ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 1, Winter 1989

JACKSON TYPE THEOREMS IN APPROXIMATION BY RECIPROCALS OF POLYNOMIALS

A.L. LEVIN[1] AND E.B. SAFF[2]

ABSTRACT. It was previously shown by the authors that Jackson type theorems hold for the case of approximating a continuous real-valued function f on a real interval by the *reciprocals* of complex polynomials. In this paper we extend these results to the general case when f is complex-valued.

1. Statement of results. Let $C^*[-\pi,\pi]$ denote the set of 2π-periodic continuous complex-valued functions and let $C[-1,1]$ denote the set of continuous complex-valued functions on [-1,1]. For any $f \in C^*[-\pi,\pi]$ (resp. $f \in C[-1,1]$) we denote by $E^*_{0n}(f)$ (resp. by $E_{0n}(f)$) the error in best uniform approximation of f on $[-\pi,\pi]$ (resp. on [-1,1]) by *reciprocals* of trigonometric (resp. algebraic) polynomials of degree $\leq n$ with complex coefficients.

Our goal is to prove the following Jackson type theorems.

THEOREM 1. *There exists a constant M such that for any $f \in C^*[-\pi,\pi]$,*

$$E^*_{0n}(f) \leq M\omega(f;n^{-1}), \quad n = 1,2,3,\dots,$$

where $\omega(f;\delta)$ denotes the modulus of continuity of f on $[-\pi,\pi]$.

THEOREM 2. *There exists a constant M such that, for any $f \in C[-1,1]$,*

$$E_{0n}(f) \leq M\omega(f;n^{-1}), \quad n = 1,2,3,\dots,$$

where $\omega(f;\delta)$ denotes the modulus of continuity of f on [-1,1].

---

1 The research of this author was conducted while visiting the Institute for Constructive Mathematics at the University of South Florida.
2 The research of this author was supported, in part, by the National Science Foundation.
AMS *Subject Classification*: 41A20, 41A17.
Received by the editors on September 19, 1986.

For the case of real-valued f, these theorems (with slightly different notation) were proved in our paper [**1**]. Although the idea of the proof remains the same, the passage to a complex-valued f is not straightforward (in contrast with polynomial approximation). It requires a preliminary construction (see Lemma 1 below) that is trivial in the case of real f but rather complicated in general.

2. Proofs. We first formulate two lemmas. In these results, $||\cdot||$ denotes the sup norm on $[-\pi,\pi]$ and ω is the modulus of continuity on $[-\pi,\pi]$.

LEMMA 1. *For any $f \in C^*[-\pi,\pi]$, for any positive integer n, and for any $A > 0$, there exists a function $g \in C^*[-\pi,\pi]$ such that*

(1) $||f-g|| \leq 4A\omega(f;n^{-1})$,

(2) $|g(x)| \geq \frac{1}{2}A\omega(f;n^{-1}), \quad -\pi \leq x \leq \pi$, *and*

(3) $\omega(g;n^{-1}) \leq (1+8\pi)\omega(f;n^{-1})$.

Also, if f is even, then g may be chosen even as well.

LEMMA 2. *There exist absolute constants $A_0 > 0$, $A_1 > 0$ such that, for any $g \in C^*[-\pi,\pi]$ that satisfies* (2) *with $A = A_0$ and* (3), *one can find a trigonometric polynomial P_n of degree $\leq n$ such that*

$$||g - 1/P_n|| \leq A_1\omega(f;n^{-1}).$$

Also, if g is even, then P_n may be chosen even as well.

Theorem 1 is an immediate consequence of these lemmas. Indeed, applying Lemma 1 with $A = A_0$ and Lemma 2 we obtain that

$$E^*_{0n}(f) \leq ||f-g|| + ||g-1/P_n|| \leq M\omega(f;n^{-1}),$$

where $M := 4A_0 + A_1$. Theorem 2 follows from Theorem 1 by a standard argument (notice the last assertions of the lemmas).

PROOF OF LEMMA 1. Set

$$K_1 := \{x \in [-\pi,\pi] : |f(x)| \geq A\omega(f;n^{-1})\}, \tag{4}$$

$$K_2 := \{x \in [-\pi,\pi] : |f(x)| < A\omega(f;n^{-1})\}. \tag{5}$$

We assume first that $\pm\pi \in K_1$. In this case we can represent K_2 as a union $\cup(a_k,b_k)$ of disjoint open intervals in $(-\pi,\pi)$ with

$$|f(a_k)| = |f(b_k)| = A\omega(f;n^{-1}). \tag{6}$$

Further, we write K_2 as a union $K_2' \cup K_2''$, where

$$K_2' := \cup\Big\{(a_k,b_k) : |f(x)| \geq 1/2 A\omega(f;n^{-1}), \text{ all } x \in (a_k,b_k)\Big\}, \tag{7}$$

(8)
$$K_2'' := \cup\Big\{(a_k,b_k) : |f(x)| < 1/2 A\omega(f;n^{-1}) \text{ for some } x \in (a_k,b_k)\Big\}.$$

Then, for the length $\Delta_k := b_k - a_k$ of any interval (a_k,b_k) in K_2'', we have the estimate

$$\omega(f;\Delta_k) \geq ||f(b_k)| - \min_{(a_k,b_k)} |f|| \geq \frac{1}{2}A\omega(f;n^{-1}), \tag{9}$$

by (6) and (8).

