_i$ , $i \in I_u$ is an element of $J(u)$. A generating family $(U_i)_{i\in I}$ of a topological space $N$ generates the topology of the category $\mathrm{Top}\,(N)$.

**Definition 2.2.** Let $(N, J)$ be a category $N$ endowed with a topology $J$. A *sheaf of categories* defined on $(N, J)$ is a correspondence $C$:

$$U \longrightarrow C(U)$$

where $C(U)$ is a category, and $U$ an object of $N$, which verifies the following properties:

(i) For each map $U \to V$, there exists a restriction map $r_{U,V}$ : $C(V) \to C(U)$ such that

$$r_{U_1,U_2} \circ r_{U_2,U_3} = r_{U_1,U_3}.$$

In fact, while the previous equality is verified in many examples, only an isomorphism between $r_{U_1,U_2} \circ r_{U_2,U_3}$ and $r_{U_1,U_3}$ is needed. The last relation defined the notion of *presheaf of categories.*

The following properties need to be verified to complete the notion of sheaf of categories.

(ii) **Gluing properties for objects.** Let $(U_i)_{i\in I}$ be a covering family of the object $U$ of $N$, and let $e_i$ be an object of $C(U_i)$. We denote abusively by $N$ the final object of $N$. Suppose there are morphisms

$$g_{ij} : r_{U_i\times_U U_j,U_j}(e_j) \longrightarrow r_{U_i\times_U U_j,U_i}(e_i)$$

such that on $U_{i_1} \times_N U_{i_2} \times_N U_{i_3}$, the restrictions of the morphisms $g_{i_1i_2}g_{i_2i_3}$, and $g_{i_1i_3}$ between the respective restrictions of $e_{i_3}$ and $e_{i_1}$ to

$U_{i_1} \times_N U_{i_2} \times_N U_{i_3}$ are equal. Then there exists an object $e_U$ of $U$ such that $r_{U_i,U}(e_U) = e_i$.

(iii) **Gluing conditions for maps.** For each of the objects $e$, $e'$ of $C(U)$, the correspondence defined on the category above $U$ by

$$V \longrightarrow \mathrm{Hom}\,(r_{U,V}(e), r_{U,V}(e'))$$

is a sheaf of sets.

A correspondence $C$ which satisfies properties (i), (ii) and (iii) is a sheaf of categories. A *gerbe* is a sheaf of categories which satisfies the following conditions:

(iv) There exists a covering family $(U_i)_{i\in I}$ of $N$ such that $C(U_i)$ is not empty for each $i$,

(v) **Local connectivity.** For each object $U$ of $N$, there exists a covering family $(U_i)_{i\in I}$ of $U$ such that, for each pair of elements $e$ and $e'$ of $C(U)$, $r_{U_i,U}(e)$ and $r_{U_i,U}(e')$ are isomorphic.

(vi) There exists a sheaf $L$ on $N$ such that, for each object $e_U$ of $C(U)$, $\mathrm{Hom}\,(e_U, e_U) = L(U)$, and this identification commutes with restrictions an arrows. The sheaf $L$ is called the *band* of the gerbe $C$, or we say that the gerbe $C$ is *bounded* by $L$ •.

**The classifying cocycle of a gerbe.** Let $(U_i)_{i\in I}$ be a covering family of $N$ such that, for each $i$, $C(U_i)$ is not empty, and $e_i$ is an object of $C(U_i)$. Choose maps $g_{ij} : r_{U_i\times_N U_j,U_j}(e_j) \to r_{U_i\times_N U_j,U_i}(e_i)$ for all $i,j$. Denote by ${g_{i_1i_2}}^{i_3}$ the restriction of $g_{i_1i_2}$ between the restrictions of $e_{i_2}$ and $e_{i_1}$ to $U_{i_1} \times_N U_{i_2} \times_N U_{i_3}$. Then the map

$$c_{i_1i_2i_3} = {g_{i_1i_2}}^{i_3}{g_{i_2i_3}}^{i_1}{g_{i_3i_1}}^{i_2}$$

is an automorphism of $r_{U_{i_1}\times_N U_{i_2}\times_N U_{i_3},U_{i_1}}(e_1)$ which may be thought of as an element of $L(U_{i_1} \times_N U_{i_2} \times_N U_{i_3})$. The assignment $U_{i_1} \times_N U_{i_2} \times_N U_{i_3} \to c_{i_1i_2i_3}$ is called the classifying cocycle of the gerbe. If the band is commutative, it is a Cech-cocycle in the classical sense. It has been shown by Giraud [**6**] that the isomorphism classes of gerbes bounded by the sheaf $L$ is one to one with the Cech cohomology group $H^2(N, L)$, when $L$ is commutative. If the band is not commutative $H^2(N, L)$ is defined to be set the of equivalence classes of gerbes bounded by $L$. The trivial gerbe is a gerbe such that $C(N)$ is not empty. The elements of $C(N)$ are called global sections. They are one-to-one with $H^1(N,L)$ when the gerbe is trivial.

**2.2 Notations.** Let $U_{i_1},\dots,U_{i_p}$ be objects of a topos $N$, and let $C$ be a presheaf defined on $N$. We will denote by $U_{i_1,\cdots,i_p}$ the fiber product of $U_{i_1},\dots,U_{i_p}$ on the final object. If $e_{i_1}$ is an object of $C(U_{i_1})$, ${e_{i_1}}^{i_2\cdots i_p}$ will be the restriction of $e_{i_1}$ to $U_{i_1\cdots i_p}$. For a map $h: e\to e'$ between two objects of $C(U_{i_1\cdots i_p})$, we denote by $h^{i_{p+1}\cdots i_n}$ the restriction of $h$ to a morphism between $e^{i_{p+1}\cdots i_n}\to {e'}^{i_{p+1}\cdots i_n}$.

Now we provide details on the classic example of sheaf of categories given at the beginning. Consider an extension:

$$1\longrightarrow H\longrightarrow G'\longrightarrow G\longrightarrow 1$$

such that $H$ is a central group in $G'$, and the map $G'\to G$ has local sections. Supposed a $G$-principal bundle $p_G$ is defined over $N$. The obstruction of the existence of a $G'$-principal bundle over $N$, whose quotient by $H$ is $p_G$, is the cohomology class of the classifying cocycle of the following gerbe $C_H$ defined on the categories of open subsets of $N$ as follows: for each open subset $U$ of $N$, we define $C_H(U)$ to be the category whose objects are principal $G'$-bundles over $U$ whose quotient by $H$ is the restriction of $p_G$ to $U$. To make explicit the classifying cocycle $c_H$ of this gerbe, consider an open covering $(U_i)_{i\in I}$ of $N$, which trivializes the bundle $p_G$. We denote by $g_{ij}: U_i\cap U_j\to G$ the transition functions. Since the projection $G'\to G$ has local sections, we can suppose that we can lift each map $g_{ij}$ to a map $\hat{g}_{ij}: U_i\cap U_j\to G'$. The classifying cocycle of $C_H$ is defined by:

$$c_{i_1i_2i_3}=\widehat{g}^{i_3}_{i_1i_2}\widehat{g}^{i_1}_{i_2i_3}\widehat{g}^{i_2}_{i_3i_1}.$$

This situation applies to the case where $H$ is $\mathbf{Z}/2$, $G'$ the spin group, and $G$ the orthogonal group $O(n)$. The $O(n)$-bundle is the orthogonal reduction of the bundle of linear frames of the $n$-dimensional manifold $N$, defined by a Riemannian metric. The gerbe represents the geometric obstruction of the existence of a spin structure on $N$. The cocycle in this case is the second Stiefel-Whitney class.

**2.3 Connective structures on gerbes.** The notion of a connective structure on a gerbe has been defined by Deligne, see [**3**]. It is analogous to the notion of a connection on a principal bundle.

**Definition 2.3.1.** Consider a gerbe $C$ defined on a manifold whose band is $L$. A *connective structure* on $C$ is a correspondence which associates to each object $e_U$ of $C(U)$ a torsor $Co(e_U)$, called the torsor of connections, that is, an affine space whose underlying vector space is a subset of the set of 1-forms defined on $U$. The following properties are supposed to be satisfied by this assignment:

(i) The correspondence $e_U \to Co(e_U)$ is functorial with respect to restrictions to smaller subsets.

(ii) For every isomorphism $h : e_U \to e'_U$ between objects of $C(U)$, there exists an isomorphism of torsors $h^* : Co(e_U) \to Co(e'_U)$ compatible with the composition of morphisms of $C(U)$, and the restrictions to smaller subsets.

Suppose now that the band of the gerbe is a $T^1$-sheaf, where $T^1$ is the circle. Then, for each morphism $g$ of the object $e_U$ of $C$, and $\nabla_{e_U}$ a connection of $Co(e_U)$,

$$g^*\nabla_{e_U} = \nabla_{e_U} + g^{-1}dg.$$

A *curving* of a connective structure $Co$ is an assignment to each object $e_U$, and each element $\nabla$ of $Co(e_U)$, a 2-form $D(e_U, \nabla)$ defined on $U$ such that for each morphism $h : e'_U \to e_U$, $D(e_U, \nabla) = D(e'_U, h^*\nabla)$.

If $\alpha$ is a 1-form on $U$ such that $\nabla + \alpha$ is an element of $Co(e_U)$, then

$$D(e_U, \nabla + \alpha) = D(e_U, \nabla) + d\alpha.$$

The assignment $e_U \to D(e_U, \nabla)$ is compatible with restrictions to smaller subsets.

The *curvature of the curving* is the form whose restriction to each open subset such that $C(U)$ is not empty is $dD(e_U, \nabla)$, where $e_U$ is an object of $C(U)$, and $\nabla$ an element of $Co(e_U)$. □

**2.4 The gerbe associated to a closed 2-form.** Let $(N, \omega)$ be a manifold $N$, endowed with a closed 2-form $\omega$; $(N, \omega)$ is often called a *Dirac manifold*. There exists a Cech-Weil isomorphism between the De Rham cohomology groups of $N$ and the Cech-cohomology groups of the sheaf of locally constant $\mathbf{R}$-functions defined on $N$. Thus, using the theorem of Giraud [**6**], we deduce that the cohomology class $[\omega]$ of $\omega$ classifies a gerbe $C'(\omega)$ defined on $N$ and bounded by the sheaf of locally constant $\mathbf{R}$-functions.

In this part, we present the construction of the classifying cocycle of this gerbe. This is in fact the classic explanation of the Cech-Weil isomorphism. This gerbe is the fundamental gerbe used to define many of the geometric obstructions involved in this paper.

Let $N$ be a manifold, $\omega$ a closed 2-form defined on $N$, and $(U_i)_{i\in I}$ a cover of $N$ by contractible open subsets. Without loss of generality, we can suppose that $U_i \cap U_j$ is connected. The Poincare lemma implies the existence of a family of 1-forms $(\alpha_i)_{i\in I}$ such that

$$d(\alpha_i) = \omega_{|U_i},$$

where $\omega_{|U_i}$ is the restriction of $\omega$ to $U_i$. Let $\alpha_j^i$ and $\alpha_i^j$ be the respective restrictions of $\alpha_j$ and $\alpha_i$ to $U_i \cap U_j$. Denote by $\alpha_{ij}$, the form $\alpha_j^i - \alpha_i^j$ on $U_i \cap U_j$. The form $\alpha_{ij}$ is closed. By applying the Poincare lemma to $\alpha_{ij}$, we obtain a family of real-valued functions $u_{ij}$ defined on $U_i \cap U_j$ such that

$$d(u_{ij}) = \alpha_{ij}.$$

On $U_{i_1 i_2 i_3}$, the differential of $c_{i_1 i_2 i_3} = u_{i_2 i_3} - u_{i_1 i_3} + u_{i_1 i_2}$ is zero. This implies that it is a constant map. The family of functions $c_{i_1 i_2 i_3}$ is a 2-Cech cocycle of the sheaf of locally constant $\mathbf{R}$-functions.

If $c_{i_1 i_2 i_3} \in \mathbf{Z}$, the functions $h_{ij} = \exp(2i\pi u_{ij})$ defines a line bundle over $N$. This bundle is the well known Kostant-Weil construction. In this case, the cohomology class $[\omega]$ of $\omega$ is an element of $H^2(N, \mathbf{Z})$.

Suppose that $[\omega]$ is not necessarily an element of $H^2(N, \mathbf{Z})$. Using Giraud's theorem concerning the classification of gerbes, we can associate to $\omega$ a gerbe $C'(\omega)$ bounded by the sheaf of locally constant $\mathbf{R}$-functions, whose classifying cohomology class is the image of $[\omega]$ by the De Rham Cech isomorphism. This gerbe represents the obstruction of the class $[\omega]$ to be zero. The objects of $C'(\omega)(U)$ when it is not empty, can be represented by flat $\mathbf{R}$-bundles by using the reconstruction theorem of Giraud presented in Brylinsky [**3**]. We denote by $c'_\omega$ the classifying cocycle of $C'(\omega)$.

The following proposition describes a gerbe bounded by $T^1$ which will play a fundamental role in this paper.

**Proposition 2.4.1.** *Let $U$ be an open subset of $N$, and denote by $C(\omega)(U)$ the category whose objects are circle bundles over $U$, endowed with a connection whose curvature is $\omega_{|U}$ the restriction of $\omega$ to $U$. We will denote by $(e_U, \nabla_{e_U})$ an object of $C(\omega)(U)$; $e_U$ represents a $T^1$-bundle and $\nabla_{e_U}$ the connection on $e_U$ whose curvature is the restriction of $\omega$ to $U$. The set of morphisms between two objects $(e_U, \nabla_{e_U})$, and $(e'_U, \nabla_{e'_U})$ is the set of morphisms of differential bundles over the identity $f : e_U \to e'_U$ such that $f^*(\nabla_{e'_U}) = \nabla_{e_U}$. The correspondence $U \to C(\omega)(U)$ is a gerbe bounded by the sheaf of locally constant $T^1$-valued functions. The class of its classifying cocycle is the obstruction of $[\omega]$ to be integral.*

*Proof.* First, we show that $C(\omega)$ is a sheaf of categories.

*Gluing conditions for objects.* Let $(U_i)_{i\in I}$ be an open cover of an open set $U$ of $N$, $(e_i, \nabla_{e_i})$ an object of $C(\omega)(U_i)$ and $g_{ij} : e^i_j \to e^j_i$ a morphism such that, on $U_{i_1i_2i_3}$, $g^{i_3}_{i_1i_2} g^{i_1}_{i_2i_3} = g^{i_2}_{i_1i_3}$. Since the elements of the family $(e_i)_{i\in I}$ are bundles, we deduce that there exists a bundle $e$ over $U$ whose restriction to $U_i$ is $e_i$. The bundle $e$ is endowed with a connection whose curvature is the restriction of $\omega$ to $U$ since the restriction of this curvature to $U_i$ is the restriction of $\omega$ to $U_i$.

*Gluing conditions for arrows.* Let $e, e'$ be a pair of elements of $C(\omega)(U)$, the correspondence defined on the category of open subsets of $U$ by $V \to \mathrm{Hom}\,(r_{U,V}(e), r_{U,V}(e'))$ is a sheaf of sets, since it is the sheaf of morphisms between two bundles.

It remains to be verified that the sheaf of categories is a gerbe.

Let $(U_i)_{i\in I}$ be an open covering of $N$ by contractible open subsets. For each pair of objects $(e, \nabla_e)$ and $(e', \nabla_{e'})$ of $C(\omega)(U)$ we have to show that these objects are locally isomorphic.

