ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 38, Number 3, 2008

SOME NEAR-RINGS IN WHICH ALL IDEALS ARE INTERSECTIONS OF NOETHERIAN QUOTIENTS

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ABSTRACT. For every near-ring, Noetherian quotients are one source of ideals, but usually not all ideals can be obtained from such quotients. In this paper, we show that every ideal of a zero symmetric ring-free tame near-ring with identity is dense in the intersection of the Noetherian quotients that contain it. In many cases, we are able to determine the ideal lattice of the near-ring of those functions on a group that are compatible with a given subset of the set of all normal subgroups. In particular, let G be a finite group, and let $\{0\} = A_1 < A_2 < \cdots < A_{n-1} < A_n = G$ be a chain of normal subgroups of G with $|A_i/A_{i-1}| \geq 3$ for all $i \in \{2, \dots, n\}$. Then the lattice of ideals of the near-ring of zero-preserving functions compatible with A_i for all i is shown to consist entirely of intersections of Noetherian quotients. The unique minimal ideal of these near-rings is explicitly determined.

1. Motivation. We will compute the ideal lattice of certain finite near-rings. For most of the well-studied function near-rings, such as the inner automorphism near-ring on a given finite group, the lattice of ideals is not known. However, using *Noetherian quotients* [**10**, Definition 1.41], one obtains many ideals of a given function near-ring, and often all maximal ideals [**1**, Theorem 1.2]. In [**11**, Problem 5], Scott proposed the problem to find all ideals for a certain type of function near-rings. He conjectured that all ideals of these near-rings could be found as intersections of Noetherian quotients. In [**3, 4**] this problem was solved. In most cases, all ideals were in fact intersections of Noetherian quotients, and in one case, one additional ideal appeared. In this paper, we will exhibit a large class of near-rings in which all ideals are intersections of Noetherian quotients.

2000 AMS *Mathematics subject classification.* Primary 16Y30.
Keywords and phrases. Near-rings, congruence preserving functions, ideals, Noetherian quotients.
Supported by project P-15691 of the Austrian Science Fund (FWF). The authors would like to thank the Department for International Cooperation of the Johannes Kepler University Linz for the financial support of a one week stay of the second, fourth and fifth authors at Linz in July, 2003.
Received by the editors on August 19, 2004.

DOI:10.1216/RMJ-2008-38-3-713

2. Notation. Throughout this section, let $\langle G;+\rangle$ be a (not necessarily abelian) group, and let N be a subnear-ring of $\langle M_0(G);+,\circ\rangle$. For $f, g \in M_0(G)$, $f \circ g$ denotes the function that maps $\gamma \in G$ to $f(g(\gamma))$; this implies that we use right near-rings, i.e., near-rings that satisfy the distributive law $(f+g)\circ h = f\circ h + g \circ h$.

A subset I of N is called a *left ideal* of the near-ring N if I is a normal subgroup of $\langle N;+\rangle$, and for all $i \in I$, $f, g \in N$, we have $f \circ (g+i) - f \circ g \in I$. A left ideal is an *ideal* if in addition, we have $i \circ f \in I$ for all $i \in I$, $f \in N$. A near-ring N is called *ring-free* if there is no ideal I of N with $I \neq N$ such that the quotient N/I is a ring.

A subgroup S of G is an *N-subgroup* of G if for all $n \in N$, $n(S) \subseteq S$. A normal subgroup H of G is called an *N-ideal* of G if for all $n \in N$, $\rho \in H$, and $\gamma \in G$, we have $n(\gamma+\rho) - n(\gamma) \in H$. If M is a subset of N and $\gamma \in G$, we write $M * \gamma$ for $\{m(\gamma) \mid m \in M\}$. We say that N is *tame* on G if $\mathrm{id}_G \in N$, and for all $n \in N$, $\gamma, \alpha \in G$, we have

$$n(\alpha+\gamma) - n(\alpha) \in N * \gamma.$$

For two subsets A, B of G, we define the *Noetherian quotient* $(A : B)_N$ by

$$(A : B)_N = \{n \in N \mid n(B) \subseteq A\}.$$

If A is an N-ideal of G, then $(A : B)_N$ is a left ideal of the near-ring N. If A is an N-ideal and B is an N-subgroup of G, then $(A : B)_N$ is an ideal of N.

Let S and T be two subsets of N. We say that S is *dense* in T if $S \subseteq T$, and for every $t \in T$ and every finite subset F of G, there is an $s \in S$ such that $s|_F = t|_F$.

3. Near-rings with a distributive lattice of left ideals.

Lemma 3.1. *Let G be a group, let N be a subnear-ring of $M_0(G)$, and let L be a left ideal of N. We assume that the lattice of left ideals of N is distributive. Then L is dense in*

$$M := \bigcap_{\gamma\in G} (L * \gamma : \gamma)_N.$$

Proof. It is easy to see that L is a subset of M. In order to show that L is dense in M, we fix $m \in M$, and a finite subset $Y = \{\gamma_1, \dots, \gamma_n\}$ of G.