For every interval (a_k,b_k) in K_2'', write (cf. (6)) $f(a_k) = A\omega(f;n^{-1})\exp(i\alpha_k)$, $f(b_k) = A\omega(f;n^{-1})\exp(i\beta_k)$, with $|\beta_k - \alpha_k| \leq \pi$ and let $L_k(x)$ be the linear function that satisfies

$$L_k(a_k) = \alpha_k, \quad L_k(b_k) = \beta_k.$$

Then, for any $h > 0$,

$$|L_k(x+h) - L_k(x)| \leq \frac{\pi}{\Delta_k}h, \quad \text{where } \Delta_k := b_k - a_k. \tag{10}$$

Now define the function g on $[-\pi,\pi]$ by

$$g(x) := f(x), \quad x \in K_1 \cup K_2', \tag{11}$$

$$g(x) := A\omega(f;n^{-1})\exp(iL_k(x)), \quad x \in (a_k,b_k) \subset K_2''. \tag{12}$$

From the construction of g it follows that $g \in C^*[-\pi,\pi]$ and satisfies

$$\|f-g\| \le 2A\omega(f;n^{-1}), \tag{13}$$

$$|g(x)| \ge \frac{1}{2}A\omega(f;n^{-1}), \quad -\pi \le x \le \pi. \tag{14}$$

To estimate the modulus of continuity of g we make use of the well-known inequality

$$\frac{\omega(f;h)}{h} \le 2\frac{\omega(f;h')}{h'}, \qquad \text{for } h \ge h' > 0. \tag{15}$$

Let $x, x+h (h>0)$ be any two points in $[-\pi,\pi]$.

Case 1. $x,\ x+h \in K_1 \cup K_2'$. Then (cf. (11)) $|g(x+h)-g(x)| \le \omega(f;h)$.

Case 2. $x,\ x+h \in (a_k,b_k) \subset K_2''$. *Since* $|\exp(it)-\exp(is)| \le |t-s|$, *we obtain, from* (12) *and* (10):

$$\begin{aligned} |g(x+h)-g(x)| &\le A\omega(f;n^{-1})\frac{\pi}{\Delta_k}h \\ &\le 2\pi\frac{\omega(f;\Delta_k)}{\Delta_k}h \quad \text{(by (9))} \\ &\le 4\pi\frac{\omega(f;h)}{h}h \quad \text{(by (15), since } \Delta_k \ge h) \\ &= 4\pi\omega(f;h). \end{aligned}$$

Case 3. $x \in (a_k,b_k) \subset K_2''$, $x+h \in K_1 \cup K_2'$. Write

$$\begin{aligned} |g(x+h)-g(x)| &\le |g(b_k)-g(x)| + |g(x+h)-g(b_k)| \\ &= |g(b_k)-g(x)| + |f(x+h)-f(b_k)| \\ &\le |g(b_k)-g(x)| + \omega(f;h). \end{aligned}$$

Since $|b_k - x| < \Delta_k$, we obtain as in Case 2, that

$$\begin{aligned} |g(b_k)-g(x)| &\le 2\pi\frac{\omega(f;\Delta_k)}{\Delta_k}|b_k-x| \\ &\le 4\pi\frac{\omega(f;|b_k-x|)}{|b_k-x|}\cdot|b_k-x|, \text{ (by (15))} \\ &= 4\pi\omega(f;|b_k-x|) \le 4\pi\omega(f;h). \end{aligned}$$

Hence

$$|g(x+h)-g(x)| \leq (1+4\pi)\omega(f;h). \tag{16}$$

Case 4. $x \in K_1 \cup K_2'$, $x+h \in K_2''$. Just as in Case 3, it can be shown that inequality (16) holds.