To show this, consider two objects $(e_i, \nabla_{e_i})$ and $(e'_i, \nabla_{e'_i})$ of $C(\omega)(U_i)$. The bundle $e_i$ and $e'_i$ are isomorphic to the trivial bundle $U_i \times T^1$. Let $d$ be the differential, $\nabla_{e_i} = d + \alpha_i$, and $\nabla_{e'_i} = d + \alpha'_i$. For each section $u : U_i \to i\mathbf{R}$ of the Lie algebra bundle associated to this bundle, and each automorphism $g$ defined by a differentiable map $U_i \to T^1$, we have:

$$(1)\qquad g^*(d+\alpha_i)(u) = g^{-1}(d+\alpha_i)(gu)) = (g^{-1}dg + d + \alpha_i)(u).$$

Since the connections $\nabla_{e_i}$ and $\nabla_{e'_i}$ have the same curvature, there exists a function $v_i$ such that $\alpha'_i = \alpha_i + dv_i$. We can suppose (by shrinking $U_i$ if needed) that the logarithm is defined on $U_i$, thus $g^{-1}dg = d\log(g)$. If we take $g = \exp(v_i)$,where $\alpha'_i = \alpha_i + dv_i$, then $g^*(d+\alpha_i) = d+\alpha'_i$. We obtain that the respective restrictions $(e_i, \nabla_{e_i})$ and $(e'_i, \nabla_{e'_i})$ of $(e, \nabla_e)$ and $(e', \nabla_{e'})$ to $U_i$ are isomorphic.

The automorphism group of the object $(e, \nabla_e)$ of $C(\omega)(U)$ is the group of gauge transformations which preserve the connection $\nabla_e$. These gauge transformations are necessarily constant maps, as is shown by (1).

Now we have to interpret geometrically the vanishing of the cohomology class $[c_\omega]$, of the classifying cocycle $c_\omega$ of $C(\omega)$. The theorem of Giraud [**6**] implies that this is equivalent to the existence of a global object of the gerbe, that is, a $T^1$-bundle over $N$ whose curvature is $\omega$. The Kostant-Weil construction implies that this is equivalent to the fact that the class $[\omega]$ of $\omega$ is integral. □

Now we establish the relation between the gerbes $C'(\omega)$ and $C(\omega)$.

**Proposition 2.4.2.** *Consider the exact sequence of sheaves of locally constant functions*:

$$(2)\qquad 1 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{R} \longrightarrow T^1 \longrightarrow 1$$

*where the map $\mathbf{Z} \to \mathbf{R}$ is the canonical injection, and $\mathbf{R} \to T^1$ is the exponential map of the Lie group $T^1$, that is, the composition of the multiplication by $2\pi i$ and the usual exponential. We obtain the following exact sequence in cohomology*:

$$H^1(N, T^1) \longrightarrow H^2(N, \mathbf{Z}) \longrightarrow H^2(N, \mathbf{R}) \longrightarrow H^2(N, T^1) \cdots .$$

*The class $[c_\omega]$ is the image of the class $[c'_\omega]$, by the map $H^2(N, \mathbf{R}) \to H^2(N, T^1)$ of this sequence.*

*Proof.* Consider an open covering $(U_i)_{i\in I}$ of $N$, such that for each $i$, $U_i$ is contractible and $U_{i_1\cdots i_p}$ is connected (using a theorem of Weil, we can suppose $U_{i_1\cdots i_p}$ to be connected). Let $c_{i_1i_2i_3}$ be the classifying cocycle of the gerbe $C'_\omega$. The image of $[c'_\omega]$ by the map $H^2(N,\mathbf{R}) \to H^2(N,T^1)$ is represented by the cocycle $\exp(2i\pi c_{i_1i_2i_3})$. Recall that to construct the cocycle $c_{i_1i_2i_3}$ we have considered the restriction $\omega_{|U_i}$ of $\omega$ to $U_i$. There exists a form $\alpha_i$ such that $d\alpha_i = \omega_{|U_i}$. We can define the object $e_i = (U_i \times T^1, d+\alpha_i)$ of $C(\omega)(U_i)$. Let $\alpha^i_j$ be the restriction of $\alpha_j$ to $U_{ij}$; then there exists a function $u_{ij}$ such that $d(u_{ij}) = \alpha^i_j - \alpha^j_i$. The functions $\exp(2i\pi u_{ij})$ defines a morphism between $e^i_j$ and $e^j_i$ (see the proof of 2.4.1). The classifying cocycle of $C'(\omega)$ is $c_{i_1i_2i_3} = u_{i_2i_3} - u_{i_1i_3} + u_{i_1i_2}$, and the classifying cocycle of $C(\omega)$ is $\exp(2i\pi u_{i_2i_3})\exp(-2i\pi u_{i_1i_3})\exp(2i\pi u_{i_1i_2}) = \exp(2i\pi c_{i_1i_2i_3})$.

Now, we are going to endow the gerbe $C(\omega)$ with a connective structure.

**Proposition 2.4.3.** *For each open set $U$ of $N$, and the object $e_U$ of $C(\omega)(U)$, the set $Co(\omega)(e_U)$ of connections defined on $e_U$ whose curvature is the restriction of $\omega$ to $U$ defines a connective structure on $C(\omega)$. The restriction $\omega_{|U}$ of $\omega$ to $U$, is the curving of each object $e_U$ of $C(\omega)(U)$. The curvature of this curving is zero.*

*Proof.* Let $\alpha$ and $\alpha'$ be two elements of $Co(e_U)$, and $(U_i)_{i\in I}$ a contractible open cover of $U$. It is a well-known fact that there exists a 1-form $v$ such that $\alpha' = \alpha + v$. The restriction of $e_U$ to $U_i$ is diffeomorphic to the trivial $T^1$-bundle. This implies that, under this identification, the respective restrictions $\alpha_i$ and $\alpha'_i$ of the connections $\alpha$ and $\alpha'$ to $U_i$, have the form $d+u_i$, and $d+u_i+v_{|U_i}$ where $u_i$ is a 1-form defined on $U_i$, and $v_{|U_i}$ is the restriction of $v$ to $U_i$. The respective curvatures of $d+u_i$ and $d+u_i+v_{|U_i}$ are the 2-forms $du_i$ and $d(u_i+v)$. Since they coincide with the restriction of $\omega$ to $U_i$, we deduce that $dv = 0$; thus, $Co(\omega)(e_U)$ is an affine space whose underlying vector space is the vector space of closed 1-forms. We deduce that it is a torsor.

The fact that, for each automorphism $g$ of $e_U$, $g^*\nabla_{e_U} = \nabla_{e_U} + g^{-1}dg$ results from the fact that $\nabla_{e_U}$ is a connection.

For each map $h : e_U \to e'_U$, we define the map $h^* : Co(\omega)(e_U) \to Co(\omega)(e'_U)$, to be the pull-back of connections by $h^{-1}$. This implies that $h^*$ behave naturally in respect to restrictions to smaller subsets and compositions.

Let $\nabla_{e_U}$ be an element of $Co(e_U)$, the curvature of $\nabla_{e_U}$ is the restriction of $\omega$ to $U$, $\omega_{|U}$. It is also the curvature of ${h^{-1}}^*(\nabla e_U)$. This can be shown using (1). This implies that $\omega$ defines a curving for this connective structure. The fact that the curvature of this connective structure is zero follows from the fact that the form $\omega$ is closed. □

At the end of this paper, we will present a quantization of symplectic manifolds using the gerbe $C(\omega)$. This gerbe thus appears to be fundamental in symplectic geometry.

**2.5 Symplectic fibrations and gerbes.** Let $p : P \to N$ be a symplectic fibration, whose fiber $F$ is the closed symplectic manifold $(F, \omega)$. We study the following problem: extend $[\omega]$ to a class $[\Omega]$ defined on $P$, that is, find a cohomology class $[\Omega] \in H^2(P, \mathbf{R})$ such that for every $u \in N$, consider the canonical embedding $i_u : F \to F_u \to P$, $i_u^*([\Omega]) = [\omega]$. A result of Thurston [**18**] implies that in this situation there exists a form $\Omega$ such that ${i_u}^*\Omega = \omega_u$ for all $u \in N$.

To use the theory of gerbes, we must suppose that the class $[\omega]$ of the symplectic form $\omega$ is integral. Thus, it is the Chern class of a circle bundle $h_F$ over $F$. In the general case, we will use the gerbe $C'(\omega)$ to define a 2-gerbe which represents the geometric obstruction of the class $[\omega]$ to be lifted to $P$. We have the following proposition:

**Proposition 2.5.1.** *Suppose that $[\omega]$ is integral, and consider for each open set $U$ of $N$ the category $C_F(\omega)(p^{-1}(U))$ of circle bundles over $p^{-1}(U)$ whose Chern class is $[\Omega_U]$, an element of $H^2(p^{-1}(U), \mathbf{R})$ which extends $[\omega]$. The correspondence defined on the category of open subsets of $P$ by $p^{-1}(U) \to C_F(p^{-1}(U))$, defines a gerbe on $P$, where $P$ is endowed with the topology structure generated by $p^{-1}(U)$, where $U$ is an open subset of $N$, and its differential structure is modeled on $\mathbf{R}^n \times F$, where $n$ is the dimension of $N$. The cohomology class of the classifying cocycle of this gerbe is the obstruction to extend $[\omega]$.*

*Proof. Gluing conditions for objects.* Recall that for each object $e_U$ and $e'_U$ of $C_F(\omega)(U)$, $\mathrm{Hom}\,(e_U, e'_U)$ are morphisms of circle bundles which project to the identity. Let $(U_i)_{i\in I}$ be an open covering of the open set $U$ of $N$ by open subsets, $e_i$ an object of $C_F(\omega)(p^{-1}(U_i))$, and $u_{ij} : e^i_j \to e^j_i$ a morphism which verifies $u_{i_1i_2}{}^{i_3}u_{i_2i_3}{}^{i_1} = u_{i_1i_3}{}^{i_2}$. Then there exists a bundle $e_U$ on $p^{-1}(U)$ whose restriction to each $U_i$ is $e_i$. This is deduced from the classical definition of a $T^1$-bundle $e_U$ over $p^{-1}(U)$. Consider a 2-closed form $\Omega$ which represents the Chern class of $e_U$, since the restriction of $e_U$ to $p^{-1}(U_i)$ is $e_i$, its Chern class which is the restriction of the class $[\Omega]$ of $\Omega$ to $p^{-1}(U_i)$ is the Chern class of $e_i$. This implies that $[\Omega]$ extends to $p^{-1}(U)$ the class $[\omega]$.

*Gluing condition for arrows.* The correspondence defined on the category of open subsets of $U$ by $V \to \mathrm{Hom}\,(e_{U|V}, e'_{U|V})$ defines a sheaf on this category, since it is a sheaf of morphisms between two bundles.

This shows that $C_F(\omega)$ is a sheaf of categories. It remains to prove that it is a gerbe.

Let $(U_i)_{i\in I}$ be a cover of $N$ by contractible open subsets, $p^{-1}(U_i) = U_i \times F$. This implies that $H^*(p^{-1}(U_i)) = H^*(F)$; thus, there exists a class $[\Omega_i]$ on $p^{-1}(U_i)$ which extends $[\omega]$, and which is integral. Thus, $C_F(\omega)(p^{-1}(U_i))$ is not empty.

We deduce from (1) that the group of automorphisms of the objects of $C_F(\omega)(p^{-1}(U))$ are sections of the sheaf circle valued functions defined on $p^{-1}(U)$.

*Connectivity.* Let $e_U$ and $e'_U$ be a pair of objects of $C_F(\omega)(U)$. Denote respectively by $e_i$ and $e'_i$ the respective restrictions of $e_U$ and $e'_U$ to $p^{-1}(U_i)$, where $(U_i)_{i\in I}$ is an open cover of $U$ by contractible open subsets. Since $U_i$ is contractible, the Chern class of the differentiable bundle $e_i$ and $e'_i$ are mapped to $[\omega]$ by the isomorphism $H^2(U_i\times F, \mathbf{R}) \to H^2(F, \mathbf{R})$. This implies that they are isomorphic since they have the same Chern class.

If the classifying cocycle of the gerbe $C_F(\omega)$ has a trivial cohomology class, then by a theorem of Giraud [**6**], the gerbe $C_F(\omega)$ has a global section $e$. Let $u$ be an element of the contractible open subset $U_i$ of $N$. The restriction of $e$ to $p^{-1}(U_i)$ is an element $e_i$ of $C_F(\omega)(U_i)$, by definition, its restriction to $F_u$ has Chern class $[\omega]$. $\square$

*Remark.* Denote the classifying cocycle of the gerbe $C_F(\omega)$ by $c_F(\omega)$. Its cohomology class is an element of the sheaf cohomology group of differentiable functions $H^2(P,T_1)$. We can consider the exact sequence of sheaves of differentiable functions:

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{R} \longrightarrow T^1 \longrightarrow 1. \tag{2}$$

We deduce an isomorphism between $H^2(P,T^1)$ and $H^3(P,\mathbf{Z})$, since $H^*(P,\mathbf{R})$ the cohomology of the sheaf of $\mathbf{R}$-differentiable functions is zero, because there exist partitions of unity. Thu, the gerbe $C_F(\omega)$ is classified by an element of $H^3(P,\mathbf{Z})$.

In [**3**] Brylinski has studied the following problem: Suppose there is defined on $P$ a 2-form $\Omega$ whose restriction to each fiber is closed, integral and symplectic. Find obstructions to build a closed 2-form whose restriction to a fiber $F_u$ above $u$ coincides with the restriction of $\Omega$ on the fiber $F_u$. If $H^1(F,\mathbf{R}) = 0$, the obstruction to find such a class is a gerbe $C_p(\omega)$ defined on $N$.

Recall the construction of $C_p(\omega)$. For every open set $U$, $C_p(\omega)(U)$ is the category whose objects are $T^1$-bundles over $p^{-1}(U)$, endowed with a connection such that the restriction of its curvature to a fiber $F_u$ above $u$, coincide with the restriction of $\Omega$ to $F_u$. A morphism between two objects $(e_U, \nabla_{e_U})$ and $(e'_U, \nabla_{e'_U})$ is a morphism $f : e_U \to e'_U$ of $T^1$-bundles such that $f^*(\nabla_{e'_U}) = \nabla_{e_U}$. The group of automorphisms of $(e_U, \nabla_U)$ is the set of $T^1$-differentiable functions defined on $U$. This gerbe is trivial, since as remarked by McDuff in [**16**], in this case the Guillemin-Lerman-Sternberg method allows to construct a closed form which extends $[\omega]$ if $H^1(F,\mathbf{R}) = 0$.

*Remark.* Suppose that the symplectic bundle $p : P \to N$ has a Hamiltonian reduction. Then there exists an extension $\Omega$ of $\omega$, see [**12**], which defines the distribution $D^\Omega$ on $P$ as follows: let $u$ be an element of $P$, $T_uP$ and $TF_{p(u)}$ the respective tangent spaces of $P$ at $u$ and at the fiber of $p(u)$.

$$D^{\Omega}{}_u = \{v \in T_uP : \Omega_u(v,y) = 0, y \in TF_{p(u)}\}.$$

When the bundle is Hamiltonian, we can suppose that the holonomy of the closed connection form is Hamiltonian. And, using a standard process, we can reduce the structural group of this connection to its holonomy and thus obtain the Hamiltonian reduction.

**Proposition 2.5.2.** *Suppose that there exists an extension* $[\Omega]$ *of the class* $\omega$. *Let* $\Omega$ *be a fixed representative. Then the set of cohomology classes of closed* 2*-forms* $\Omega'$ *whose restriction to any fiber* $F_u$ *coincides with the restriction of* $\Omega$ *to the fiber* $F_u$ *and such that* $D^{\Omega} = D^{\Omega'}$ *is isomorphic to* $H^2(N, \mathbf{R})$.

*Proof.* We remark that, while this proposition is very similar to the problem of the Brylinski's book [**3**] mentioned above, we cannot apply the result obtained by Brylinski since we do not suppose that the class $[\omega]$ is integral and $H^1(F, \mathbf{R})$ may not vanish.

