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# THE ZARISKI TOPOLOGY FOR DISTRIBUTIVE LATTICES

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ABSTRACT. Our purpose in this paper is to study an intrinsic topology for distributive lattices which by its very definition is analogous to the classical Zariski topology on rings. As in the case of rings, the Zariski topology is the coarsest topology making solution sets of polynomials closed. In other words, the Zariski closed sets are generated from a subbase consisting of all sets of the form $\{z \in L : p(z) = c\}$ where $p(x)$ is a polynomial over $L$ and $c$ is an arbitrary but element from $L$.

Although the name of this topology for lattices is new, the Zariski topology has appeared, usually unnamed and implicitly, in a variety of guises and formulations in lattice theory over the years. Recently, under different formulations, it was used effectively by R. Ball in [**3**] and by H. Bunch in [**7**].

Out context for studying intrinsic topologies on distributive lattices was set by Frink in [**9**] where he stated, "Many mathematical systems are at the same time lattices and topological spaces. It is natural to inquire whether the topology in such systems in definable in terms of the order relation alone." Seeking systems which are at the same time lattices and topological spaces, one must begin with $R$, the real line with its usual topology and order, along with two of its substuctures I, the closed unit interval of $R$, and 2, the chain consisting of the numbers 0 and 1, and then go on to form all finite and infinite Cartesian products of such systems using the product order and the Tychonoff topology. Given this collection of mathematical systems, is it possible to find one intrinsic topology–really, topology definition scheme–which will define the topology exclusively from the order?

For $R, I, 2$ and any chain, the interval topology gives the correct

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topology. It also works well for products of complete chains and for complete sublattices of products of complete chains (i.e., completely distributive lattices). In each of these cases it provides a compact, Hausdorff topology relative to which the operations of meet and join are continuous. When one moves away from complete chains, the interval topology is more of a point of reference than a useful and usable topology. On $R \times R$, for example, it is not even Hausdorff. To salvage the situation for finite dimensional Euclidean spaces, the Frink ideal topology and Birkhoff's new interval topology were created. (Cf. [**10**], [**5**], [**1**] and [**2**].) However these topologies do not return the correct topology for infinite products of complete chains–one of the cases handled well by the interval topology.

Unlike the interval topology, the new interval topology and the Frink ideal topology, the Zariski topology is both finitely and infinitely productive which enables it to provide the correct topology for all spaces mentioned above. For complete sublattices of products of complete chains (i.e., completely distributive lattices), it is equivalent to the interval topology and from [**12**] we know that for $R^n$ and any other distributive lattice of finite breadth, it is equivalent to the Frink ideal topology.

Much of our discussion will center on subspaces and sublattices where the Zariski topology has some interesting properties. Our discussion requires extensive and heavy use of the theory of essential extensions of distributive lattices.

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**1. Preliminaries.** By the usual topology on $R^n$ we shall mean the Cartesian product (or Tychonoff) topology, and by the usual order we shall mean the product ordering. Let $L$ be a distributive lattice. For $x \in L$ and $A \subseteq L$ we define $\uparrow X = \{z \in L : z \geq X\}$ and $\uparrow A = \bigcup\{\uparrow a : a \in A\}$. The sets $\downarrow X$ and $\downarrow A$ are defined dually. The interval topology on $L$ is defined by declaring sets of the form $\uparrow X$ and $\downarrow X$ to be closed for all $X \in L$. For$\mathcal{F}$, a filter of subsets of $L$, lim inf $\mathcal{F}$ is defined to be $\sup\{\inf A : A \in \mathcal{F}\}$ and lim sup $\mathcal{F}$ is defined dually.

A subset $D$ of a lattice $L$ is up–directed if for each pair $c, d \in D$, there is an $e \in D$ such that $c, d \leq e$. A complete lattice $L$ is said to be meet–continuous if for every up–directed subset $D$ of $L$ and ever element $x \in L$, $x \wedge \sup D = \sup\{x \wedge d : d \in D\}$. Down–directed sets and join–continuity are defined dually. A complete distributive lattice which is meet–continuous and joint–continuous is called infinitely distributive.