Now we consider the following system of congruences.

$$\begin{aligned} x &\equiv 0 \pmod{L} \\ x &\equiv m \pmod{(0:\gamma_1)_N} \\ &\vdots \\ x &\equiv m \pmod{(0:\gamma_n)_N}. \end{aligned} \tag{3.1}$$

We will now prove that each subsystem of (3.1) that consists of exactly two congruences has a solution x in N. Let us first assume that such a subsystem is of the form

$$\begin{aligned} x &\equiv 0 \pmod{L} \\ x &\equiv m \pmod{(0:\gamma_j)_N} \end{aligned} \tag{3.2}$$

for some $j \in \{1, \dots, n\}$. We know that $m(\gamma_j)$ lies in $L{*}\gamma_j$. Hence, there is an $l \in L$ such that $l(\gamma_j) = m(\gamma_j)$. Therefore, $l \equiv m \pmod{(0:\gamma_j)_N}$. Thus, $x := l$ is a solution of (3.2). As a second case, let us assume that we have a subsystem of the form

$$\begin{aligned} x &\equiv m \pmod{(0:\gamma_i)_N} \\ x &\equiv m \pmod{(0:\gamma_j)_N} \end{aligned} \tag{3.3}$$

where $i, j \in \{1, 2, \dots, n\}$. Then $x := m$ is a solution of (3.3) that lies in N. Since the lattice of left ideals of N is distributive, the Chinese remainder theorem [**14**, Folgerung 12.2] (see also [**7**, Theorem 2.2.1]) yields that system (3.1) has a solution f in N. Hence, $f \in L$ and $f|_Y = m|_Y$. $\square$

For near-rings with the descending chain condition on left ideals, "density" means "equality." This is made precise in the following proposition.

Proposition 3.2. *Let G be a group, and let N be a subnear-ring of $M_0(G)$ that satisfies the descending chain condition on left ideals. Let S and T be subsets of N such that S is dense in T. Then $S = T$.*

Proof. We call $D \subseteq G$ a *base of equality* for N if for all $n \in N$ with $n|_D = 0$ we have $n = 0$. We will show that N has a finite base of equality. To this end, we define subsets $D_1, D_2, \dots$ of G as follows: let $D_1 := \varnothing$. If D_i is not a base of equality, then there is a function $f \in N$ with $f|_{D_i} = 0$ and $f \neq 0$. Let $d \in G \setminus D_i$ be such that $f(d) \neq 0$, and let $D_{i+1} := D_i \cup \{d\}$. Clearly, $f \in (0 : D_i)_N \setminus (0 : D_{i+1})_N$. Now we have $(0 : D_1)_N \supset (0 : D_2)_N \supset \cdots$. Since N has the descending chain condition on left ideals, there will be a $k \in \mathbf{N}$ such that D_k is a base of equality.

Now let D be a finite base of equality for N, and let $t \in T$. Since S is dense in T, there is an $s \in S$ with $s|_D = t|_D$. Since D is a base of equality for N, we obtain $s = t$, and hence $t \in S$. □

Lemma 3.3. *Let G be a group, let N be a subnear-ring of $M_0(G)$ with $\mathrm{id}_G \in N$, and let I be an ideal of N. Then we have*

$$\bigcap_{\gamma \in G} (I * \gamma : \gamma)_N = \bigcap_{\delta \in G} (I * \delta : N * \delta)_N. \tag{3.4}$$

Proof. To prove $\subseteq$, let f be on the lefthand side of (3.4). To show that f lies on the righthand side, let $\delta \in G$ and $n \in N$. We compute $f(n(\delta))$. Since f lies on the lefthand side, we have an $i \in I$ such that $f(n(\delta)) = i(n(\delta))$. Since I is a right ideal, $i \circ n \in I$, and thus $f(n(\delta)) \in I * \delta$. This completes the proof of $\subseteq$. To prove $\supseteq$, we fix f on the righthand side of (3.4), and $\gamma \in G$. Then we have $f(\gamma) = f(\mathrm{id}_G(\gamma))$. Since f lies on the righthand side, we obtain $f(\mathrm{id}_G(\gamma)) \in I * \gamma$. □

Theorem 3.4. *Let G be a finite group, and let N be a subnear-ring of $M_0(G)$ with $\mathrm{id}_G \in N$. We assume that N is tame on G and that N is ring-free. Let $\mathrm{Id}_N G$ be the set of all N-ideals of G, and let I be an ideal of N. Then there is a mapping $\Phi_I : \mathrm{Id}_N G \to \mathrm{Id}_N G$ such that*

(1) $I = \cap_{A \in \mathrm{Id}_N G} (\Phi_I(A) : A)_N$,

(2) *for all $A, B \in \mathrm{Id}_N G$ with $A \leq B$, we have $\Phi_I(A) \leq \Phi_I(B)$, and*