Let $\Omega'$ be a representative of a cohomology class whose restriction to the fiber $F_u$ of $p : P \to N$ coincides with the restriction of $\Omega$ to $F_u$ and such that $D^{\Omega} = D^{\Omega'}$. Then the form $\Omega - \Omega'$ projects to a closed 2-form $p(\Omega - \Omega')$ defined on the base; we have thus defined a map between the set of extensions of $[\omega]$ which has a representative whose restriction to a fiber $F_u$ coincides with the restriction of $\Omega$ to $F_u$ and also defines the distribution $D^{\Omega}$ and $H^2(N, \mathbf{R})$ by assigning to the class of $\Omega'$ the class of $\Omega - \Omega'$. We have to show that this map is an isomorphism.

Suppose that the class of $p(\Omega - \Omega')$ is trivial. Then there exists a 1-form $\alpha$ on $N$ such that $d(\alpha) = p(\Omega - \Omega')$. We denote by $p^*(\alpha)$ the pulls-back of $\alpha$ to $P$. This implies that $\Omega' = \Omega + d(p^*(\alpha))$; thus, the classes of $\Omega$ and $\Omega'$ coincide. This shows that the map $[\Omega'] \to [p(\Omega - \Omega')]$ is injective.

To show that this map is surjective, consider a closed 2-form $v$ of $N$, $p(\Omega - (\Omega - p^*(v))) = v$. □

The initial problem studied by Mc Duff was to find a Hamiltonian reduction of the bundle $p : P \to N$, that is, a symplectic bundle isomorphic to $p$, whose transition functions take their values in the Hamiltonian group of $(F, \omega)$. This problem can be studied by a sheaf of categories. The definition of this sheaf of category uses the following result of Lalonde-McDuff [**13**], which allows to define its band:

**Proposition 2.5.3.** *Let $p : P \to N$ be a symplectic bundle. Suppose that there exists a Hamiltonian reduction of $p$. Then there exists an extension $\Omega$ of $\omega$, such that the Hamiltonian reduction is defined by the holonomy of the closed connection form $\Omega$. A Hamiltonian automorphism of the bundle $p : P \to N$ is a diffeomorphism $\phi$ of $P$ which covers the identity, such that the restriction of $\phi$ to the fiber over $n \in N$ is a Hamiltonian automorphism of $(F, \omega_n)$, and such that $\phi^*(\Omega) = \Omega$. We denote by* $\mathrm{Aut}\,(P,\Omega)$ *the group of Hamiltonian automorphisms of the Hamiltonian reduction $(P,\Omega)$. The group* $\mathrm{Aut}\,(P,\Omega)$ *does not depend of the Hamiltonian reduction.*

*Remark.* In fact a more general result is shown in Lalonde-McDuff [**12**], that is, the group of diffeomorphisms $G(P,\omega)$ which cover the identity and such that the restriction of each of its element $\phi$ to a fiber $F_u$ belongs to the connected component of the group of symplectic diffeomorphisms of $(F_u, \omega_u)$, and which preserves the symplectic class which defines the Hamiltonian reduction does not depend of the chosen Hamiltonian reduction. This result implies the one stated in the proposition above since this group $G(P,\omega)$ contains $\mathrm{Aut}\,(P,\Omega)$. The elements of $\mathrm{Aut}\,(P,\Omega)$ are the elements of $G(P,\Omega)$ which when restricted to $(F_u, \omega_u)$ are Hamiltonian. We see that this last condition is independent of the chosen Hamiltonian connection $\Omega$ which defines any Hamiltonian reduction of $p : P \to N$. A morphism $f : P \to P'$ between the Hamiltonian bundles $P$ and $P'$ is a morphism of fiber bundles $f$ such that $f^*(\Omega') = \Omega$, where $\Omega$ and $\Omega'$ are the closed connections forms whose holonomy define respectively the Hamiltonian reduction of $P$ and $P'$.

Now we can show the following:

**Proposition 2.5.4.** *Let $p : P \to N$ be a symplectic fibration. For any open set $U$ of $N$, we define $C^1_F(\omega)(U)$ to be the category whose objects are Hamiltonian structures on the symplectic bundle $p^{-1}(U) \to U$. A morphism between the objects $(e_U, \Omega_U)$, and $(e', \Omega'_U)$ of $C^1_F(\omega)(U)$ is a morphism of bundles $f : e_U \to e'_U$ such that $f^*(\Omega'_U) = \Omega_U$. The correspondence defined on the category of open subsets of $N$ by $U \to C^1_F(\omega)(U)$ is a gerbe whose band $L$ is the sheaf induced by the presheaf of Hamiltonian automorphisms such that, for each open set*

*$U$ of $N$, and each $e_U$ of $C^1_F(\omega)(U)$, $L(U)$ is the group of Hamiltonian automorphisms of $e_U$, see Proposition* 2.5.3. *The cohomology class of the classifying cocycle of $C^1_F(\omega)$ is the obstruction for the existence of a Hamiltonian reduction of $p : P \to N$.*

*Proof. Gluing conditions of objects.* Consider $(U_i)_{i\in I}$ an open cover of the open subset $U$ of $N$, such that $C^1_F(\omega)(U_i)$ is not empty, and $(e_i, \Omega_i)$ an object of $C^1_F(\omega)(U_i)$. Suppose that there exists a family of morphisms $u_{ij} : e^i_j \to e^j_i$ such that $u_{i_1 i_2}{}^{i_3} u_{i_2 i_3}{}^{i_1} = u_{i_1 i_3}{}^{i_2}$. Then there exists an $F$-bundle $e$ over $U$ whose restriction to $U_i$ is $e_i$. We have to show that this bundle is Hamiltonian. Since $u_{ij}{}^*(\Omega_i) = \Omega_j$, the forms $\Omega_i$ glue together to define on $e$ an extension $\Omega$ of $\omega$. Consider a path $c : [0,1] \to N$; we can suppose that $[0,1]$ is a union of intervals $I_l$ such that $I_l$ is contained in $U_l$, an open set of the above cover. The holonomy of the connection form $\Omega$ along $I_l$ is the holonomy of $\Omega_l$ along $I_l$. We conclude that the holonomy of $\Omega$ along $I$ is Hamiltonian, since each closed form $\Omega_l$ define an Hamiltonian reduction on $e_l$.

*Gluing conditions of arrows.* Let $e_U$ and $e'_U$ be a pair of objects of $C^1_F(\omega)(U)$. The correspondence defined on the category of open subsets of $U$ which associates to $V$ the set of Hamiltonian morphisms $\mathrm{Ham}\,(e_U, e'_U)$ is a sheaf since it is the subsheaf of the sheaf of morphisms between two bundles.

*Connectivity.* Let $e_U$ and $e'_U$ be a pair of objects of $C^1_F(\omega)$. We can suppose that the open cover $(U_i)_{i\in I}$ of $U$ is a Hamiltonian trivialization of the both bundles $e_U$ and $e'_U$. This implies that the restrictions of $e_U$ and $e'_U$ to $U_i$ are isomorphic as Hamiltonian bundles to the trivial Hamiltonian bundle $U_i \times (F, \omega)$. We deduce that these Hamiltonian bundles are locally isomorphic.

Let $(U_i)_{i\in I}$ be a symplectic trivialization of $p : P \to N$. The trivial symplectic bundle $U_i \times (F, \omega)$ is an element of $C^1_F(\omega)(U_i)$, which is not empty.

The result of Lalonde and McDuff [**12**] recalled above shows that the group $\mathrm{Aut}\,(e_U, \Omega_U)$ of Hamiltonian automorphisms of the Hamiltonian reduction of the restriction of $p$ to $p^{-1}(U)$ does not depend of the chosen object in $C^1_F(\omega)(U)$. This implies that the correspondence defined on the category of open subsets of $N$ by $U \to \mathrm{Aut}\,(e_U, \Omega_U)$ defines a presheaf $L'$ on $U$. We denote by $L$ the sheaf associated to this presheaf.

We remark that if $C^1_F(\omega)(U)$ is not empty, then $L(U) = \mathrm{Aut}\,(e_U, \Omega_U)$ for each object $e_U$ of $C^1_F(\omega)(U)$. This implies that the gerbe is bounded by $L$. $\square$

**2.6 The McDuff construction of** $\mathrm{Ham}^s$**, and closed connection forms.** The existence of a closed connection form $\Omega$ on the symplectic bundle $p : P \to N$ does not insure the existence of a Hamiltonian reduction of this bundle. This has motivated McDuff to introduce the group denoted $\mathrm{Ham}^s$, such that the existence of a closed connection form is equivalent to the existence of a $\mathrm{Ham}^s$-reduction. We will now present the construction of the group $\mathrm{Ham}^s$ and show using gerbe theory that a $\mathrm{Ham}^s$-reduction implies the existence of a closed connection form on a symplectic bundle.

**Definition 2.6.1** (McDuff). Let $H_1(F, \omega, \mathbf{Z})$ be the first homology group of $F$ with integral coefficients, we define $SH_1(F, \omega, \mathbf{Z})$ to be the quotient of the integral 1-cycles of $F$ by the image under the boundary of 2-cycles with zero symplectic area. We denote $SH_1(F, \omega, \mathbf{Q})$ to be the tensor product $SH_1(F, \omega, \mathbf{Z}) \otimes \mathbf{Q}$. Often we will respectively denote $SH_1(F, \omega, \mathbf{Z})$, and $SH_1(F, \omega, \mathbf{Q})$ by $SH_1(F, \mathbf{Z})$ and $SH_1(F, \mathbf{Q})$. Let $P_\omega$ be the values of $\omega$ on rational cycles. We have the exact sequence:

$$0 \longrightarrow \mathbf{R}/P_\omega \longrightarrow SH_1(F, \mathbf{Q}) \longrightarrow H_1(F, \mathbf{Q}) \longrightarrow 0.$$

Consider a section $s$ of $H_1(F, \mathbf{Q}) \to SH_1(F, \mathbf{Q})$. Then we can define on $\mathrm{Symp}\,(F, \omega)$ the group of symplectomorphisms of $(F, \omega)$, the map $F_s : \mathrm{Symp}\,(F, \omega) \to H^1(F, \mathbf{R}/P_\omega) = H^1(F, \mathbf{R})/H^1(F, P_\omega)$ by:

$$F_s(g)(u) = g(su) - s(gu). \qquad \square$$

Recall that the group $\mathrm{Symp}\,(F, \omega)$ acts canonically on $SH_1(F, \mathbf{Q})$ and $H_1(F, \mathbf{Q})$. McDuff has shown that the application $F_s$ is a 1-cocycle for the canonical representation defined on $\mathrm{Symp}\,(F, \omega)$ which takes its values in the group of linear automorphisms of $H^1(F, \mathbf{R})/H^1(F, P_\omega)$ and has defined $\mathrm{Ham}^s$ to be the kernel of this cocycle $F_s$.

**Theorem 2.6.2** (McDuff). *A symplectic bundle $p : P \to N$ has a* $\mathrm{Ham}^s$*-reduction if and only if there exists a closed connection form.*

*Moreover, the group* $\mathrm{Ham}^s$ *intersects every connected component of* $\mathrm{Symp}\,(F,\omega)$.

We will now present a proof of the first part of this theorem using gerbe theory. In fact this problem can be reformulated in a more general situation: Let $G$ be a Lie group whose dimension can be infinite and $H$ a subgroup of $G$; we suppose that $G/H$ is a $K(\pi,1)$ space, that is, its universal cover is contractible and its fundamental group is $\pi$. We are looking for conditions which insure the existence of a $H$-reduction. This problem can be formulated using gerbe theory. We have:

**Theorem 2.6.3.** *Let* $p: P \to N$ *be a* $G$*-principal bundle defined on* $N$. *Suppose either*:

(i) *the transition functions* $u_{ij} : U_i \cap U_j \to G$ *take their values in the normalizer* $\mathrm{Nor}\,(H)$ *of* $H$ *in* $G$, *where* $H$ *is a subgroup of* $G$, *and* $G/H$ *is a* $K(\pi,1)$ *space, or*

(ii) *there exists a continuous representation* $h : G \to L$ *where* $L$ *is an abelian group isomorphic to the quotient of a vector space* $V$ *by a discrete subgroup* $\pi$ *such that the restriction of* $h$ *to the connected component of the identity* $G_0$ *of* $G$ *is trivial, a continuous surjective* 1*-cocycle for this representation whose kernel* $H$ *intersects every connected component of* $G$.

*Then there exists a gerbe* $C_H$ *defined on* $N$, *bounded by the locally constant sheaf defined on* $N$ *by* $\pi$ *which represents the obstruction of the bundle* $p: P \to N$ *to have an* $H$*-reduction.*

*Proof.* The proof is a corollary of the following lemmas:

**Lemma 2.6.4.** *Suppose first that there exists a subgroup* $H$ *of* $G$, *such that the transition functions* $u_{ij}$ *of* $p : P \to N$ *are contained in the normalizer* $\mathrm{Nor}\,(H)$ *of* $H$ *in* $G$. *Then the right quotient fiber by fiber of the bundle* $p$ *by* $H$ *defines a* $G/H$*-bundle* $\bar{p} : \overline{P} \to N$. *Let* $\widehat{G/H}$ *be the universal cover of* $G/H$. *For each open subset* $U$ *of* $N$, *define the category* $C_H(U)$ *to be the category whose objects are* $\widehat{G/H}$*-bundles whose quotient fiber by fiber by* $\pi$ (*recall that* $\pi$ *is the fundamental group of* $G/H$) *is the restriction of* $\bar{p}$ *to* $U$, *a morphism* $f : e_U \to e'_U$ *between* *two objects $e_U$ and $e'_U$ of $C_H(U)$ is a morphism of $\widehat{G/H}$-bundles which projects to the identity on their quotient by $\pi$. Then the correspondence defined on the category of open subsets of $N$, $U \to C_H(U)$ defines a gerbe, whose classifying cocycle is the obstruction for reducing the structural group $G$ of the bundle $p$ to $H$.*

*Proof.* We first have to show the existence of the bundle $\bar{p}$. Let $(U_i, u_{ij})$ be a trivialization of the bundle $p$. Since $u_{ij}$ take their values in Nor$(H)$, for each element $x$ of $U_i \cap U_j$, the right multiplication by $u_{ij}(x)$ of $G$ gives rise to a $G/H$-action of $u_{ij}(x)$ on $G/H$. We denote by $\overline{u_{ij}}(x)$ this induced action. The map $\overline{u_{ij}} : U_i \cap U_j \to G$, $x \to \overline{u_{ij}}(x)$ verified the Chasle relation and thus defines a $G/H$-bundle $\bar{p}$ over $N$. Now we show that the correspondence $U \to C_H(U)$ is a gerbe.

*Gluing condition for objects.* Let $U$ be an open set of $N$, $(U_i)_{i\in I}$ an open cover of $U$, and $e_i$ an object of $C_H(U_i)$. We suppose that there exist maps $g_{ij} : e^i_j \to e^j_i$ such that ${g_{i_1 i_2}}^{i_3} {g_{i_2 i_3}}^{i_1} = {g_{i_1 i_3}}^{i_2}$. Since $e_i$ is a bundle, there exists a bundle $e$ over $U$ whose restriction to $U_i$ is $e_i$. Since the restriction to $U_i$ of the quotient fiber by fiber, of $e$ by $\pi$ is the quotient fiber by fiber of $e_i$ by $\pi$, we deduce that $e$ is an element of $C_H(U)$.

*Gluing condition for arrows.* For each pair of objects $e$ and $e'$, of $C_H(U)$ the correspondence defined on the category of open subsets of $U$ by $V \to \mathrm{Hom}\,(e_V, {e'}_V)$ where $e_V$ and ${e'}_V$ are the respective restrictions of $e$ and $e'$ to $V$ defines a sheaf, since it is the sheaf of morphisms between two bundles.

This shows that the correspondence defined on the category of open subsets of $N$ by $U \to C_H(U)$ is a sheaf of categories. Now we show that it is a gerbe.

Let $(U_i)_{i\in I}$ be a trivialization of the bundle $\bar{p}$. Then we can lift the restriction of $\bar{p}$ to $U_i$ to a bundle $U_i \times \widehat{G/H}$. This shows that $C_H(U_i)$ is not empty.