Case 5. $x \in (a_k, b_k) \subset K_2''$, $x+h \in (a_l, b_l) \subset K_2''$, with $k \neq l$. In this case we write (assume $b_k \leq a_l$)

$$|g(x+h)-g(x)| \leq |g(b_k)-g(x)| + |g(a_l)-g(b_k)| + |g(x+h)-g(a_l)|,$$

and proceeding as in Case 3 we conclude that

$$|g(x+h)-g(x)| \leq (1+8\pi)\omega(f;h).$$

Putting all the cases together we obtain

$$\omega(g;h) \leq (1+8\pi)\omega(f;h), \quad h>0. \tag{17}$$

The inequalities (13), (14) and (17) prove Lemma 1 for the case $\pm\pi \in K_1$. If $\pm\pi \in K_2$, that is if $|f(\pm\pi)| < A\omega(f;n^{-1})$, we replace f by $\tilde{f} := f + 2A\omega(f;n^{-1})$ and apply the above argument to construct the function g that satisfies (13), (14), and (17) with $\tilde{f}$ instead of f. Since $\omega(\tilde{f};h) = \omega(f;h)$ and $||f-\tilde{f}|| \leq 2A\omega(f;n^{-1})$, the same function g will satisfy the requirements (1), (2), and (3) of Lemma 1.

Finally, if f is even, then each of the sets K_1, K_2', and K_2'' is symmetric with respect to the origin. From this and from the definition (11), (12) of g it follows easily that g is also even. □

REMARK. If f is real, the function g can be constructed in a much simpler way, namely we can set $g(x) := f(x) + iA\omega(f;n^{-1})$.

PROOF OF LEMMA 2. The proof is essentially contained in our paper [**1**]. For the reader's convenience we reproduce it briefly.

Let $K_n(t)$ be the Jackson kernel (cf. Lorentz [**2**, p. 55]). Then, for any $g \in C^*[-\pi, \pi]$,

$$\int_{-\pi}^{\pi} |g(x+t) - g(x)|^j \; K_n(t)dt \le c[\omega(g; n^{-1})]^j, \quad j = 1, 2, \tag{18}$$

where c is an absolute constant. Define

$$A_0 := 4c(1 + 8\pi) \tag{19}$$

and let g be the function from Lemma 1 with $A = A_0$. Further, define the trigonometric polynomial P_n of degree $\le n$ by

$$P_n(x) := \int_{-\pi}^{\pi} \frac{1}{g(x+t)} K_n(t)dt. \tag{20}$$

Then

$$\begin{aligned}
|1 - P_n(x)g(x)| &= \left| \int_{-\pi}^{\pi} \frac{g(x+t) - g(x)}{g(x+t)} K_n(t)dt \right| \\
&\le \frac{2}{A_0\omega(f; n^{-1})} \cdot c\omega(g; n^{-1}), \quad \text{(by (2), (18))} \\
&\le \frac{2c(1 + 8\pi)}{A_0} = \frac{1}{2}, \quad \text{(by (3), (19))}.
\end{aligned}$$

Hence,

$$|P_n(x)g(x)| \ge 1/2, \quad -\pi \le x \le \pi. \tag{21}$$

Now,

$$\begin{aligned}
&|g(x) - 1/P_n(x)| \\
&\quad \le \int_{-\pi}^{\pi} \left| \frac{g(x+t) - g(x)}{g(x)g(x+t)} \right| \cdot \left| \frac{g(x)}{P_n(x)} \right| \cdot K_n(t)dt \\
&\quad \le 2 \int_{-\pi}^{\pi} |g(x+t) - g(x)| \cdot \left| \frac{g(x)}{g(x+t)} \right| \cdot K_n(t)dt \quad \text{(by (21))} \\
&\quad \le 2 \int_{-\pi}^{\pi} |g(x+t) - g(x)| K_n(t)dt + 2 \int_{-\pi}^{\pi} \frac{|g(x+t) - g(x)|^2}{|g(x+t)|} K_n(t)dt \\
&\quad \le 2c\omega(g; n^{-1}) + 4c(\omega(g; n^{-1}))^2 / A_0\omega(f; n^{-1}) \quad \text{(by (2),(18))} \\
&\quad \le (2c + 1)(1 + 8\pi)\omega(f; n^{-1}) =: A_1\omega(f; n^{-1}) \quad \text{(by (2), (3), and (19))}.
\end{aligned}$$

Finally, if $g \in C^*[-\pi, \pi]$ is even, then (cf. (20)) P_n is an even trigonometric polynomial. □

REFERENCES

1. A.L. Levin and E.B. Saff, *Degree of approximation of real functions by reciprocals of real and complex polynomials*, SIAM J. Math. Anal. **19** (1988), 233-245.

2. G.G. Lorentz, *Approximation of functions*, Holt, Rinehart and Winston, New York, 1966.

DEPARTMENT OF MATHEMATICS, EVERYMAN'S UNIVERSITY, 16 KLAUSNER ST., POB 39328, TEL-AVIV 61392, ISRAEL

INSTITUTE FOR CONSTRUCTIVE MATHEMATICS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FL 33620