(3) *for all $A \in \mathrm{Id}_N G$, we have $\Phi_I(A) \leq A$.*

Proof. For each $A \in \mathrm{Id}_N G$, we define $\Phi_I(A)$ as the N-ideal of G that is generated by $\{i(\alpha) \mid i \in I, \alpha \in A\}$. From this definition, one can immediately verify $\subseteq$ of assertion (1). Next, we prove for all $\gamma \in G$:

$$\Phi_I(N * \gamma) = I * \gamma. \tag{3.5}$$

To prove $\supseteq$ of (3.5), we let $i \in I$ and $\gamma \in G$. Clearly, we have $\gamma = \mathrm{id}_G(\gamma) \in N*\gamma$, and thus $i(\gamma)$ is among the generators of $\Phi_I(N*\gamma)$. To prove $\subseteq$ of (3.5), we observe that $I*\gamma$ is an N-ideal of G. Hence, it is sufficient to show that each of the generators of $\Phi_I(N*\gamma)$ lies in $I*\gamma$. To this end, let $n(\gamma) \in N * \gamma$, and let $i \in I$. Then $i(n(\gamma)) = (i \circ n)\,(\gamma)$. Since $i \circ n \in I$, we have $i(n(\gamma)) \in I * \gamma$, which completes the proof of (3.5).

Now we are ready to prove $\supseteq$ of assertion (1). Since N is a ring free near-ring with identity, Wielandt's lemma yields that its lattice of left ideals is distributive, see [**2**, Lemma 2.9], [**10**, Corollary 2.25]. Hence, by Lemma 3.1, we have

$$I = \bigcap_{\gamma \in G} (I * \gamma : \gamma)_N.$$

From Lemma 3.3, we obtain

$$I = \bigcap_{\delta \in G} (I * \delta : N * \delta)_N.$$

By (3.5), we obtain

$$I = \bigcap_{\delta \in G} (\Phi_I(N * \delta) : N * \delta)_N. \tag{3.6}$$

Since for every $\delta \in G$, $N*\delta$ is an N-ideal of G, we have $\cap_{A \in \mathrm{Id}_N G}(\Phi_I(A) : A)_N \subseteq \cap_{\delta \in G}(\Phi_I(N * \delta) : N * \delta)_N$. Hence, we have proved $\supseteq$ of assertion (1). Items (2) and (3) follow from the definition of Φ_I. $\square$

4. Near-rings of compatible functions. Let G be a group, let $\mathbf{N}(G)$ be the set of normal subgroups of G, and let $H \in \mathbf{N}(G)$. Let X be a subset of G. A function $f : X \to G$ is *compatible with* H if for all $\gamma_1, \gamma_2 \in X$ with $\gamma_1 - \gamma_2 \in H$, we have $f(\gamma_1) - f(\gamma_2) \in H$. If $\mathcal{L}$ is a subset of $\mathbf{N}(G)$, we say that $f : X \to G$ is *compatible with* $\mathcal{L}$ if f is compatible with each $H \in \mathcal{L}$. For a subset $\mathcal{L}$ of $\mathbf{N}(G)$ we define two near-rings:

$$\begin{aligned} C(G, \mathcal{L}) &:= \{f : G \to G \mid f \text{ is compatible with } \mathcal{L}\}, \\ C_0(G, \mathcal{L}) &:= \{f \in C(G, \mathcal{L}) \mid f(0) = 0\}. \end{aligned}$$

For every $\mathcal{L} \subseteq \mathbf{N}(G)$, the near-ring $C_0(G, \mathcal{L})$ is tame on G. This implies that every $C_0(G, \mathcal{L})$-subgroup of G is a $C_0(G, \mathcal{L})$-ideal of G. Furthermore, if $A \in \mathcal{L}$, A is a $C_0(G, \mathcal{L})$-subgroup of G: if $a \in A$ and $f \in C_0(G, \mathcal{L})$, then $f(a) = f(a) - f(0)$ lies in A because f is compatible with A. However, given $\mathcal{L} \subseteq \mathbf{N}(G)$, there may be $C_0(G, \mathcal{L})$-subgroups of G that do not lie in $\mathcal{L}$; for example, $\{0\}$ and G will always be $C_0(G, \mathcal{L})$-subgroups of G even if they do not lie in $\mathcal{L}$. Furthermore, the $C_0(G, \mathcal{L})$-subgroups of G form a sublattice of $\mathbf{N}(G)$. In the sequel, we will give conditions that imply that every $C_0(G, \mathcal{L})$-subgroup of G lies in $\mathcal{L}$, Proposition 4.2. In the proof, we will use the following consequence of Kaarli's extension principle for compatible functions, which we quote here as Lemma 4.1. We notice that the lattice of normal subgroups $\langle \mathbf{N}(G); \cap, + \rangle$ is a complete lattice. If $\mathcal{A} \subseteq \mathbf{N}(G)$, the least upper bound of $\mathcal{A}$ will be denoted with $\vee_{X \in \mathcal{A}} X$.