Let $U$ be an open set of $N$. Consider two objects $e_U$, and $e'_U$ of $C_H(U)$. The restriction of $e_U$ and $e'_U$ to $U_i \cap U$ are isomorphic to $U_i \cap U \times \widehat{G/H}$. This implies that the connectivity property holds.

The definition of $\mathrm{Hom}\,(e_U, e_U)$, the group of automorphisms of an object $e_U$, shows that its elements coincide with the action of $\pi$, which thus defines a locally constant sheaf on $N$, which is the band of $C_H$.

It remains to show that the triviality of the gerbe $C_H$ is equivalent to the existence of a $H$-reduction of $G$. Let $\widehat{G}$ and $\widehat{H}$ be respectively the universal cover of $G$ and $H$. The homotopy sequence applied to the fibration $\widehat{H} \to \widehat{G} \to \widehat{G}/\widehat{H}$ implies that $\widehat{G}/\widehat{H}$ is simply connected. The map $\widehat{G}/\widehat{H} \to G/H$ is a covering map, thus $\widehat{G}/\widehat{H}$ is the universal cover of $G/H$. Suppose that the gerbe $C_H$ is trivial. Then a global object of this gerbe is a $\widehat{G}/\widehat{H}$-bundle. Since $\widehat{G}/\widehat{H}$ is contractible, we deduce that this bundle is trivial, and thus have a global section. This section projects to a section of $\bar{p}$. This implies that the bundle $p$ has a $H$-reduction. $\square$

**Lemma 2.6.5.** *Suppose that there exists a continuous representation $h : G \to L$ (where $L$ is a quotient of a vector space $V$ by a discrete subgroup $\pi$), whose restriction to the connected component $G_0$ of $G$ is trivial. Suppose also the existence of a continuous cocycle $F$, surjective, for this representation whose kernel $H$ intersects every connected component of $G$. Then for every principal $G$-bundle $p : P \to N$, there exists a gerbe $C_H$, which represents the geometric obstruction for the bundle $p$ to have a $H$-reduction.*

*Proof.* For each of the elements $g \in G$ and $h \in H$, we have $F(gh) = F(g) + gF(h) = F(g)$. This implies that the cocycle $F$ defines a map $\overline{F} : G/H \to L$. The map $\overline{F}$ is surjective since $F$ is surjective. Let $[g]$ and $[g']$ be two elements of $G/H$, suppose that $\overline{F}([g]) = \overline{F}([g'])$. Since $H$ intersects every connected component of $G$, we can choose two elements $g$ and $g'$ in $G_0$, and respectively in the class $[g]$ and $[g']$ such that $F(g) = F(g')$.

$$F(g{g'}^{-1}) = F(g) + h(g)F({g'}^{-1}) = F(g) + F({g'}^{-1}) = 0$$

since $g \in G_0$, and the restriction of $h$ to $G_0$ is trivial. We deduce that $F(g) = F(g')$, thus $\overline{F}$ is a diffeomorphism.

We remark that $H \cap G_0 = H_0$ is a normal subgroup of $G_0$. Since $\overline{F}$ is a diffeomorphism, we deduce that $G/H = G_0/H_0$ is diffeomorphic to $L$.

Consider now a $G$-bundle $p : P \to N$, defined by the trivialization $u_{ij} : U_i \cap U_j \to G$. Then $F(u_{ij})$ defines a $L = G/H$-bundle $\bar{p}$ over $N$. The action of $F(u_{ij}(x))$ on an element $[g]$ of $G/H$ is defined by $F(gu_{ij}(x))$ where $g$ is an element of $[g]$ in $G_0$. For each open subset $U$ of $N$, we define $C_H(U)$ to be the category whose objects are $V$-bundles over $U$, whose quotient fiber by fiber by $\pi$ is the restriction of $\bar{p}$ to $U$. Recall that $L$ is the quotient of $V$ by $\pi$. The set of morphisms $\mathrm{Hom}\,(e, e')$ between two objects $e$ and $e'$ of $C_H(U)$ is the set of morphisms of $L$-bundles which project to the identity on the restriction of $\bar{p}$ to $U$. We are going to show that the correspondence defined on the category of open subsets of $N$, by $U \to C_H(U)$ is a gerbe which represents the geometric obstruction to reduce $G$ to $H$.

*Gluing property for objects.* Consider an open subset $U$ of $N$ and an open covering $(U_i)_{i\in I}$ of $U$. Let $e_i$ be an element of $C_H(U_i)$. Consider a morphism $g_{ij} : e_j^i \to e_i^j$ such that $g_{i_1i_2}{}^{i_3} g_{i_2i_3}{}^{i_1} = g_{i_1i_3}{}^{i_2}$. Since $e_i$ are bundles, there exists a bundle $e$ over $U$ whose restriction to $U_i$ is $e_i$. Since the restriction to $U_i$ of the quotient fiber by fiber of $e$ by $\pi$ is the quotient fiber by fiber of $e_i$ by $\pi$, we deduce that $e$ is an element of $C_H(U)$.

*Gluing condition for arrows.* For each pair of objects $e$ and $e'$, the correspondence defined on the category of open subsets of $U$ by $V \to \mathrm{Hom}\,(e_V, {e'}_V)$, where $e_V$ and ${e'}_V$ are the respective restrictions of $e$ and $e'$ to $V$ defines a sheaf, since it is the sheaf of morphisms between two bundles.

Let $(U_i)_{i\in I}$ be a trivialization of the bundle $\bar{p}$. Then we can lift the restriction of $\bar{p}$ to $U_i$, to a bundle $U_i \times \widehat{G/H}$. This shows that $C_H(U_i)$ is not empty.

Consider two objects $e_U$ and $e'_U$ of $C_H(U)$. The restrictions of $e_U$ and $e'_U$ to $U_i \cap U$ are isomorphic to $U_i \cap U \times \widehat{G/H}$. This implies that the connectivity property holds.

The definition of $\mathrm{Hom}\,(e_U, e_U)$, the group of automorphisms of the bundle $e_U$, shows that it can be identified with $\pi$, which thus defines a locally constant sheaf on $N$ which is the band of $C_H$.

It remains to show that the triviality of the classifying cocycle of the gerbe $c_H$ implies the existence of an $H$-reduction. Let $\widehat{G}$ and $\widehat{H}$ be respectively the universal cover of $G$ and $H$. The homotopy sequence applied to the fibration $\widehat{H} \to \widehat{G} \to \widehat{G}/\widehat{H}$ implies that $\widehat{G}/\widehat{H}$ is simply connected. The map $\widehat{G}/\widehat{H} \to \widehat{G/H}$ is a covering map, thus $\widehat{G}/\widehat{H}$ is the universal cover of $G/H$. Suppose that the gerbe $C_H$ is trivial. Then a global object of this gerbe is a $\widehat{G}/\widehat{H}$-bundle. Since $\widehat{G}/\widehat{H}$ is contractible, we deduce that this bundle is trivial and thus have a global section. This section projects to a section of $\bar{p}$. This implies that the bundle $p$ has a $H$-reduction. □

*Remark.* In the case of Lemma 2.6.5, above, the cohomology class of the classifying cocycle $c_H$ is the obstruction for the bundle $\bar{p}$ to be flat. This implies that it is the Chern class of this bundle.

We are going to apply the above result to study the problem of the existence of $\mathrm{Ham}^s$-reductions.

**Theorem 2.6.6.** *Let $p : P \to N$ be a symplectic bundle whose typical fiber is $(F, \omega)$. Then there exists a gerbe $C_{\mathrm{Ham}^s}$ such that the cohomology class $[c_{\mathrm{Ham}^s}] \in H^2(N, H^1(F, \mathbf{R})/H^1(F, P_\omega))$ of its classifying cocycle $c_{\mathrm{Ham}^s}$ is the obstruction to reduce the structural group of the bundle to* $\mathrm{Ham}^s$*. If the coordinate changes of the bundle take their values in the connected component* $\mathrm{Symp}\,(F, \omega)_0$ *of* $\mathrm{Symp}\,(F, \omega)$*, then there exists a gerbe $C_{\mathrm{Ham}}$ whose classifying cocycle is the obstruction for reduce the structural group to* $\mathrm{Ham}\,(F, \omega)$*.*

*Proof.* The group $\mathrm{Ham}^s$ is the kernel of the continuous surjective 1-cocycle $F_s$, and it intersects every connected component of $\mathrm{Symp}\,(F, \omega)$. The quotient of $\mathrm{Symp}\,(F, \omega)$ by $\mathrm{Ham}^s$ is $H^1(F, \mathbf{R})/H^1(F, P_\omega)$. We can apply Theorem 2.6.4.

Suppose that the coordinate changes take their values in $\mathrm{Symp}\,(F, \omega)_0$, since $\mathrm{Ham}\,(F, \omega)$ is a normal subgroup of $\mathrm{Symp}\,(F, \omega)_0$, and the flux homomorphism allow us to identify $\mathrm{Symp}\,(F, \omega)_0/\mathrm{Ham}\,(F, \omega)$ with $H^1(F, \Gamma)/H^1(F, \Gamma)$, where $\Gamma$ is the flux group, we can apply Theorem 2.6.4. □

*Remark.* In differential geometry, as in the theory of $G$-structures, the question of finding reductions of a $G$-bundle is intensively studied. Let $H$ be a subgroup of $G$; if the left quotient $H/G$ is a $K(\pi, 1)$ space, it is possible to write a similar theorem to the one above and obtain an obstruction cocycle whose cohomology class decides upon the existence of a $H$-reduction. This can be, for example, applied to the existence of a Riemannian structure on a manifold and also to solve differential equations defined on jet-bundles, since in many cases the existence of solutions is equivalent to the existence of reductions of jet-bundles.

We will give now another proof of the theorem of McDuff mentioned above which says that the existence of a closed connection form on a symplectic bundle $p : P \to N$ implies the existence of a $\mathrm{Ham}^s$-reduction.

**Theorem 2.6.7** (see McDuff [**16**]). *Let $p : P \to N$ be a symplectic bundle endowed with a closed connection form. Then there exists on $P$ a $\mathrm{Ham}^s$-reduction.*

*Other proof.* Suppose the existence of a closed connection form defined on the bundle $p : P \to N$. We have to show that the cohomology class $[c_{\mathrm{Ham}^s}]$ is trivial. It has been shown by McDuff-Lalonde [**12**], that the holonomy around a contractible loop is Hamiltonian. Consider the reduction of the symplectic bundle to the holonomy of the closed connection form. Since the Hamiltonian group is the connected component of $\mathrm{Ham}^s$, we deduce that the composition of the transitions functions $u_{ij}$ and of $F_s$, $F_s(u_{ij})$ is constant, if needed, we shrink the open set $U_i$ such that $u_{ij}(U_i \cap U_j)$ is contained in the same connected component of $\mathrm{Symp}\,(F, \omega)$. This implies that the bundle $\bar{p}$ (defined in the proof of Lemma 2.6.5) is flat. Thus, its Chern class is a torsion class. Since the lattice $\pi$ in this case is a $\mathbf{Q}$-vector space, we deduce that the Chern class of this bundle is zero. □

*Sketch of the proof of McDuff* [**16**]. McDuff defines for each symplectic bundle $p : P \to N$ of fiber $(F, \omega)$, a cohomology 2-class in $H^2(N, H^1(F, P_\omega))$ (in fact it is the class of 2.6.5) as follows: The bundle $p$ is defined by a classifying map $p' : N \to B\mathrm{Symp}\,(F, \omega)$. The map $F_s$ induces a map $F'_s : B\mathrm{Symp}\,(F, \omega) \to BH^1(F, \mathbf{R})/H^1(F, P_\omega)$. There exists a $\mathrm{Ham}^s$-reduction if and only if the composition $F'_s \circ p'$ is null homotopic, since we have an exact sequence

$$1 \longrightarrow \mathrm{Ham}^s \longrightarrow \mathrm{Symp}\,(F, \omega) \longrightarrow H^1(F, \mathbf{R})/H^1(F, P_\omega) \longrightarrow 1.$$

The space $BH^1(F,\mathbf{R})/H^1(F,P_\omega)$ is a $K(H^1(F,P_\omega),2)$-space, and the set of homotopy classes of maps $N \to K(H^1(F,P_\omega),2)$ is one-to-one with $H^2(N,P_\omega)$. The obstruction class of McDuff is defined to be the homotopy class of $F'_s \circ p'$.

The proof of McDuff of the previous result is done by showing that the previous class vanishes on the 2 sub-complex of the $CW$-complex $N$. In this regard, she shows that it is the image by a null connecting homomorphism related to an exact sequence of a one class. □

**2.7. The universal obstruction of McDuff.** In this part, we will show how the universal class defined by McDuff can be defined using gerbe theory.

Let $E\mathrm{Symp}\,(F,\omega) \to B\mathrm{Symp}\,(F,\omega)$ be the universal bundle of the group $\mathrm{Symp}\,(F,\omega)$. The 1-cocycle $F_s : \mathrm{Symp}\,(F,\omega) \to H^1(F,\mathbf{R})/H^1(F,P_\omega)$ defined by McDuff induces an $H^1(F,\mathbf{R})/H^1(F,P_\omega)$-bundle on $B\mathrm{Symp}\,(F,\omega)$. See Lemma 2.6.5. The Chern class $U_F$ of this bundle is the universal class $U_M$ defined by McDuff; it can be viewed as the cohomology class of the classifying cocycle of the gerbe which represents geometrically the obstruction for the previous $H^1(F,\mathbf{R})/H^1(F,P_\omega)$-bundle to be trivial. Since each $(F,\omega)$-symplectic bundle $p : P \to N$, is classified by a classifying map $f : N \to B\mathrm{Symp}\,(F,\omega)$, the obstruction class to obtain a $\mathrm{Ham}^s$-reduction is $f^*(U_F)$. This is the class defined in the sketch of the proof of McDuff in 2.6.

**2.8. Generalizations to topoi.** The previous construction applied to symplectic bundles can be generalized to other situations; algebraic geometry, arithmetic, etc. In this regard we will adapt this result to topoi.

**Definition 2.8.1.** Let $G$ be a group endowed with a topology. The topology can be the Zariski, etale, the Lie topology, etc. A continuous right $G$-action of $G$ on the topos $(P,J_P)$ is a continuous functor $d_G : P \times G \to P$ such that if $u$ is the multiplication of $G$ by $g$, $d_G \circ (\mathrm{Id}_P \times u) = d_G(d_G \times \mathrm{Id}_G)$.

A *$G$-torsor* defined on a topos $N$ is a continuous functor $p : (P,J_P) \to (N,J_N)$ such that:

(i) $(P, J_P)$ is endowed with an action of $G$, $p$ commutes with the action of $G$ that is, the composition $P \times G \to P \to N$, (where $P \times G \to P$ is the canonical projection, and $P \to N$ is $p$) and $P \times G \to P \to N$ (where $P \times G \to P$ is the multiplication $d_G$ and $P \to N$ is $p$) coincide.

(ii) The canonical map $P \times G \to P \times P \times G \to P \times P$ which is the composition of the canonical embedding $P \times G \to P \times P \times G$, and the product of the identity on the first factor, and the multiplication $d_G$ on the second and third factor is an isomorphism. We suppose that the quotient of $P$ by $G$ is $N$. Recall that the quotient of $P$ by $G$ is the initial element in the category of maps $p' : P \to N'$ such that $p'$ commutes with the action of $G$. $\square$

We will assume that the torsor is locally trivial. This means that there exists a covering family of $N$, $(U_i)_{i \in I}$ such that: There exists an isomorphism $u_i : P_{|U_i} \to U_i \times G$ between the restriction $P_{|U_i}$ of $P$ to $U_i$ and $U_i \times G$. We can thus define $u'_{ij} = u_i \circ {u_j}^{-1}{}_{|U_i \times_N U_j \times G} : U_i \times_N U_j \times G \to U_i \times_N U_j \times G$. Let $e' : G \to G$, $g \to e$, where $e$ is the neutral of $G$ and $e_{ij} : U_i \times_N U_j \times G \to G$ the canonical projection. We can define $u_{ij} : U_i \times_N U_j \to G$ by $e_{ij} \circ u'_{ij} \circ (\mathrm{Id}_{U_i \times_N U_j} \times e')$. We have ${u'_{i_1 i_2}}^{i_3} {u'_{i_2 i_3}}^{i_1} = {u'_{i_1 i_3}}^{i_2}$; $P$ is obtained by gluing the family of $(U_i \times G)_{i \in I}$ using $u'_{ij}$.