A topological space is quasi–compact if it has the Heine–Borel property, i.e., every open cover has a finite subcover. A quasicompact Hausdorff space is called compact. The next result collects well known information about the interval topology

PROPOSITION 1.1 *Let $L$ be a distributive lattice.*

(i) *If $L$ is complete, then it is quasicompact in the interval topology. If $\hat{\mathcal{U}}$ is an ultrafilter of subsets of $L$, then in the interval topology, $\hat{\mathcal{U}}$ converges to every point in the interval [lim inf $\hat{\mathcal{U}}$, lim inf $\hat{\mathcal{U}}$].*

(ii) *$L$ is compact in the interval topology if and only if it is complete and lim inf $\hat{\mathcal{U}}$ = lim sup $\hat{\mathcal{U}}$ for every ultrafilter $\hat{\mathcal{U}}$ of subsets of $L$.*

(iii) *If $L$ is infinitely distributive, then $L$ is compact in the interval topology of and only if $L$ is completely distributive.*

(iv) *If $L$ is completely distributive, then $L$ is a compact topological lattice in the interval topology, i.e., the maps $\wedge, \vee : L \times L \to L$ are jointly continuous.*

**2. The Zariski topology.** A polynomial in the variable $x$ over the distributive lattice $L$ is any expression which can be formed by taking any finite combination of meets and joins of elements from $L \bigcup \{x\}$. Appeal to the distributivity law reduces the range of such expressions to a collection of very elementary polynomials: $x, a, a \vee x, b \wedge x$ and $a \vee (x \wedge b)$ where $a$ and $b$ are arbitrary elements in $L$. The *Zariski topology* is the coarsest topology on $L$ for which sets of the form $\{z \in L : p(z) = c\}$ are closed where $p(x)$ ranges over all polynomials in $x$ and where $c$ is an arbitrary but fixed element of $L$.

A further reduction of complexity in the definition and use of the Zariski topology is achieved by means of two specified types of sets which normally are used in calculating relative pseudo–complements in Heyting algebras (cf. [**6**, p. 45]).

DEFINITION 2.1. Let $L$ be a distributive lattice and let $a, b \in L$. Set

$$b\backslash a = \{x \in L : a \vee x \geq b\}$$

and

$$a/b = \{x \in L : b \wedge x \leq a\}.$$

When working with sets of the form $a/b$ and $b\backslash a$, we may assume that $a \leq b$ since $a \vee x \geq b$ holds if and only if $a \vee x \geq b \vee a$ and the inequality

$b \wedge x \leq a$ holds if and only if $b \wedge x \leq a \wedge b$, whence $a/b = (a \wedge b)b$ and $b\backslash a = (a/b)/a$.

PROPOSITION 2.2. *The Zariski topology is the coarsest topology on a distributive lattice which makes all sets of the form $a/b$ and $b$ " $a$ closed.*

PROOF. The sets $b\backslash a$ may be written as $\{x \in L : (a \vee x) \wedge b = b\}$ while the set $a/b$ may be rewritten as $\{x \in L : (b \wedge x) \vee a = a\}$. Thus the topology generated by the sets $a/b$ and $b\backslash a$ is coarser than in the Zariski topology.

Conversely suppose that $\{x \in L : p(x) = c\}$ is one of the defining sets for the Zariski topology with $p(x) = a \vee (x \wedge b)$. Since $L$ is distributive, we have $a \vee (x \wedge b) = a \vee (x \wedge (a \vee b))$ which allows us to assume that $a \leq b$. Then from the equation $a \vee (x \wedge b) = c$, we have $a \leq c$ and from $c = a \vee (x \wedge b) = (a \vee x) \wedge b$, we have $c \leq b$. Thus $a \leq c \leq b$.

We contend that $\{x : p(x) = c\} = c/b \bigcap c\backslash a$. If $x \in c/b \bigcap c\backslash a$ then $x \wedge b \leq c$ and $a \vee x \leq c$. Hence $c \leq a \vee (x \wedge b) = (a \vee x) \wedge b \leq c$ from which it follows that $p(x) = c$. One the other hand, suppose that $p(x) = c$. Then $a \vee (x \wedge b) = c$, which implies that $x \wedge b \leq c$ and upon rewriting $p(x)$ as $b \wedge (a \vee x)$, we have $a \vee x \geq c$. Thus $x \in c/b \bigcap c\backslash a$.

At this point it might be useful to explore a few of the connections between the Zariski topology and the interval topology. For an element $a$ of a distributive lattice $L$, define two polynomials in the variable $x$, namely $p_a(x) = a \wedge x$ and $q_a(x) = a \vee x$. Then $\{z \in L : p_a(z) = a\}$ and $\{z \in L : q_a(z) = a\}$ are subbasic closed sets in the Zariski topology. However, a calculation shows that $\{z \in L : p_a(z) = a\} = \ \uparrow a$ and $\{z \in L : q_a(z) = a\} = \ \downarrow a$. Thus the interval topology will always be coarser than the Zariski topology.