Lemma 4.1 [**6**], [**7**, page 69]. *Let G be a group such that the cardinality of G is at most countably infinite, and let $\mathcal{L}$ be a complete sublattice of $\mathbf{N}(G)$. We assume that the lattice $\mathcal{L}$ is distributive. Let X be a finite subset of G, and let $f : X \to G$ be a function that is compatible with $\mathcal{L}$. Then there exists a function $F \in C(G, \mathcal{L})$ such that $F|_X = f$.*

Proposition 4.2. *Let G be a group such that the cardinality of G is at most countably infinite, and let $\mathcal{L}$ be a complete sublattice of $\mathbf{N}(G)$ such that $\{0\} \in \mathcal{L}$ and $G \in \mathcal{L}$. We assume that the lattice $\mathcal{L}$ is distributive. Let A be a $C_0(G, \mathcal{L})$-subgroup of G. Then we have $A \in \mathcal{L}$.*

Proof. We will show

$$A = \bigvee_{a \in A} \bigcap_{X \in \mathcal{L} \text{ with } a \in X} X. \tag{4.1}$$

For $\subseteq$, we fix $a \in A$ and observe that we have $a \in \cap_{X \in \mathcal{L} \text{ with } a \in X} X$. For $\supseteq$, it is sufficient to prove that for every $a \in A$, we have

$$\bigcap_{X \in \mathcal{L} \text{ with } a \in X} X \leq A. \tag{4.2}$$

To prove this, we fix $a \in A$. In the case $a = 0$, we observe that since $\{0\} \in \mathcal{L}$, the lefthand side of (4.2) is $\{0\}$, and thus the required inclusion holds. Now we consider the case $a \neq 0$. We fix $x \in \cap_{X \in \mathcal{L} \text{ with } a \in X} X$, and we define a function $f : \{0, a\} \to G$ by $f(0) = 0$, $f(a) = x$. We will show that f is compatible with $\mathcal{L}$. To this end, we let $H \in \mathcal{L}$ be such that $a - 0 \in H$. Since $x \in \cap_{X \in \mathcal{L} \text{ with } a \in X} X$, we have $x \in H$. Hence $f(a) - f(0) \in H$. Therefore, f is compatible with $\mathcal{L}$. By Lemma 4.1, there is an $F \in C_0(G, \mathcal{L})$ such that $F(a) = x$. Hence $x \in C_0(G, \mathcal{L}) * a$. Since A is a $C_0(G, \mathcal{L})$-subgroup of G, we obtain $x \in A$. This completes the proof of (4.2), and hence also of $\supseteq$ of (4.1). Now $\mathcal{L}$ is a complete sublattice of $\mathbf{N}(G)$, and $\{0\}$ and G are elements of $\mathcal{L}$. Hence, the righthand side of (4.1) lies in $\mathcal{L}$. Thus, (4.1) yields $A \in \mathcal{L}$. □

If $\mathcal{L}$ is a sublattice of $\mathbf{N}(G)$, and $A, B \in \mathcal{L}$, we write $A \prec_{\mathcal{L}} B$ if $A < B$ and there is no $I \in \mathcal{L}$ with $A < I < B$.

Proposition 4.3. *Let G be a group such that the cardinality of G is at most countably infinite, and let $\mathcal{L}$ be a complete sublattice of $\mathbf{N}(G)$ such that $\{0\} \in \mathcal{L}$ and $G \in \mathcal{L}$. We assume that the lattice $\mathcal{L}$ is distributive and satisfies the descending chain condition. Furthermore, we assume that for all $A, B \in \mathcal{L}$ with $A \prec_{\mathcal{L}} B$, we have $|B/A| \geq 3$. Then, if the near-ring $C_0(G, \mathcal{L})$ satisfies the descending chain condition on left ideals, it is ring-free.*

Proof. Since $C_0(G, \mathcal{L})$ is a near-ring with 1, every ideal is contained in a maximal ideal. It is therefore sufficient to show that for each maximal ideal I of $C_0(G, \mathcal{L})$, the quotient $C_0(G, \mathcal{L})/I$ is not a ring. To this end, we fix a maximal ideal I of $C_0(G, \mathcal{L})$. By Proposition 4.2 and the remarks after the definition of $C_0(G, \mathcal{L})$, the lattice of $C_0(G, \mathcal{L})$-subgroups of G is equal to $\mathcal{L}$. By Theorem 1.2 of [**1**] and Proposition 3.2, there are $E, F \in \mathcal{L}$ with $E \prec_{\mathcal{L}} F$ such that $I = (E : F)_{C_0(G, \mathcal{L})}$. Now we choose A minimal in $\mathcal{L}$ with $A \leq F$ and $A \not\leq E$. Then A is a strictly join irreducible element of $\mathcal{L}$. We denote its unique subcover by A^-.