Let $H$ be a subgroup of $G$. We say that the torsor $P \to N$ has an $H$-reduction if and only if it is isomorphic to a torsor whose transition functions $u_{ij}$ take their values in $H$.

**Theorem 2.8.2.** *Let $p : P \to N$ be a $G$-torsor. Suppose that either*

1. *there exists a subgroup $H$ of $G$ such that $G/H$ is a $K(\pi, 1)$ space, and the torsor has a* $\mathrm{Nor}\,(H)$*-reduction, or*

2. *there exists a* 1*-cocycle surjective and continuous $F : G \to L$ for a representation $h$ of $G$, where $L$ is the quotient of a vector space by a discrete subgroup $\pi$, such that the restriction of $h$ to the connected component $G_0$ of $G$ is trivial, and the kernel $H$ of $F$ intersects every connected component of $G$, or*

3. *the left quotient $H/G$ is a $K(\pi, 1)$-space.*

*Then there exists a gerbe $C_H$ defined on $N$ such that the cohomology class of its classifying cocycle is the obstruction for reduce $G$ to $H$.*

*Proof.* We will only give the proof in the first case. The fact that the torsor has a $\mathrm{Nor}\,(H)$-reduction implies the existence of a $G/H$-torsor $\overline{P}$, which is the right quotient of $P$ by $H$.

For each object $U$ of $N$, we define $C_H(U)$ to be the category whose objects are $\widehat{G/H}$-torsors whose quotient by $\pi$ is the restriction of $\overline{P}$ to $U$. A morphism between two objects of $C_H(U)$ is a morphism of torsors which projects to the identity on the restriction of $\overline{P}$ to $U$.

Now we show that the correspondence $U \to C_H(U)$ is a gerbe.

*Gluing condition for objects.* Let $U$ be an object of $N$, $(U_i)_{i\in I}$ a covering family of $U$, $e_i$ an object of $C_H(U_i)$. We suppose that there exist maps $g_{ij} : e^i_j \to e^j_i$ such that $g_{i_1i_2}{}^{i_3}g_{i_2i_3}{}^{i_1} = g_{i_1i_3}{}^{i_2}$. Since $e_i$ are torsors, there exists a torsor $e$ over $U$ whose restriction to $U_i$ is $e_i$. Since the restriction to $U_i$ of the quotient of $e$ by $\pi$ is the quotient of $e_i$ by $\pi$, we deduce that $e$ is an element of $C_H(U)$.

*Gluing condition for arrows.* For each of the objects $e$ and $e'$, the set of morphims defined on the sub-topos over $U$, by $V \to \mathrm{Hom}\,(e_V, e'_V)$, where $e_V$ and $e'_V$ are the respective restrictions of $e$ and $e'$ to $V$ defines a sheaf of sets, since it is the sheaf of morphisms between two torsors.

Let $(U_i)_{i\in I}$ be a trivialization of the torsor $\overline{P}$. We can lift the restriction of $\overline{P}$ to $U_i$ to the torsor $U_i \times \widehat{G/H}$. This shows that $C_H(U_i)$ is not empty.

Consider two objects $e_U$ and $e'_U$ of $C_H(U)$. The restrictions of $e_U$ and $e'_U$ to $U_i \times_N U$ are isomorphic to $U_i \times_N U \times \widehat{G/H}$, this implies that the connectivity property holds.

The group $\mathrm{Hom}\,(e_U, e_U)$ is the group of automorphisms of the torsor $e_U$ which project to the identity isomorphism of the restriction of $\bar{p}$ to $U$. This group is identified to $\pi$, which thus defines a locally constant sheaf on $N$ which is the band of $C_H$. $\square$

*Remark.* The triviality of the gerbe $C_H$ does not necessarily imply the existence of an $H$-reduction, if $N$ is not a manifold. Since, for other categories, homotopy is not well-understood, there are no precise definitions of null-homotopic maps.

**3. The group** $\mathrm{Ham}^s$ **and the etale topos of a manifold.** The group $\mathrm{Ham}^s$ introduced by McDuff allows to characterize symplectic bundles whose closed connection forms are the symplectic bundles endowed with a $\mathrm{Ham}^s$-reduction. In [**16**] it is shown that a $\mathrm{Symp}\,(F,\omega)_0$-bundle $p: P \to N$ is endowed with a closed connection form if and only if there exists a finite cover $\widehat{N}$ of $N$, such that the pull-back of $p$ to $\widehat{N}$ has a Hamiltonian reduction. This motivates to define $\mathrm{Symp}\,(F,\omega)$-bundles on the etale topos of $N$. The motivation is due to this historical remark: In algebraic geometry, algebraic principal bundles are locally trivial up to a finite etale cover. This has motivated the definition of the etale topology.

**Definition 3.1.** The *etale topos* of a manifold $N$ is the category whose objects are differentiable maps $c : U \to N$ which are finite covering maps onto their images. A morphism between two objects is a covering map.

A covering family of the etale topos, $\mathrm{Et}\,(N)$ of $N$, is a family $(U_i)_{i\in I}$ such that the arrow $u_i : U_i \to N$ is a finite etale cover, and the union of $(u_i(U_i))_{i\in I}$ is $N$.

A symplectic bundle $p : P \to \mathrm{Et}\,(N)$ whose typical fiber is the symplectic manifold $(F,\omega)$ defined by a covering family $(U_i)_{i\in I}$ of $\mathrm{Et}\,(N)$ for the etale topology. The transition functions are symplectic bundles isomorphisms of the trivial symplectic bundle $U_i \times_N U_j \times \mathrm{Symp}\,(F,\omega)$, defined by $u_{ij} : U_i \times_N U_j \to \mathrm{Symp}\,(F,\omega)$ such that ${u_{i_2 i_3}}^{i_1} {u_{i_1 i_2}}^{i_3} = {u_{i_1 i_3}}^{i_2}$.

A closed connection form on the symplectic bundle is defined by a family of closed connections forms $\Omega_i$ of the bundle $e_i : U_i \times (F,\omega)$ (recall that $\Omega_i$ is a 2-form which extends $\omega$), such that on $U_i \times_N U_j$, we have: ${u_{ij}}^*(\Omega_{i|U_i\times_N U_j}) = \Omega_{j|U_i\times_N U_j}$, where $\Omega_{i|U_i\times_N U_j}$ and $\Omega_{j|U_i\times_N U_j}$ are the respective restrictions of $\Omega_i$ and $\Omega_j$ to $U_i \times_N U_j$.

A symplectic bundle defined on $N$ induces canonically a symplectic bundle on $\mathrm{Et}\,(N)$, since an open covering of $N$ defines an etale covering of $N$. □

**Proposition 3.2.** *Let $P$ be a symplectic bundle defined on the etale topos of a manifold $N$. Then there exists a symplectic bundle $\widehat{P}$ defined on a covering space $\widehat{N}$ of $N$, such that the symplectic bundle induced by $\widehat{P}$ on $\mathrm{Et}\,(\widehat{N})$, is the pull-back of $P$ by the covering map $\widehat{N} \to N$. If $N$ is compact we can suppose that $\widehat{N}$ is a finite cover.*

*Proof.* Let $(d_i : U_i \to N)_{i \in I}$ be the etale covering family of $N$ which defines the symplectic bundle. Then we can define a manifold $\widehat{N}$ as follows: $\widehat{N}$ is the quotient of the union of $U_i$ by identifying the elements $u_i \in U_i$, and $u_j \in U_j$ such that $d_i(u_i) = d_j(u_j)$. We denote by $l_i : U_i \to \widehat{N}$ the canonical map. The manifold $\widehat{N}$ is a cover of $N$ since the restriction of the canonical projection $\widehat{N} \to N$ to $l_i(U_i)$ is $d_i {l_i}^{-1}$.

There exists a diffeomorphism $l_{ij} : l_i(U_i) \cap l_j(U_j) \to U_i \times_N U_j$, such that on

$$l_{i_1}(U_{i_1}) \cap l_{i_2}(U_{i_2}) \cap l_{i_3}(U_{i_3}), {l_{i_2 i_3}}^{{i_1}^{-1}} {l_{i_1 i_2}}^{i_3} = \mathrm{Id}_{l_{i_1}(U_{i_1}) \cap l_{i_2}(U_{i_2}) \cap l_{i_3}(U_{i_3})};$$

thus, we can define the symplectic bundle $\widehat{P}$ on $\widehat{N}$ by gluing $l_i(U_i) \times (F, \omega)$ using $u'_{ij} = u_{ij} \circ l_{ij}$, where $u_{ij}$ are the transition functions of $P$.

The construction of $\widehat{P}$ shows that the induced bundle on $\mathrm{Et}\,(\widehat{N})$, by $\widehat{P}$, is the pull-back of $P$ by the canonical map $\widehat{N} \to N$. If $N$ is compact, then we can suppose that there exists a finite number of $U_i$. This implies that $\widehat{N}$ is compact and therefore is a finite cover of $N$. □

We can rewrite the theorem of McDuff [**16**] as follows:

**Theorem 3.3.** *Let $p : P \to \mathrm{Et}\,(N)$ be an $\mathrm{Sym}\,(F, \omega)_0$-bundle defined on the etale topos of a compact manifold $N$. Then $P$ has a closed connection form if and only if it has a Hamiltonian reduction.*

*Proof.* The previous proposition shows that there exists a finite cover $\widehat{N}$ of $N$ and an induced symplectic bundle $\widehat{P}$ over $\widehat{N}$. Suppose that the closed connection form is defined on $P$ by the family of 2-forms

$\Omega_i$ defined on the etale cover $(l_i : U_i \to N)_{i\in I}$. As in the previous proposition, we can show that there exists a finite cover $N'$ of $N$ such that the pull-back $P'$ of $\widehat{P}$ to $N'$ is endowed with a closed connection form, such that the closed connection form induced on its etale cover is defined on $U'_i = P' \times_N U_i$ by the pull-back of $\Omega_i$ by $P' \times_N U_i \to U_i$. We can apply the result of McDuff and obtain a Hamiltonian reduction $P'' \to N''$ on the pull-back of $P''$ of $P'$ to a finite cover $N''$ of $N'$. We denote by $l''_i$ the canonical map $l''_i : U''_i = U'_i \times_{N'} P'' \to N''$. There exists a family of maps $u''_i : l''_i(U''_i) \to \mathrm{Symp}\,(F,\omega)$ such that $u''_i u''_{ij} {u''_j}^{-1} \in \mathrm{Ham}\,(F,\omega)$, where $u''_{ij}$ are the coordinate changes of $P''$; thus, $u''_i u''_{ij} {u''_j}^{-1} {l''_j}^{-1}$ defined a Hamiltonian reduction of $P$. Since the family $(U''_i \to N'')_{i\in I}$ is an etale cover of $N''$, $(U''_i \to N'' \to N)_{i\in I}$ is also an etale cover of $N$. □

**4. Flux and holonomy of gerbes.** In this part, we will relate the flux of a symplectic manifold $(F,\omega)$ to the holonomy of the gerbe $C(\omega)$ defined in subsection 2.4.

Let $E$ be a $T^1$-gerbe defined on $P$, that is, a gerbe such that for each open set $U$ of $P$, $E(U)$ is a category of $T^1$-bundles defined on $U$. Consider an open covering $(U_i)_{i\in I}$ of $P$ such that $U_i$ is contractible. Let $e_i^j$ be the restriction of an object $e_i$ of $E(U_i)$ to $U_i \cap U_j$. There exists a morphism $u_{ij} : e_j^i \to e_i^j$. We denote by $c_{ijl}$, the automorphism $u_{li}u_{ij}u_{jl}$ of the restriction of $e_l$ to $U_i \cap U_j \cap U_l$. It is defined by a $T^1$-differentiable function. Since $c_{ijl}$ is the classifying 2-cocycle of $E$, there exists a 1-chain $h_{ij}$ of 1-forms such that:

$$h_{jl} - h_{il} + h_{ij} = -\frac{i}{2\pi} d(\mathrm{Log}\,(c_{ijl}));$$

since $d(h_{ij})$ is a 1-cocycle, there exists a 0-chain of 2-forms $L_i$ such that

$$L_j - L_i = d(h_{ij}).$$

**Definition 4.1.** The family of forms $h_{ij}$ is a called a *connection of the gerbe*, and the family of forms $(L_i)_{i\in I}$ is the curving of the gerbe. This means that there exists a related connective structure $Co$ defined on the gerbe, and elements $\alpha_i$ of $Co(e_i)$, such that $h_{ij} = \alpha_j - {u_{ij}}^* \alpha_i$.

The 3-form whose restriction to $U_i$ is $dL_i$ is the curvature of the connective structure. Suppose that the curvature is zero. Then $L_i = d(L'_i)$, $h_{ij} = L'_j - L'_i + d(h'_{ij})$, and we denote by $c'_{i_1i_2i_3} = -i/(2\pi)\mathrm{Log}\,(c^{-1}_{i_1i_2i_3}) + h'_{i_2i_3} - h'_{i_1i_3} + h'_{i_1i_2}$ to be the *holonomy of the connection*, $c'_{i_1i_2i_3}$ is constant and is a 2-cocycle. □

**Definition 4.2.** For each map $l : N_2 \to P$, where $N_2$ is a surface without a boundary, the pull-back of the gerbe, and its connective structure to $N_2$, by $l$ has a vanishing curving. Using the Cech-de Rham isomorphism, we can identify the holonomy cocycle of this gerbe with a 2-form $\mathrm{Hol}\,(h_{ij}, N_2)$. The holonomy of the connection on $N_2$ is

$$\int_{N_2} \mathrm{Hol}\,(h_{ij}, N_2). \qquad \square$$

Let $(F, \omega)$ be a symplectic manifold, and let $C_F(\omega)$, the $T^1$-gerbe representing the obstruction of $[\omega]$, be integral. If the band of this gerbe is extended to the sheaf of differentiable $T^1$-functions, it becomes trivial and flat.

For each open set $U$ of $F$, the set of connections defined on an object $e_U$ of $C_F(\omega)(U)$ whose curvature is the restriction of $\omega$ to $U$ defines a connective structure, the curving of the connective structure is the restriction of $\omega$ to $U_i$. The cocycle representing the holonomy of this connective structure is the image of $\omega$ by the Cech-de Rham isomorphism. This can be deduced from subsection 2.4.3.

Let $l : N_2 \to F$ be a differentiable map defined on the surface $N_2$; the holonomy of this connective structure around $N_2$ is:

$$\int_{N_2} l^*(\omega).$$

This definition is related to the definition of the flux, since for each path $\gamma = c_t$ of $N$, and each path $\phi_t$ of the connected component of $\mathrm{Symp}\,(F, \omega)$, $\phi_t(\gamma)$ is a map from $h : I_2 \to F$, the flux of $\phi_t(\gamma)$ is nothing but the half of the holonomy around the sphere $S^2$ obtained by gluing two copies of $I_2$ along their boundaries. The map $f : S^2 \to N$

is obtained by restricting $h$ to each copy of $I_2$. The holonomy of $f$ is defined to be the limit of the holonomy of a sequence of differentiable maps which converges towards $f$.

## 5. A geometric interpretation of a section of $H_1(M,\mathbf{R}) \to SH_1(M,\mathbf{R})$.

In [**16**], McDuff gives a geometric interpretation of a section $p : H_1(M,\mathbf{R}) \to SH_1(M,\mathbf{R})$, when the cohomology class $[\omega]$ is integral. In this section we generalize this interpretation when $[\omega]$ is not necessarily integral. We denote by $\pi : SH_1(M,\mathbf{R}) \to H_1(M,\mathbf{R})$ the projection map. Suppose that the class $[\omega]$ of the symplectic manifold $(F,\omega)$ is not necessarily integral. Consider a cycle $[\gamma]$ represented by the chain $h : T^1 \to F$, where $T^1$ is the circle, and the pull-back by $h$, of the gerbe $C_F(\omega)$, to $T^1$ is trivial.

**Proposition 5.1.** *Consider an object $e$ of $h^*(C_F(\omega))$ which is the pull-back of an object $e'$ of a tubular neighborhood of $h(T^1)$. Let $L$ be a connection in $h^*Co(e')$. Denote by $h'_L(\gamma)$ the holonomy around $\gamma$ of $L$. It does not depend on the element chosen in $h^*(Co(e'))$.*

*Proof.* To show this, consider another connection $L'$ in $h^*(Co(e'))$. We can suppose that $h(T^1)$ is covered by $(U_i)_{i\in I}$, the union of $U_i$ is a tubular neighborhood of $h(T^1)$, and $C_F(\omega)(U_i)$ is not empty. The fact that the union of $U_i$ is a tubular neighborhood of $h(T_1)$ implies that the restrictions of $L$ and $L'$ to $I_i = h^{-1}(h(T^1)\cap U_i)$ can be supposed to be the pull-back of elements $d+\alpha_i$ and $d+\alpha'_i$ of $Co(e_i)$, where $d$ is the differential $e_i$ is an object of $C_F(\omega)(U_i)$ and $\alpha_i$ and $\alpha'_i$ are 1-forms defined on $U_i$. We have $\alpha'_i = \alpha_i + df'_i$ where $f'_i$ is a function defined on $U_i$ since the curving of the gerbe is the closed form $\omega$. Denote by $L_i$ and $L'_i$ the restrictions of $L$ and $L'$ to $I_i$. On $I_i$, $L_i = d + du_i$, the coordinate changes $v_{ij}$ of the bundle $e$ are defined by $du_j - du_i = -i(1/2\pi)d\mathrm{Log}\,(v_{ij})$, the holonomy cocycle of $L$ is given by $-(i/2\pi)\mathrm{Log}\,(v_{ij}^{-1}) - u_j + u_i)$. Since $L'_i = d + d(u_i + f_i)$, where $f_i$ is the pull-back of $f'_i$ by $h$. We deduce that the holonomy cocycles of $L$ and $L'$ coincide up to a boundary. Thus, their cocycles have the same cohomology class. $\square$

We can define $h_L(\gamma)$ to be the image of the holonomy of this connection in $\mathbf{R}/P_\omega$. Let $[\gamma] \in H_1(F,\mathbf{R})$ define the section $p([\gamma])$ to be the class of elements $\gamma'$ in $\pi^{-1}([\gamma])$ such that the holonomy around $\gamma'$ is in $P_\omega$.

**6. Existence of symplectic bundles and gerbes.** Let $F$ be the flux, and let $\Gamma_\omega$ be the flux group. The flux conjecture has been shown recently by Ono, thus $\Gamma_\omega$ is a discrete subgroup of $H^1(F, \mathbf{R})$. There exists an exact sequence

$$1 \longrightarrow \mathrm{Ham}\,(F,\omega) \longrightarrow \mathrm{Symp}_0(F,\omega) \longrightarrow H^1(F,\mathbf{R})/\Gamma_\omega \longrightarrow 1.$$

Let $p : P \to N$ be a symplectic bundle defined by the coordinate changes $g_{ij} : U_i \cap U_j \to \mathrm{Symp}_0(F,\omega)$ on the trivialization $(U_i)_{i\in I}$. We can project the cocycle $g_{ij}$ to maps $F(g_{ij}) = {g'}_{ij} : U_i \cap U_j \to H^1(F,\mathbf{R})/\Gamma_\omega$ and obtain an $H^1(F,\mathbf{R})/\Gamma_\omega$-bundle as in subsection 2.6.5. A natural question is the following: given an $H^1(F,\mathbf{R})/\Gamma_\omega$-bundle $\bar{p}$, is there a symplectic bundle which gives rise to $\bar{p}$?

This problem is an example of the basic examples which have motivated the definition of gerbes theory. Consider an open set $U$ of $N$, and $C(U)$ a category of symplectic bundles whose transition functions take their values in $\mathrm{Symp}\,(F,\omega)_0$ and which induces the restriction of $\bar{p}$ to $U$. Suppose that $\bar{p}$ is defined by the transition functions $g'_{ij}$, and there exist elements $g_{ij}$ over $g'_{ij}$ such that the conjugation by $g_{ij}$ in $\mathrm{Ham}\,(F,\omega)$ defined a bundle over $N$ whose typical fiber is $\mathrm{Ham}\,(F,\omega)$. We suppose also that the automorphism group of an object $e_U$ of $C(U)$ are the sections of the previous $\mathrm{Ham}\,(F,\omega)$-bundle. We denote by $L_1$ the sheaf of those sections. The correspondence $U \to C(U)$ is a gerbe bounded by $L_1$.

Denote by $l$ the rank of the group $\Gamma_\omega$, the torus $T^l$ is the maximal compact subgroup of $H^1(M,\mathbf{R})/\Gamma_\omega$. The bundles defined over $N$, whose fiber is $T^l$, are classified by their first Chern class. This can enable to construct symplectic bundles which does not admit Hamiltonian reductions if the Chern class is not zero.

**7. 2-gerbes, 2-gerbed towers.** The notion of 2-gerbe has been defined by Lawrence Breen [**2, 3**]. It allows one to represent geometrically 3-cohomology classes. In the preprint [**20**], Tsemo has defined the notion of gerbed towers; this is a recursive definition of geometric representations of cohomology classes. We will now present the notion of 2-gerbes and 2-gerbed towers, which enable us to cope with the extension problem when $[\omega]$ is not necessarily an integral class. An alternative discussion has been presented above using the group $\mathrm{Ham}^s$;

the construction given in this section allows to show the existence of a connection on a bundle which has a Hamiltonian reduction without using the Guillemin-Lerman-Sternberg construction. The definition of the sheaf of 2-categories uses the definition of 2-categories or bicategories which has been defined by Benabou.

**Definition.** A *bicategory* $C$ is defined by a class of objects $C$, for each pair of objects $u$ and $v$ of $C$, a category $\operatorname{Hom}(u,v)$. The objects of $\operatorname{Hom}(u,v)$ are called the 1-arrows, and the arrows of $\operatorname{Hom}(u,v)$ are the 2-arrows. There exists a composition map:

$$\operatorname{Hom}(u_2,u_3)\times\operatorname{Hom}(u_1,u_2)\longrightarrow\operatorname{Hom}(u_1,u_3).$$

For each quadruple $(u_1,u_2,u_3,u_4)$, there exists an isomorphism between the functors

$$(\operatorname{Hom}(u_3,u_4)\times\operatorname{Hom}(u_2,u_3))\times\operatorname{Hom}(u_1,u_2)\longrightarrow\operatorname{Hom}(u_1,u_4)$$

and

$$\operatorname{Hom}(u_3,u_4)\times(\operatorname{Hom}(u_2,u_3)\times\operatorname{Hom}(u_1,u_2))\longrightarrow\operatorname{Hom}(u_1,u_4)$$

which satisfies more compatibility axioms which can be found in Breen [**2**]. □

**Definition.** Let $N$ be a manifold; a *sheaf of* 2*-categories* is a correspondence $C$ defined on the category of open subsets of $N$ by:

$$U\longrightarrow C(U)$$

where $C(U)$ is a 2-category, which verifies the following properties: for each embedding map $U\to V$, there exists a restriction functor $r_{U,V}:C(V)\to C(U)$, such that

$$r_{U_1,U_2}\circ r_{U_2,U_3}=r_{U_1,U_3}.$$

**Gluing properties for objects.** Let $(U_i)_{i\in I}$ be a covering family of an open set $U$ of $N$, $e_i$ an object of $C(U_i)$, and a 1-arrow $g_{ij}$ :

$r_{U_i\cap U_j,U_j}(e_j) \to r_{U_i\cap U_j,U_i}(e_i)$. Suppose that there exists a 2-arrow $h_{i_1i_2i_3} : {g_{i_1i_2}}^{i_3}{g_{i_2i_3}}^{i_1} \to {g_{i_1i_3}}^{i_2}$ which satisfies:

$$ {h_{i_1i_2i_4}}^{i_3}(\mathrm{Id} \circ {h_{i_2i_3i_4}}^{i_1}) = {h_{i_1i_3i_4}}^{i_2}({h_{i_1i_2i_3}}^{i_4} \circ \mathrm{Id}). $$

Then there exists an object $e$ of $C(U)$ whose restriction to $U_i$ is $e_i$.

**Gluing conditions for arrows.** For each pair of objects $e$ and $e'$ of $U$, the correspondence defined on the category of open sets contained in $U$ by $V \to \mathrm{Hom}\,(r_{U,V}(e), r_{U,V}(e'))$ defines a sheaf of categories.

A 2-*gerbe* is a sheaf of bicategories which satisfies the following:

1. The bicategory $C(U)$ is a 2-groupoid; this means that 1-arrows are invertible up to 2-arrows, and 2-arrows are invertible.

2. For every point $x$ of $N$, there exists a neighborhood $U_x$ of $x$, such that $C(U_x)$ is not empty.

3. Any pair of objects $e$ and $e'$ of $C(U)$ is locally isomorphic. This means that there exists an open covering $(U_i)_{i\in I}$ of $U$ such that the restrictions $e_i$ and $e'_i$ of respectively $e$ and $e'$ to $U_i$ are isomorphic.

We say that a 2-gerbe is bounded by the sheaf of abelian groups $L$, if the following two conditions are satisfied:

4. Any pair of 1-arrows can be joined by a 2-arrow.

5. Let $e_U$ and $e'_U$ be a pair of objects of $C(U)$. For any 1-arrow $h : e_U \to e'_U$, there is a specified isomorphism $L(U) \to \mathrm{Aut}\,(h)$, compatible with compositions and with restrictions $\bullet$. We say that the sheaf $L$ is the band of the 2-gerbe, or that the gerbe is bounded by $L$.

**7.2. Classifying cocycle of a 2-gerbe.** Let $(U_i)_{i\in I}$ be an open covering of $N$ such that $C(U_i)$ is not empty. Consider an object $e_i$ of $C(U_i)$, and $g_{ij} : r_{U_{ij},U_j}(e_j) \to r_{U_{ij},U_i}(e_i)$. There exists a 2-arrow $h_{i_1i_2i_3} : {g_{i_1i_2}}^{i_3}{g_{i_2i_3}}^{i_1} \to {g_{i_1i_3}}^{i_2}$, and on $U_{i_1i_2i_3i_4}$ a 2-arrow $u_{i_1i_2i_3i_4}$ which verifies:

$$ {h_{i_1i_2i_4}}^{i_3}(\mathrm{Id} \circ {h_{i_2i_3i_4}}^{i_1}) = u_{i_1i_2i_3i_4}({h_{i_1i_3i_4}}^{i_2}({h_{i_1i_2i_3}}^{i_4} \circ \mathrm{Id})). $$

The family $u_{i_1i_2i_3i_4}$ is the classifying 2-cocycle of $C$; if the sheaf $L$ is commutative, it is a Cech cocycle in the classical sense, and the set of isomorphic classes of 2-gerbes bounded by $L$ is isomorphic to $H^3(N, L)$. If $L$ is not commutative, we define $H^3(N, L)$ to be the set of isomorphic classes of 2-gerbes bounded by $L$.

In [**20**] we have given a simplified version of 2-gerbes, which we have named 2-gerbed towers.

**Definition 7.2.1.** A 2-*gerbed tower* defined on $N$, is defined by a gerbe $C$ on $N$ and, for each object $e_U$ of $C(U)$, a gerbe $C_1(e_U)$ defined on $U$ such that the following conditions are satisfied:

(i) For each embedding map $U \to V$, there exists a restriction functor $r^1_{U,V} : C_1(e_U) \to C_1(r_{U,V}(e_U))$ such that $r^1_{V,W} \circ r^1_{U,V} = r^1_{U,W}$, where $r$ is the restriction functor of the gerbe $C$.

(ii) There exists a commutative sheaf $L_1$ defined on $N$, such that, for each object $e_U$ of $C(U)$, the band of $C_1(e_U)$ is the restriction of $L_1$ to $U$.

(iii) For each morphism $h : e_U \to e'_U$ of objects of $C(U)$, there exists a functor $h^* : C_1(e_U) \to C_1(e'_U)$ which is compatible with restrictions, such that for a morphism $h' : e'_U \to e''_U$, there exists a natural transformation between the functors $(h'h)^*$ and $h'^*h^*$. We suppose also the functors $(h'h)^*$ and $h'^*h^*$ coincide on objects. This implies the existence of an element $l_{h',h}$ of $L_1(U)$ such that $(h'h)^* = l_{h',h} \circ h'^*h^*$.

**7.3. The classifying cocycle of a 2-gerbed tower.** We can associate to a 2-gerbed tower, a 3-Cech cocycle defined as follows: Consider an object $e_i$ of $C(U_i)$ and a morphism $g_{ij} : r_{U_{ij},U_j}(e_j) \to r_{U_{ij},U_i}(e_i)$. The arrow $c_{i_1i_2i_3} = {g_{i_3i_1}}^{i_2}{g_{i_1i_2}}^{i_3}{g_{i_2i_3}}^{i_1}$ is the Cech classifying cocycle of the gerbe $C$. It can be identified to an element of the band of $C$.

The classifying cocycle of the 2-gerbed tower is defined by considering the family of automorphisms

$$c_{i_1i_2i_3i_4} = ({c_{i_2i_3i_4}}^{i_1})^*(-{c_{i_1i_3i_4}}^{i_2})^*({c_{i_1i_2i_4}}^{i_3})^*(-{c_{i_1i_2i_3}}^{i_4})^*.$$

Property (iii) implies that $c_{i_1i_2i_3i_4}$ is an element of $L_1(U_{i_1} \cdots i_4)$. Contrary to the case of 2-gerbes, it is after having defined the classifying cocycle that we set the axiom concerning the gluing property for objects:

**Gluing property for objects.** Suppose that the cohomology class of the classifying cocycle of a 2-gerbed tower is zero. Let $(U_i)_{i\in I}$ be the open covering of $N$ used to construct the cocycle. Then there exists a gerbe $C_0$, such that for each open subset $U$ of $N$, the restriction of $C_0$ to $U\cap U_i$ is $C_1(e_{iU})$, where $e_{iU}$ is the restriction of $e_i$ to $U_i\cap U$, and $e_i$ is the object of $C(U_i)$ used to construct the 2-cocycle.

**Proposition 7.3.1.** *Let $(C, C_1)$ be a 2-gerbed tower defined on $N$, the correspondence defined on the category of open subsets of $N$ as follows*:

*To each open set $U$ of $N$, $C'(U)$ is the bicategory whose objects are gerbes $E_U$ such that, for every open covering $(U_i)_{i\in I}$ of $U$ such that $C(U_i)$ is not empty, the restriction of $E_U$ to $U_i$ is isomorphic to a gerbe $C_1(e_i)$ where $e_i$ is an object of $C(U_i)$. A 1-arrow $h : E_U = C_1(e_U)\to E'_U = C_1(e'_U)$ between two objects of $C'(U)$ is a functor $h^*$, where $h : e_U \to e'_U$ is an arrow. A 2-arrow is a natural transformation $l_U$ between two 1-arrows which coincide on objects.*

*Proof. Gluing property for objects.* Consider an open covering family $(U_i)_{i\in I}$ of $N$. Let $E_i = C_1(e_i)$ be an object of $C'(U_i)$, a morphism between $g_{ij} : E^i_j \to E^j_i$ is a functor ${h_{ij}}^* : C_1(e^i_j) \to C_1(e^j_i)$, where $h_{ij} : e^i_j \to e^i_j$ is an arrow. A 2-arrow between ${h_{i_1i_2}}^{i_3 *}{h_{i_2i_3}}^{i_1 *}$ and ${h_{i_1i_3}}^{i_2 *}$ is a natural transformation

$$c_{i_1i_2i_3} : {h_{i_1i_2}}^{i_3 *}{h_{i_2i_3}}^{i_1 *} \longrightarrow {h_{i_1i_3}}^{i_2 *},$$

defined by an element of $L_1(U_{i_1i_2i_3})$. The fact that:

$${c_{i_1i_3i_4}}^{i_2}({c_{i_1i_2i_3}}^{i_4}\circ \mathrm{Id}) = {c_{i_1i_2i_4}}^{i_3}(\mathrm{Id}\circ {c_{i_2i_3i_4}}^{i_1})$$

is equivalent to the gluing property of objects of a 2-gerbed tower. This implies by definition the existence of an object $E_U$ whose restriction to $U_i$ is $E_i$.