We have $A \cap E = A^-$ and $A + E = F$. Thus, by the isomorphism theorem, the $C_0(G, \mathcal{L})$-group F/E is isomorphic to the $C_0(G, \mathcal{L})$-group A/A^-. Hence, $(E : F)_{C_0(G,\mathcal{L})} = (A^- : A)_{C_0(G,\mathcal{L})}$. We will now show that $C_0(G, \mathcal{L})/(A^- : A)_{C_0(G,\mathcal{L})}$ is not a ring. Since $|A/A^-| \geq 3$, we can choose $a_1 \in A \setminus A^-$ and $a_2 \in A \setminus A^-$ such that $a_1 + a_2 \notin A^-$. We define a function $f : \{0, a_1, a_2, a_1 + a_2\} \to G$ by $f(0) = f(a_1) = f(a_2) = 0$ and $f(a_1 + a_2) = a_1$. We prove

$$f \text{ is compatible with } \mathcal{L}. \tag{4.3}$$

We first show

$$a_1 \in C_0(G, \mathcal{L}) * (a_1 + a_2) \cap C_0(G, \mathcal{L}) * a_2 \cap C_0(G, \mathcal{L}) * a_1. \tag{4.4}$$

We note that for $b \in \{a_1 + a_2, a_2, a_1\}$, the choice of a_1 and a_2 yields $b \in A \setminus A^-$. Since $C_0(G, \mathcal{L})$ is tame on G, $C_0(G, \mathcal{L}) * b$ is a $C_0(G, \mathcal{L})$-ideal of G. Thus we have $C_0(G, \mathcal{L}) * b \leq A$ and $C_0(G, \mathcal{L}) * b \not\leq A^-$. Hence, for $b \in \{a_1 + a_2, a_2, a_1\}$, we have $C_0(G, \mathcal{L}) * b = A$. This completes the proof of (4.4).

Now we are ready to prove (4.3). We first show that for $\gamma_1 := a_1 + a_2$ and $\gamma_2 := a_1$, and for every $C_0(G, \mathcal{L})$-subgroup H with $\gamma_1 - \gamma_2 \in H$, we also have $f(\gamma_1) - f(\gamma_2) \in H$. To this end, let H be a $C_0(G, \mathcal{L})$-subgroup such that $(a_1 + a_2) - a_1 \in H$. Since H is normal, we obtain $a_2 \in H$. Hence $C_0(G, \mathcal{L}) * a_2 \subseteq H$. Now (4.4) yields $a_1 \in H$. Similarly, each of the conditions $(a_1 + a_2) - 0 \in H$, $(a_1 + a_2) - a_2 \in H$, $0 - (a_1 + a_2) \in H$, $a_1 - (a_1 + a_2) \in H$, $a_2 - (a_1 + a_2) \in H$ implies $a_1 \in H$. This completes the proof of (4.3).

By Lemma 4.1, there is an $F \in C_0(G, \mathcal{L})$ such that $F(0) = F(a_1) = F(a_2) = 0$ and $F(a_1 + a_2) = a_1$. Then $F(a_1 + a_2)$ is not congruent to $F(a_1) + F(a_2)$ modulo A^-. Thus, $C_0(G, \mathcal{L})/(A^- : A)_{C_0(G,\mathcal{L})} = C_0(G, \mathcal{L})/I$ is not a ring. $\square$

5. The near-ring that is compatible with a chain of normal subgroups. Let G be a group, let H be a normal subgroup of G, and let $\mathcal{L} = \{\{0\}, H, G\}$. In this case, the ideal lattice of $C_0(G, \mathcal{L})$ has been determined in [**3, 4**]. The techniques developed in the present note allow us to describe the ideal lattice of $C_0(G, \mathcal{L})$ in the following case:

(1) G is finite,

(2) $\mathcal{L}$ is a chain, and

(3) for all $A, B \in \mathcal{L}$ with $A \prec_{\mathcal{L}} B$, we have $|B/A| \geq 3$.

For a natural number n, we let $\underline{n}$ be the set $\{1, 2, \ldots, n\}$, and we let $S(n)$ be the set of all functions $f : \underline{n} \to \underline{n}$ that satisfy $f(i) \leq i$ for all $i \in \underline{n}$, and $f(i) \leq f(j)$ for all $i, j \in \underline{n}$ with $i \leq j$. The set $S(n)$ can be ordered by $f \leq g$ if $f(i) \leq g(i)$ for all $i \in \underline{n}$. Then $\langle S(n); \leq \rangle$ is a lattice ordered set. Let $\mathcal{S}(n)$ be the corresponding lattice.

Theorem 5.1. *Let G be a finite group, let $n \in \mathbf{N}$, and let $A_1, A_2, \ldots, A_n$ be normal subgroups of G with $\{0\} = A_1 < A_2 < \cdots < A_{n-1} < A_n = G$. We assume that for all $i \in \{2, \ldots, n\}$, we have $|A_i/A_{i-1}| \geq 3$. Let $\mathcal{L} := \{A_1, A_2, \ldots, A_n\}$, and let $\mathcal{I}$ be the set of ideals of the near-ring $C_0(G, \mathcal{L})$. Then the mapping Ψ defined by*

$$\begin{aligned} \Psi : \mathcal{S}(n) &\longrightarrow \mathcal{I} \\ f &\longmapsto \bigcap_{i \in \underline{n}} (A_{f(i)} : A_i)_{C_0(G,\mathcal{L})} \end{aligned}$$

is a lattice isomorphism.

Proof. We note that by Proposition 4.2, every $C_0(G, \mathcal{L})$-ideal of G must be equal to some A_k with $k \in \underline{n}$. We first show

$$\Psi \text{ is surjective.} \tag{5.1}$$

Let $I \in \mathcal{I}$. By Proposition 4.2, $C_0(G, \mathcal{L})$ is ring-free. Hence by Theorem 3.4, there is a mapping $f : \underline{n} \to \underline{n}$ such that $I = \cap_{i \in \underline{n}} (A_{f(i)} : A_i)_{C_0(G,\mathcal{L})}$, $f(i) \leq f(j)$ for $i, j \in \underline{n}$ with $i \leq j$, $f(i) \leq i$ for all $i \in \underline{n}$. Thus $I = \Psi(f)$. This completes the proof of (5.1).

Now, we define a mapping $\Phi : \mathcal{I} \to \mathcal{S}(n)$ as follows. For $I \in \mathcal{I}$ and $j \in \underline{n}$, we define

$$\begin{aligned} \Phi(I)\,(j) := {} & \text{ the } k \in \underline{n} \text{ such that the } C_0(G, \mathcal{L})\text{-ideal} \\ & \text{of } G \text{ generated by } \{p(\gamma) \mid p \in I, \gamma \in A_j\} \text{ is equal to } A_k. \end{aligned}$$

Now we prove

$$\Phi \circ \Psi = \mathrm{id}_{S(n)}. \tag{5.2}$$

We let $f \in S(n)$ and $j \in \underline{n}$ and prove

$$\Phi(\Psi(f))\,(j) = f(j). \tag{5.3}$$

We compute the lefthand side of (5.3). By the definition of Φ, $\Phi(\Psi(f))\,(j)$ is the $r \in \underline{n}$ such that the $C_0(G,\mathcal{L})$-ideal of G generated by $\{p(\gamma) \mid p \in \Psi(f), \gamma \in A_j\}$ is equal to A_r. We will now show

$$\{p(\gamma) \mid p \in \Psi(f), \gamma \in A_j\} \subseteq A_{f(j)}. \tag{5.4}$$

Fix $p \in \cap(A_{f(i)} : A_i)_{C_0(G,\mathcal{L})}$ and $\gamma \in A_j$. Then, clearly $p(\gamma) \in A_{f(j)}$. This completes the proof of (5.4). Now we show

$$\{p(\gamma) \mid p \in \Psi(f), \gamma \in A_j\} \not\subseteq A_{f(j)-1}. \tag{5.5}$$

To prove (5.5), we let $B_1 := \{0\}$, and $B_k := A_k \setminus A_{k-1}$ for $k \in \{2,\dots,n\}$. We note that $(B_1, B_2, \dots, B_n)$ is a partition of G. Now for each $j \in \underline{n}$, we choose an element $\beta_j \in B_j$. We define a mapping $g : G \to G$ by

$$g(\gamma) = \beta_{f(k)}$$

for all $k \in \underline{n}$ and $\gamma \in B_k$. We show

$$g \text{ is compatible with } \mathcal{L}. \tag{5.6}$$

Fix $\xi, \eta \in G$, $m \in \underline{n}$, and assume that $\xi - \eta$ lies in A_m. Let k, l be such that $\xi \in B_k$ and $\eta \in B_l$. We assume $k \geq l$. If $k = l$, then $g(\xi) = g(\eta)$, and thus $g(\xi) - g(\eta) = 0 \in A_m$. If $k > l$, then we notice that $\xi \in A_k \setminus A_{k-1}$, and since $l < k$, we have $\eta \in A_{k-1}$. Hence, $\xi - \eta \notin A_{k-1}$. Thus, $m \geq k$. Since $g(\xi) - g(\eta) = \beta_{f(k)} - \beta_{f(l)} \in A_{f(k)} \subseteq A_k \subseteq A_m$, the difference $g(\xi) - g(\eta)$ lies in A_m, which completes the proof of (5.6). From the definition of g, we obtain that g lies in $\cap_{k \in \underline{n}}(A_{f(j)} : A_j)_{C_0(G,\mathcal{L})} = \Psi(f)$. Since $g(\beta_j) = \beta_{f(j)} \notin A_{f(j)-1}$, we obtain (5.5). Hence, the r that yields the value of the lefthand side of (5.3) is equal to $f(j)$. This completes the proof of (5.3), and hence of (5.2). Equation (5.2) implies that Ψ is injective. Next, we prove that for all $f, g \in S(n)$, we have

$$f \leq g \quad \text{if and only if} \quad \Psi(f) \leq \Psi(g). \tag{5.7}$$

Since the lattice operations $\wedge$ and $\vee$ are uniquely determined by their corresponding order, (5.7) will imply that Ψ is a lattice isomorphism from $\mathcal{S}(n)$ to $\mathcal{I}$, see [**8**, page 41]. The "only if"-direction of (5.7) is obvious. For the "if"-direction, we assume $\Psi(f) \leq \Psi(g)$. Since Φ is order preserving, we obtain $\Phi(\Psi(f)) \leq \Phi(\Psi(g))$. Thus by (5.2), we have $f \leq g$. $\square$

For a finite lattice $\mathcal{L}$, we let its *height* be $|\mathcal{C}| - 1$, where $\mathcal{C}$ is a chain of maximal cardinality in $\mathcal{L}$.