*Gluing conditions of arrows.* Let $E_U$ and $E'_U$ be two respective objects of $C'(U)$. The correspondence defined on the category of open subsets of $U$ by $V\to \mathrm{Hom}\,(E_{U|V}, E'_{U|V})$, where $E_{U|V}$ and $E'_{U|V}$ are the respective restrictions of $E_U$ and $E'_U$ to $V$ is a sheaf of categories since it is the sheaf of morphisms between two gerbes.

Let $U$ be an open subset of $N$; the objects $E_U$ and $E'_U$ of $C'(U)$ are locally isomorphic, since the restrictions of $E_U$ and $E'_U$ to an open cover of $(U_i)_{i\in I}$ of $U$ such that the objects of $C(U_i)$ are isomorphic. If we replace $U$ by $N$, and choose a covering family such that $C(U_i)$ is not empty, we obtain that $C'(U_i)$ is not empty.

The set of automorphisms of a 1-arrow is isomorphic to $L_1(U)$ by definition. □

The notion of 2-gerbed tower is easier to understand than the one of 2-gerbe, principally because we do not need the notion of bicategory to define it. In practice, many of the examples of 2-gerbed are defined using the notion of 2-gerbed tower; another advantage of this notion is the fact that the classifying cocycle of a 2-gerbed tower $(C, C_1)$ is the image of a 2-cocycle, that is, the classifying cocycle of $C$ by a connecting morphism in cohomology.

**8. The general case.** We will now describe the 2-gerbe and 2-gerbed towers bounded by the sheaf of locally constant $\mathbf{R}$-functions which represent the geometric obstruction to extend $\omega$ to $P$ when the cohomology class $[\omega]$ of $\omega$ is not necessarily integral.

Let $U$ be an open subset of $N$, and let $[\Omega_U]$ be an extension of $[\omega]$ to $p^{-1}(U)$. We cannot define a $T^1$-bundle over $p^{-1}(U)$ (as in the integral case) whose Chern class is $[\Omega_U]$.

**Definition 8.1.** We denote by $C'_F(\Omega, p^{-1}(U))$ the gerbe defined on $p^{-1}(U)$ which is the obstruction of the class $[\Omega_U]$ to be trivial. See subsection 2.4.

Let $U$ be an open subset of $N$. We define the bicategory $C^2_F(\omega)$ $(p^{-1}(U))$ to be the class whose elements are categories $C'_F(\Omega, p^{-1}(U))$. Let $e_1$ and $e_2$ be two objects of $C^2_F(p^{-1}(U))$, a 1-arrow $f : e^1 = C'_F(\Omega_1, p^{-1}(U)) \to e^2 = C'_F(\Omega_2, p^{-1}(U))$ is an isomorphism of gerbes between $e^1$ and $e^2$, and a 2-arrow is a natural transformation between those functors.

More precisely, on a contractible cover $(U'_i)_{i\in I}$ of $p^{-1}(U)$, the restrictions of the objects of $e^1$ are torsors whose objects are isomorphic to trivial $\mathbf{R}$-bundles $U'_i \times \mathbf{R}$, a 1-arrow $f$ is defined by the respective objects $e_i^1$ and $e_i^2$ of the respective restrictions of $e^1$ to $U'_i$, and of $e^2$ to $U'_i$, and an isomorphism $f_i$ between $e_i^1$ and $e_i^2$. Due to the natural properties of $f$, we can use these morphisms to completely rebuild $f$. This implies that these data satisfy the following properties:

The identification of $e_i^1$ and $e_i^2$ to $U'_i \times \mathbf{R}$ allows us to represent that $f_i$ has a morphism of the trivial torsor $U'_i \times \mathbf{R}$; the fact that $f$ behaves naturally in respect to restrictions implies the existence of a morphism $u_{ij}$ of $U'_{ij} \times \mathbf{R}$ such that $f_i = u_{ij} f_j$. We have $u_{i_1 i_2}{}^{i_3} u_{i_2 i_3}{}^{i_1} = u_{i_1 i_3}{}^{i_2}$. The map $u_{ij}$ is a translation by an element of $\mathbf{R}$. The family $(u_{ij})_{i,j\in I}$ defines a 1-cocycle, thus a closed 1-form on $p^{-1}(U)$. Conversely, a 1-cocycle of the sheaf of locally constant $\mathbf{R}$-maps defines a torsor, and a 1-arrow between $e^1$ and $e^2$ by using the previous identification of $e_i^1$ and $e_i^2$ to $U'_i \times \mathbf{R}$.

Using the identification above, a 2-arrow is defined locally by a chain of constant sections $u_i$ defined on $U'_i \times \mathbf{R}$ such that $u_i = u_{ij} u_j$. Thus, a morphism between two objects is defined by a 1-cocycle of the sheaf of locally constant $\mathbf{R}$-functions, that is, a torsor, and a 2-arrow is an element of $\mathbf{R}$.

**Theorem 8.2.** *The correspondence $p^{-1}(U) \to C_F^2(\omega)(p^{-1}(U))$ defines a 2-gerbe such that the cohomology class of its classifying cocycle is the obstruction for extending $[\omega]$ to $P$.*

*Proof. Gluing conditions for objects.* Consider an open covering $(U_i)_{i\in I}$ of an open subset $U$ of $N$, and $(e_i, [\Omega_i])$ an object of $C_F^2(\omega)(p^{-1}(U_i))$ where $[\Omega_i]$ is a cohomology class defined on $p^{-1}(U_i)$ which extends $[\omega]$. Suppose that there exist 1-arrows $h_{ij} : e_j^i \to e_i^j$ and a 2-arrow $d_{i_1 i_2 i_3} : h_{i_1 i_2}{}^{i_3} h_{i_2 i_3}{}^{i_1} \to h_{i_1 i_3}{}^{i_2}$ such that $d_{i_1 i_3 i_4}{}^{i_2}(d_{i_1 i_2 i_3}{}^{i_4} \circ \mathrm{Id}) = d_{i_1 i_2 i_4}{}^{i_3}(\mathrm{Id} \circ d_{i_2 i_3 i_4}{}^{i_1})$. The maps $d_{i_1 i_2 i_3}$ can be identified with a 2-Cech cocycle of the sheaf of locally constant $\mathbf{R}$-functions defined on $p^{-1}(U)$. We can identify it using the de Rham-Weil isomorphism with an element $[\Omega_U]$ of $H^2(p^{-1}(U), \mathbf{R})$. The class $[\Omega_U]$ is the classifying cocycle of a gerbe $e_U$ defined on $U$ bounded by the sheaf of locally constant $\mathbf{R}$-functions. We have to show now that this gerbe is an element of $C_F^2(p^{-1}(U))$.

The fact that the family $d_{i_1 i_2 i_3}$ is a 2-Cech cocycle implies that there exists a gerbe bounded by the sheaf of locally constant functions whose restriction to $U_i$ is $e_i$, (see the proof of the classifying theorem for gerbes presented in the book of Breen [**2**]). This gerbe is isomorphic to $e_U$. This implies that the restriction of $[\Omega_U]$ to $p^{-1}(U_i)$ is the classifying cocycle of $e_i$, and that $[\Omega_U]$ extends $[\omega]$, since the restriction of $[\Omega_U]$ to $U_i$ is $[\Omega_i]$. We deduce that it is the cohomology class of the classifying cocycle of an element of $C^2_F(\omega)(p^{-1}(U))$ whose restriction to $p^{-1}(U_i)$ is $e_i$.

*Gluing conditions for arrows.* Let $e$ and $e'$ be a pair of objects of $C^2_F(\omega)(p^{-1}(U))$, the correspondence defined on the category of open subsets of $U$ by $U' \to \mathrm{Hom}\,(e_{|U'}, e'_{|U'})$ is a sheaf of categories, since it is the sheaf of categories of morphisms between two gerbes.

This shows that $C^2_F(\omega)$ is a sheaf of 2-categories.

Consider an open covering $(U_i)_{i\in I}$ of $N$ by contractible open sets. Since $H^*(U_i \times F) = H^*(F)$, we can extend $[\omega]$ to $p^{-1}(U_i) = U_i \times F$, and two such extension classes are equal to the class $[\omega]$ as the identification $H^*(U_i \times F) = H^*(F)$ shows. This implies that $C^2_F(\omega)(p^{-1}(U_i))$ is not empty, and its objects are isomorphic.

The sheaf of 2-categories $C^2_F(\omega)$ is bounded by the sheaf of $\mathbf{R}$-locally constant functions defined on $P$. This is shown in the paragraph above this theorem.

Suppose that the class of the classifying cocycle of $C^2_F(\omega)$ vanishes; then the 2-gerbe has a global section $e$, its restriction to $p^{-1}(U_i)$ is an element of $C^2_F(\omega)(p^{-1}(U_i))$ whose classifying cocycle extends $[\omega]$. This implies that the classifying cocycle of $e$ extends $[\omega]$. $\square$

The cocycle defined by McDuff, and the classifying cocycle $c^2_F$ of the 2-gerbe $C^2_F(\omega)$ solve the same geometric problem: decide if the class $[\omega]$ can be extended to the total space of the symplectic bundle $p : P \to N$ whose typical fiber is $(F, \omega)$. We will show now that they are related by an isomorphism of cohomology groups.

Suppose that the family $(U_i)_{i\in I}, g_{ij}$ defines the coordinate changes of $P$. Let $\widehat{\mathrm{Symp}}\,(F,\omega)$ be the universal cover of $\mathrm{Symp}\,(F,\omega)$. Consider an element $h_{ij}$ of $\mathrm{Ham}^s$ such that $g_{ij}(x)h_{ij} = g'_{ij}(x)$ is contained in $\mathrm{Symp}\,(F,\omega)_0$, and a lift: $\widehat{g'_{ij}} : U_i \cap U_j \to \widehat{\mathrm{Symp}}\,(F,\omega)$ of the functions

$g'_{ij}$. We remark that an element $\widehat{g'_{ij}}(x)$ is an equivalence class of a path in $c : [0,1] \to \mathrm{Symp}\,(F,\omega)$. We choose a path $u_{ij}$ which represents it and set:

$$\int_0^1 \omega\left(\frac{d}{dt}\hat{u_{ij}}(x), .\right) = g''_{ij}(x).$$

**Proposition 8.3.** *The chain* $c_{i_1i_2i_3} = {g''_{i_2i_3}}^{i_1} - {g''_{i_1i_3}}^{i_2} + {g''_{i_1i_2}}^{i_3}$ *is a 2-Cech cocycle whose cohomology class is identified using the Cech-Weil isomorphism to the McDuff obstruction class.*

*Proof.* The element $g''_{ij}(x)$ is a lift of $F_s(g''_{ij})$ in $H^1(F,\mathbf{R})$, since the restriction of $F_s$ to $\mathrm{Symp}\,(F,\omega)_0$ factors by the flux. This implies that $g''_{ij}$ represents the classifying cocycle of the $H^1(F,\mathbf{R})/H^1(F,P_\omega)$-bundle (see subsection 2.6.5) whose coordinate changes are the functions $F_s(g'_{ij})$. □

We can use the Cech-Weil isomorphism to identify $c_{i_1i_2i_3}$ to a closed 2-form $\Omega'$ defined on $N$ which takes values in the vector bundle $p_\omega$ of closed $P_\omega$ 1-forms defined on $F$ induced by $g_{ij}$. Let $\Omega(F,P_\omega)$ be the vector space of closed $P_\omega$ 1-forms defined on $F$. The bundle $p_\omega$ is the quotient of the union of $U_i \times \Omega(F,P_\omega)$ by the following transition functions:

$$(x,\alpha) \longrightarrow (x, g_{ij}(x)^*(\alpha))$$

where $g_{ij}(x)^*(\alpha)(y)$ is defined by:

$$g_{ij}(x)^*(\alpha)(y) = \alpha(d(g_{ij}(x)^{-1})(y)).$$

The identification of $c_{i_1i_2i_3}$ to $\Omega'$ defines a 3-form $\Omega$ on $P$ by $\Omega(x,y,z) = \Omega'(x,y)(z)$ where $x$ and $y$ are elements of $T_nN$, the tangent space of $N$ at $n$, and $z$ is an element of the tangent space to the fiber at $n$.

Consider the Leray-Serre spectral sequence related to the fibration $p : P \to N$. The McDuff obstruction class is an element of $E_2^{2,1}$ which converges to $[c_F^2(\omega)]$.

**Theorem 8.3.** *Under the notation introduced just above, the cohomology class of $\Omega$ is the obstruction to lift $[\omega]$ to $P$. Its cohomology class can be identified to the class of the classifying cocycle of $C^2_F(\omega)$.*

*Proof.* Let $e_i$ be the gerbe defined on $U_i \times F$ whose classifying cohomology class is the image of the class of the 2-form $\Omega_i$ which is the product of 0 and $\omega$ by the Cech-Weil isomorphism. The gerbe $e_i$ is an object of $C^2_F(\omega)(U_i)$. The morphism $g''_{ij}$ defined at the proposition above is a morphism between $e^i_j$ and $e^j_i$. This implies that $c_{i_1 i_2 i_3}$ represents also the classifying cocycle of $C^2_F(\omega)$. □

**8.2 Hamiltonian reduction and closed connection forms.** We have given a gerbe formulation to the problem of the existence of a Hamiltonian reduction by defining the gerbe $C^1_F(\omega)$. Now we are going to show how the classifying cocycles of $C^1_F(\omega)$ and $C^2_F(\omega)$ are related.

The link between the classifying cocycles of $C^1_F(\omega)$ and $C^2_F(\omega)$ appears clearly by considering the 2-gerbed towers defined as follows:

**Definition 8.2.1.** Consider $U$, an open set of $N$ and $e_U$ an object of $C^1_F(\omega)(U)$; it is a Hamiltonian structure defined on the restriction of the symplectic fibration $p : P \to N$ to $U$. We deduce that there exists an extension $[\Omega_U]$ of $[\omega]$ to $p^{-1}(U)$ whose holonomy defines the Hamiltonian reduction of $e_U$. Denote by $C_2(e_U)$ the gerbe which represents the obstruction of $[\Omega_U]$ to be trivial. We have just defined a 2-gerbed tower $(C^1_F(\omega), C_2)$. □

Let $L$ be the band of the gerbe $C^1_F(\omega)$, and let $L_0$ be the sheaf of locally constant $\mathbf{R}$-functions defined on $P$. We define the following sheaf $L'$ on $P$: suppose that $e_U$ is an object of $C^1_F(\omega)(U)$, $V$ an open subset of $p^{-1}(U)$ and $e_V$ an object of $C_2(e_U)(V)$. An automorphism $g$ of $e_U$ maps $e_V$ to the object ${g^{-1}}^*(e_V)$ of $C_2(e_U)(g(V))$, given $c \in \mathbf{R}$. For each morphism $h : e_V \to e'_V$ between objects of $C_2(e_U)(V)$ we consider the morphism between ${g^{-1}}^*(e_V) \to {g^{-1}}^*(e'_V)$ defined by composing ${g^{-1}}^*(h)$ by the translation by $c$ fiber by fiber. The sheaf generated by the set of actions on the gerbe $C_2(e_U)$ that we have just defined is $L'$. (It does not depend on $e_U$.) We can suppose that $L$ is defined on $P$ by setting $L(U) = L(p(U))$, $U \subset P$. We have the exact sequence:

$$1 \longrightarrow L_0 \longrightarrow L' \longrightarrow L \longrightarrow 1.$$

This gives rise to the following exact sequence in cohomology:

$$H^2(P, L_0) \longrightarrow H^2(P, L') \longrightarrow H^2(P, L) \longrightarrow H^3(P, L_0).$$

Here, if $E$ is a sheaf defined on $P$, the space $H^2(P, E)$ is the space of isomorphism classes of gerbes bounded by $E$ defined on $P$. The space $H^3(P, E)$ is the space of isomorphism classes of 2-gerbes bounded by $E$. See Breen [**2**].