Corollary 5.2. *Let G be a finite group with $|G| \geq 2$, let $n \in \mathbf{N}$, and let $A_1, A_2, \dots, A_n$ be normal subgroups of G with $\{0\} = A_1 < A_2 < \cdots < A_{n-1} < A_n = G$. We assume that for all $i \in \{2, \dots, n\}$, we have $|A_i/A_{i-1}| \geq 3$. Let $\mathcal{L} := \{A_1, A_2, \dots, A_n\}$, and let $\mathcal{I}$ be the set of ideals of the near-ring $C_0(G, \mathcal{L})$. Then*:

(1) *The near-ring $C_0(G, \mathcal{L})$ has $C(n)$ ideals, where*

$$C(n) := \frac{1}{n+1}\binom{2n}{n}$$

is the nth Catalan number.

(2) *The lattice $\mathcal{I}$ is of height $\binom{n}{2}$.*

(3) *The near-ring $C_0(G, \mathcal{L})$ is subdirectly irreducible, and its unique minimal ideal is $(0 : A_{n-1})_{C_0(G,\mathcal{L})} \cap (A_2 : G)_{C_0(G,\mathcal{L})}$.*

(4) *$C_0(G, \mathcal{L})$ has $n-1$ maximal ideals.*

Proof. By Theorem 5.1, the number of ideals is $|S(n)|$, which is equal to $C(n)$ by [**13**, page 224]. This completes the proof of item (1). For item (2), we observe that by [**5**, Corollary II.1.14], the height of $\mathcal{I}$ is equal to the number of meet irreducible elements of $\mathcal{I}$ that are not equal to $C_0(G, \mathcal{L})$. By Theorem 5.1, this number is equal to $|M(\mathcal{S}(n))|$, where $M(\mathcal{S}(n))$ is the set of meet irreducible elements of $\mathcal{S}(n)$ that are not equal to the identity function $\mathrm{id}_{\underline{n}}$. For $i, j \in \underline{n}$ with $1 \leq i < j \leq n$, we define a function $t_{i,j}$ by

$$t_{i,j}(k) := \begin{cases} k & \text{if } k \leq i-1, \\ i & \text{if } i \leq k \leq j, \\ k & \text{if } k > j. \end{cases}$$

Then we have:

$$M(\mathcal{S}(n)) = \{t_{i,j} \mid i, j \in \underline{n}, 1 \leq i < j \leq n\}. \tag{5.8}$$

To prove $\supseteq$, let $i, j \in \underline{n}$ with $i < j$. We suppose that $t_{i,j}$ is the meet of two elements in $\mathcal{S}(n)$, and we let $f, g \in S(n)$ be such that the meet of f and g is $t_{i,j}$. We will show that $f = t_{i,j}$ or $g = t_{i,j}$. Seeking a contradiction, we suppose $f \neq t_{i,j}$ and $g \neq t_{i,j}$, and we let k be the smallest number such that $f(k) \neq t_{i,j}(k)$ or $g(k) \neq t_{i,j}(k)$. Clearly, we have $k > i$ and $k \leq j$. We assume that $f(k) > t_{i,j}(k)$ and $g(k) = t_{i,j}(k)$. We will now prove $g = t_{i,j}$. To this end, let $l \in \underline{n}$. If $l \leq k$, then $g(l) = t_{i,j}(l)$ is immediate. If $l > k$ and $l \leq j$, then we have $f(l) \geq f(k) > t_{i,j}(k) = i = t_{i,j}(l)$. Since $\min(f(l), g(l)) = t_{i,j}(l)$, we must have $g(l) = t_{i,j}(l)$. If $l > j$, then we have $t_{i,j}(l) = l$. Since $g \in S(n)$, we have $g(l) \leq l$. Thus, we obtain $g(l) = t_{i,j}(l)$ also in this case. Hence, we have proved $g = t_{i,j}$, a contradiction. Clearly, if $f(k) = t_{i,j}(k)$ and $g(k) > t_{i,j}(k)$, we obtain $f = t_{i,j}$ in the same way. Hence $t_{i,j}$ is meet irreducible, which completes the proof of $\supseteq$. For $\subseteq$, we let f be a meet irreducible element of $M(\mathcal{S}(n))$. We let $A := \{k \in \underline{n} \mid f(k) < k\}$. Since $f \neq \mathrm{id}_{\underline{n}}$, A is not empty. Now we prove that, for all $l \in \underline{n}$, we have

$$f(l) = \min\{t_{f(k),k}(l) \mid k \in A\}. \tag{5.9}$$

We fix $l \in \underline{n}$. For $\leq$, we show that

$$f(l) \leq t_{f(k),k}(l) \tag{5.10}$$

for all $k \in A$. Let $k \in A$. If $l \leq f(k)$ or $l > k$, (5.10) follows immediately from $t_{f(k),k}(l) = l$. If $l > f(k)$ and $l \leq k$, we have $f(l) \leq f(k) = t_{f(k),k}(l)$. For $\geq$ of (5.9), we observe that the claim is obvious if $f(l) = l$. Now we assume $f(l) < l$. In this case, we have $f(l) = t_{f(l),l}(l)$. Since $l \in A$, this equality implies $t_{f(l),l}(l) \geq \min\{t_{f(k),k}(l) \mid k \in A\}$, which completes the proof of (5.9). Since f is meet irreducible, we obtain that there is a $k \in A$ such that $f = t_{f(k),k}$. This completes the proof of $\subseteq$ of (5.8). From (5.8), we obtain that we have precisely $\binom{n}{2}$ meet irreducible elements in $S(n) \setminus \{\mathrm{id}_{\underline{n}}\}$. Hence the lattice $\mathcal{S}(n)$ is of height $\binom{n}{2}$. This completes the proof of item (2).

To prove item (3), we use the isomorphism Ψ given in Theorem 5.1, and obtain that the meet irreducible elements in the ideal lattice $C_0(G, \mathcal{L})$ are precisely the $\binom{n}{2}$ ideals $(A_i : A_j)C_0(G, \mathcal{L})$ with $1 \leq i < j \leq n$. For $i = 1$, $j = n$, we obtain that $0 = (A_1 : A_n)_{C_0(G,\mathcal{L})}$ is a meet irreducible ideal, and hence $C_0(G, \mathcal{L})$ is subdirectly irreducible. Using the lattice isomorphism of Theorem 5.1, it is also possible to describe the unique minimal ideal I of $C_0(G, \mathcal{L})$. Since $\min(t_{1,n-1}, t_{2,n}) \leq f$ for all $f \in S(n)$ not identically 1, we have $I = \Psi(\min(t_{1,n-1}, t_{2,n})) = (A_1 : A_{n-1})_{C_0(G,\mathcal{L})} \cap (A_2 : A_n)_{C_0(G,\mathcal{L})}$. This completes the proof of item (3).

Item (4) follows from the fact that the functions $t_{j-1,j}$ with $j \in \underline{n}\backslash\{1\}$ are the coatoms of the lattice $\mathcal{S}(n)$. $\square$

We know that for $n \in \{2, 3\}$, Corollary 5.2 is also true for infinite groups G. For $n = 2$, this follows from the fact that $M_0(G)$ is a simple near-ring [**9**, Theorem 1.42], and for $n = 3$, this is proved in [**3, 4**].

Acknowledgments. The fact that the Catalan numbers are needed in Section 5 was first noticed using [**12**].

REFERENCES

1. E. Aichinger, *On maximal ideals of tame near-rings*, Rivista di Matematica della Università di Parma **2*** (1999), 215–233.

2. G. Betsch, *Primitive near-rings*, Math. Z. **130** (1973), 351–361.

3. G.A. Cannon and L. Kabza, *The lattice of ideals of the nearring of coset preserving functions*, Quaestiones Math. **24** (2001), 1–7.

4. ———, *Corrigendum to: "The lattice of ideals of the nearring of coset preserving functions,"* Quaestiones Math. **26** (2003), 141–145.

5. G. Grätzer, *General lattice theory*, Second ed., Birkhäuser Verlag, Basel, 1998.

6. K. Kaarli, *Compatible function extension property*, Algebra Universalis **17** (1983), 200–207.

7. K. Kaarli and A.F. Pixley, *Polynomial completeness in algebraic systems*, Chapman & Hall/CRC, Boca Raton, Florida, 2001.

8. R.N. McKenzie, G.F. McNulty and W.F. Taylor, *Algebras, lattices, varieties*, Volume I, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, California, 1987.

9. J.D.P. Meldrum, *Near-rings and their links with groups*, Pitman (Advanced Publishing Program), Boston, 1985.

10. G.F. Pilz, *Near-rings*, 2nd edition, North-Holland Publishing Company, Amsterdam, 1983.

11. S.D. Scott, *Some interesting open problems*, Near-ring newsletter–number 16, Y. Fong, G.F. Pilz, A. Oswald and K.C. Smith, eds., National Cheng Kung University, Taiwan, 1995.

12. N.J.A. Sloane, *The on-line encyclopedia of integer sequences*, `http://www.research.att.com/~njas/sequences/`, 2004.

13. R.P. Stanley, *Enumerative combinatorics*, Volume 2, Cambridge Stud. Adv. Math. **62**, Cambridge University Press, Cambridge, 1999.

14. H. Werner, *Einführung in die allgemeine Algebra*, Bibliographisches Institut Mannheim-Wien-Zürich, 1978.

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