The next result shows that the class of the classifying cocycle of the 2-gerbe tower $(C^1_F(\omega), C_2)$ is the image of the class of the classifying cocycle of $C^1_F(\omega)$ by the map $H^2(P, L) \to H^3(P, L_0)$.

**Proposition 8.2.2.** *The class of the classifying cocycle $c^2_F(\omega)$ of $C^2_F(\omega)$ is the image of the class of the classifying cocycle $c^1_F(\omega)$ of $C^1_F(\omega)$, by the map $H^2(P, L) \to H^3(P, L_0)$. Suppose that there exists a Hamiltonian reduction of the bundle $P \to N$. Then we can extend $[\omega]$ to $P$.*

*Proof.* The classifying cocycle of this 2-gerbed tower is defined as follows. Consider an object $e_i$ of $C^1_F(\omega)(U_i)$, and a map $u_{ij} : e^i_j \to e^j_i$. The map $c_{i_1 i_2 i_3} = {u_{i_3 i_1}}^{i_2} {u_{i_1 i_2}}^{i_3} {u_{i_2 i_3}}^{i_1}$ is an automorphism of ${e^{i_1 i_2}}_{i_3}$. We can lift it to a map ${c^*}_{i_1 i_2 i_3}$ of $C_2({e^{i_1 i_2}}_{i_3})$. The Cech boundary $c_{i_1 i_2 i_3 i_4}$ of the chain ${c^*}_{i_1 i_2 i_3}$ is the classifying cocycle of the 2-gerbed tower. It appears that $c_{i_1 i_2 i_3 i_4}$ is the image of $c_{i_1 i_2 i_3}$ by the connecting map $H^2(P, L) \to H^3(P, L)$. Considered as a 2-gerbe, the 2-gerbed tower involved here is a subgerbe of $C^2_F(\omega)$, since for each object $e_U$ of $C^1_F(\omega)(U)$, the gerbe $C_2(e_U)$ is an object of $C^2_F(\omega)(U)$. This shows that if $[c^1_F(\omega)]$ vanishes, then $[c^2_F(\omega)]$ also vanishes. □

This result is shown by McDuff in [**16**] by using the Guillemin-Lerman-Sternberg construction.

**9. Quantization of the symplectic gerbe.** Let $(F, \omega)$ be a symplectic manifold. When the class $[\omega]$ is integral, there exists a line bundle $L$ over $F$ whose Chern class is $[\omega]$. This line bundle is endowed with an Hermitian metric. The Hermitian space of sections $L^2(F) = \{u : F \to L : \int_F |u|^2 < +\infty\}$ is the quantization of the manifold. The elements of this Banach space are used in theoretical physics to describe evolution of particles.

The goal of this part is to associate to any symplectic form, an Hermitian space endowed with a Hermitian form, which is a candidate to represent the phase space in quantum mechanic.

Let $C(\omega)$ be the symplectic gerbe defined on $F$, which represents the obstruction of $[\omega]$ to be integral, see subsection 2.4. Consider an open covering $(U_i)_{i\in I}$ of $U$, and $e_i$ an object of $C(\omega)(U_i)$. We can define the gerbe $L(\omega)$ on $F$ such that $L(\omega)(U)$ is the category, whose objects are $(e_U, e'_U)$ where $e_U$ is an object of $C(\omega)(U)$, and $e'_U$ the **C**-line vector bundle over $U$, whose transition functions are the transition functions of $e_U$. The objects of $L(\omega)(U)$ are endowed with a canonical connective structure $Co$, see subsection 2.4. An element of $Co((e_U, e'_U))$ is a connection on $e_U$ whose curvature is the restriction of $\omega$ to $U$. A morphism between two objects $(e_U, e'_U)$ and $e^1{}_U), e'^1_U)$ of $L(\omega)(U)$ is a morphism $e_U \to e^1{}_U$. The correspondence defined on the category of open subsets of $F$ by $U \to L(\omega)(U)$ is a gerbe.

To perform the quantization we need to define the notion of sections. We will propose this definition of sections of vectorial gerbes.

**Definition 9.1.** Let $(U_i)_{i\in I}$ be an open covering family of $F$, such that $L(\omega)(U_i)$ is not empty, and let $(e_i, e'_i)$ be an object $L(\omega)(U_i)$. A section $u$ of $(e'_i)_{i\in I}$ is a family of sections $u_i : U_i \to e'_i$ such that the union of supports of $u_i$ is compact.

We denote by $V((e_i)_{i\in I})$ the vector space generated by those sections of $(e'_i)_{i\in I}$. This vector space is endowed with an Hermitian metric defined by

$$\langle u, v\rangle = \sum_{i\in I} \int_{e'_i} \langle u_i, v_i\rangle_{e'_i}$$

where $\langle , \rangle_{e'_i}$ is the Hermitian metric of $e'_i$.

For each function $f$, and each section $(u_i)_{i\in I}$, we can define

$$L_f(u_i) = \nabla_{e'_i X_f} u_i + 2i\pi f u_i,$$

where $X_f$ is the Hamiltonian of $f$, and $\nabla_{e'_i}$ is a connection defined on $e'_i$ whose curvature is the restriction of $\omega$ to $U_i$. The vector field $X_f$ is the vector field such that $\omega(X_f, .) = -df$.

**Proposition 9.2.** *The family of $L_f(u_i)$ defined a section $L_f(u)$. The map*

$$f \longrightarrow L_f$$

*verifies*

$$[L_f, L_g] = L_{\{f,g\}}.$$

*Proof.* We have to show that $L_f(u_i)$ has compact support and that the union of support of the family $(L_f(u_i))_{i\in I}$ is compact. The support of $fu_i$ and $\nabla_{{e'_i}_{X_f}}(u_i)$ are contained in the support of $u_i$. The fact that $[L_f, L_g] = L_{\{f,g\}}$ is classical. $\square$

We have obtained a Souriau-Kostant quantization. This quantization can be written without using the notion of gerbe.

We can define, using the classifying theorem of Giraud [**6**], the gerbe $L'(\omega)$ on $F$, such that $L'(\omega)(U)$ is a set of flat $\mathbf{C}$-bundles defined on $U$, and the cohomology class of the classifying cocycle of $L'(\omega)$ is the obstruction of the class $[\omega]$ to be integral. This construction of this gerbe using [**3**] shows that this gerbe is flat, the objects of $L'(\omega)(U)$ are locally flat $\mathbf{C}$-bundles, and morphisms are morphisms of locally flat $\mathbf{C}$-bundles.

For $L'(\omega)$, we can also define the following space of sections. Consider an open covering $(U_i)_{i\in I}$ of $F$ by contractible subsets, $e'_i$ an object of $L'(\omega)(U_i)$, $g_{ij} : {e'}^i_j \rightarrow {e'}^j_i$ a family of isomorphisms. A section $u = (u_i)_{i\in I}$ is a family of sections $u_i : U_i \rightarrow e'_i$ such that $u_i = g_{ij}(u_j)$. We denote by $V(e_i, g_{ij})$ the set of those sections. The set $V(e_i, g_{ij})$ can be supposed to be different than zero. To see this, we consider an open cover $(U_i)_{i\in I}$, such that there exists an element $i_0 \in I$ such that $V = U_{i_0} - \cup_{i\neq i_0} U_i$ is not empty: take a section $u_{i_0}$ of $e_i$ whose support is contained in $V$, and set $u_i = 0$ if $i \neq i_0$. This vector space can be endowed with the following scalar product.

Consider a partition of the unity $p_i$ subordinate to $(U_i)_{i\in I}$. Let $u=(u_i)_{i\in I}$, and let $u'=(u'_i)_{i\in I}$ be sections of $V(e_i,u_{ij}))$. We set

$$\langle u,v\rangle \sum_{i\in I}\int \langle p_i u_i, p_i {u'}_i\rangle.$$

For each differentiable function $f$ defined on $F$ we can define the operator $L_f$ which acts on the section $u=(u_i)_{i\in I}$ by:

$$L_f(u_i)=\nabla_{X_f}u_i+2i\pi f u_i.$$

The operator $L_f$ is well defined. Since on $U_i\cap U_j$, we have $L_f(u_i)=u_{ij}L_f(u_j)$ since the gerbe $C(\omega)$ is flat, and the map $u_{ij}$ is identified using a trivialization with the multiplication by an element of $T^1$ in the trivial bundle $U_i\cap U_j\times \mathbf{C}$.

*Quantization of other structures.* The methods of quantization of Kostant-Souriau have been extended in many directions. Here we present a quantization described in [**13**].

Consider a manifold $M$, such that the ring $C^\infty(M)$ of differentiable functions of $M$ is endowed with a bracket:

$$\{\ ,\ \}: C^\infty(M)\times C^\infty(M)\longrightarrow C^\infty(M)$$

such that $(C^\infty(M),\{\ ,\ \})$ is a Lie algebra and there is an $\mathbf{R}$-linear map:

$$\begin{aligned} H: C^\infty(M) &\longrightarrow \chi(M)\\ f &\longrightarrow X_f \end{aligned}$$

where $\chi(M)$ is the space of vector fields of $M$, such that

$$X_{\{f,g\}}=[X_f,X_g].$$

The map

$$\begin{aligned} C^\infty(M) &\longrightarrow \operatorname{End}(C^\infty(M))\\ f &\longrightarrow (g\to X_f(g)) \end{aligned}$$

is a representation of the Lie algebra $C^\infty(M)$. We denote $H^*_C(M)$ the cohomology of this representation. The correspondence:

$$\begin{aligned} C^\infty(M) \times C^\infty(M) &\longrightarrow C^\infty(M) \\ \Lambda_M(f,g) &\longrightarrow X_f(g) - X_g(f) - \{f,g\} \end{aligned}$$

is a 2-cocycle of this representation.

There is a canonical map $C' : H^*_{\text{de Rham}}(M) \to H^*_C(M)$ defined on a chain by $C'(h)(f_1,\ dpts, f_p) = h(X_{f_1}, \dots, X_{f_p})$. In [**13**], it is shown that if there exists a line bundle $L \to M$ such that $C'(\Omega) = \Lambda_M$, then the structure is quantizable, that is, there exists a representation:

$$P : C^\infty(M) \longrightarrow \mathrm{End}\,(L^2(L))$$

which verifies

$$\begin{aligned} P(\{f,g\}) &= [P_f, P_g] \\ P(f) &= \nabla_{X_f} + 2i\pi f \end{aligned}$$

where $\nabla$ is the Hermitian connection of the bundle.

Let $(U_i)_{i\in I}$ be a contractible open covering of $M$ by charts. We can restrict the bracket $\{\ ,\ \}$ to $U_i$. Suppose that on 2-chains, the map $C$ restricted to $U_i$ is surjective on closed forms, that is, there exists a 2-closed form $\Omega_{U_i}$ on $U_i$ such that $C(\Omega_{U_i}) = \Lambda_{U_i}$. The form $\Omega_{U_i}$ is the Chern class of a connection defined on $U_i \times \mathcal{C}$.

We can define on $M$ the gerbe $D$, such that for each open set $U$ of $M$, $D(U)$ is the category of line bundles over $U$ endowed with a connection whose curvature $\Omega_U$ verifies:

$$C(\Omega_U) = \Lambda_U.$$

Let $e_i$ be an object of $D(U_i)$. We consider the family $(u_i)_{i\in I}$, where $u_i : U_i \to e_i$ is a section of $e_i$, whose support is compact, and the union of support of $u_i$ is compact. The family of $(u_i)_{i\in I}$ is an Hermitian space. On $e_i$ we consider the connection $\nabla_{e_i}$ whose curvature is the restriction of $\Omega_{U_i}$

The representation

$$f \longrightarrow \nabla_{e_i X_f} + 2i\pi f$$

defines a quantization of $M$.

**Acknowledgments.** The author had the idea to write this paper after a talk of Dusa McDuff on symplectic fibrations given at Montreal in November 2004. The author wants to thank Dusa McDuff who read, corrected his manuscript and helped him to improve the presentation of the paper, and for helpful discussions and comments.

## REFERENCES

**1.** A. Banyaga, *Sur la structure du groupe des automorphismes symplectiques qui preservent une forme symplectique*, Commun. Math. Helv. **53** (1978), 174–227.

**2.** L. Breen, *On the classification of* 2*-gerbes and* 2*-stacks*, Asterisque **225** (1994), Société Mathématique de France, Paris.

**3.** J.L. Brylinski, *Loops, spaces, characteristic classes and geometric quantization*, Progress Math. **107**, Birkhauser, New York, 1993.

**4.** J.L. Brylinski and D.A. McLaughlin, *The geometry of degree four characteristic classes and of line bundles on loop spaces* I, Duke Math. Journal **75** (1994), 603–637.

**5.** P. Deligne, *Théorie de Hodge* III, Institut des Hautes Etudes Scientifiques Publications **44** (1974), 5–77.

**6.** Jean Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin, 1971.

**7.** R. Godement, *Théorie des faisceaux*, Hermann, Paris, 1958.

**8.** M. Gotay, R. Lashof, J. Sniatycki and A. Weinstein, *Closed form on symplectic bundles*, Commentarii. Math. Helvetici **58** (1983), 617–621.

**9.** V. Guillemin, E. Lerman and S. Sternberg, *Symplectic fibrations and multiplicity diagrams*, Cambridge University Press, Cambridge, 1998.

**10.** A. Grothendieck, M. Artin and J.L. Verdier, *Théorie des topos et cohomologie étale des schemas*, Lecture Notes Math. **269**, Springer-Verlag, Berlin, 1972.

**11.** J. Kedra and D McDuff, *Homotopy properties of Hamiltonian group actions*, SG/0404539.

**12.** F. Lalonde and D. McDuff, *Symplectic structures on fiber bundles*, Topology **42** (2003), 309–347.

**13.** M. de Leon, J.-C. Marrero and E. Padron, *On the geometric prequantization of brackets*, Rec. Acad. Cien. Mat. **95** (2001), 65–83.

**14.** J. Leslie, *On a differentiable structure for the group of diffeomorphisms*, Topology **6** (1967), 263–271.

**15.** M. Mackaay and R. Picken, *Holonomy and parallel transport for Abelian gerbes*, Adv. Math. **170** (2001), 287–339.

**16.** D. McDuff, *Enlarging the Hamiltonian group*, J. Symplectic Geom. **3** (2006), 1–50.

**17.** ———, *Quantum homology of fibrations over* $S^2$, Internat. Math. Journal **11** (2000), 665–721.

**18.** D. McDuff and D. Salamon, *Introduction to symplectic topology*, Oxford University Press, London, 1998.

**19.** D. McDuff and D. Salamon, *J-holomorphic curves and quantum cohomology*, American Mathematical Society, Providence, 1994.

**20.** Aristide Tsemo, *Non abelian cohomology: The point of view of gerbed tower*, preprint available at arxiv.org.

**21.** ———, *Fibrés affines*, Michigan Math. Journal **49** (2001), 459–484.

College Boreal, 951 Avenue Carlaw, Toronto, Ontario, M4K 3M2, Canada
**Email address: tsemo58@yahoo.ca**