With no difficulty one sees that for chains, the Zariski topology and the interval topology coincide. On the other end of the distributive lattice spectrum, they will also be in agreement. Suppose that $L$ is either a Boolean algebra or a relatively pseudo–complemented distributive lattice. Then for $a, b \in L$ the ideal $a/b$ and the filter $b\backslash a$ are both principal and generated by the appropriate relative pseudo–complements. In the Boolean algebra case we have $b\backslash a = \ \uparrow (b \wedge a^c)$ and $a/b = \ \downarrow (a \vee b^c)$ where $a^c$ and $b^c$ are the complements in $L$ of $a$ and $b$ respectively. The information about these relationships is summarized in the next proposition.

PROPOSITION 2.3. *Let $L$ be a distributive lattice. Then the interval topology is coarser than the Zariski topology. If $L$ is a Boolean algebra or a relatively pseudo–complemented distributive lattice, then the Zariski topology and the interval topology coincide.*

For lattices such as chains and Boolean algebras in which the interval topology works best, it is equivalent to the Zariski topology. The Euclidean plane treated as a lattice is a good illustratoin of a situation in which the interval topology works poorly. Not only does the interval topology fail to yield the usual Euclidean topology on $R \times R$, but every pair of nonempty open subsets relative to the interval topology on the plane will have a nonempty intersection.

To remedy some of the deficiencies of the interval topology has with lattice such as the plane, the Frink ideal topology (cf. [**10**]) and Birkhoff's new interval topology [**5**] were brought into the picture. For finite products of $R$, these two topologies coincide with Euclidean topology. Unfortunately they go astray for larger products like $R$ (c.f. [**1**]). The Zariski topology is fully productive. Then since it yields the usual topology for $R$ it will give the Tychonoff topology on all products of $R$.

THEOREM 2.4. *Let $(L_i)_{i\in I}$ be a family of distributive lattices. Then the Zariski topology on $\prod_{i\in I} L_i$ is the product topology of the Zariski topologies on the $L_i, i \in I$.*

PROOF. As usual, let $\pi_j : \prod_{i\in I} L_i \to L_j$ be the projection onto the $j^{th}$ factor and let $f = (f_i)_{i\in I}, g = (g_i) \in \prod_{i\in I} L_i$ be given. Then

$$\begin{aligned} g/f &= \{h \in \prod L_i : f \vee \geq g\} \\ &= \{h \in \prod L_i : f_i \vee h_i \geq g_i \text{ for all } i \in I\} \\ &= \bigcap \pi_i^{-1}(g_i/f_i). \end{aligned}$$

Dually, $f/g = \bigcap_{i\in I} \pi_i^{-1}(f_i/g_i)$; hence the Zariski topology on $\prod_{i\in I} L_i$ is coarser than the product topology. In order to show that every closed set in the product is Zariski closed, it is enough to verify that $\pi_i^{-1}(a/b)$ and $\pi_i^{-1}(b\backslash a)$ are Zariski closed for every pair $a, b \in L_i$ and every $i \in I$. For every $j \in I/\{i\}$ pick an element $c_j \in L_j$. Define elements $f, g \in \pi L_i$ by

$$f_j = g_j = c_j \text{ whenever } j \in I/\{i\}$$
$$f_i = a, \ g_i = b.$$

It is routine to check that $\pi_i^{-1}(a/b) = f/g$ and that $\pi_i^{-1}(b\backslash a) = g/f$.

One more point needs to be made about the difference between the Zariski topology and the interval topology. The interval topology is a so–called compatible topology which means that given an upward directed set $D$ in a lattice $L$ such that sup $D = x$ exists (or alternatively given a downward directed set $D$ such that inf $D = x$), then a compatible intrinsic topology will require that $D$ converge to $x$. In effect a compatible topology will waste no possible limit point.

The Zariski topology is not necessarily a compatible topology nor does it make the direct translation of completeness to quasi – compact-ness that the interval topology does. The next example should point out these distinctions.

EXAMPLE 2.5. Let $L = \{(x, y) \in R \times R : 0 < x < 1, 0 < y < 1\} \bigcup \{(0,0),(1,1)\}$ and let $D = \{(x,y) \in L : x = \frac{1}{2}\}$. Then $\sup_L D = (1,1)$ and $\inf_L D = (0,0)$. For any compatible topology, the closure of $D$ will contain both $(0,0)$ and(1,1). The Zariski topology on $L$ is the same as the restriction of the usual topology of $R \times R$ to $L$.

The interval topology on $L$ will be compact since $L$ is complete, but it will not be Hausdorff. The Zariski topology on $L$ will be Hausdorff, but it will not be compact.

In [**12**] the authors, together with J.D. Lawson, show that for distributive lattices of finite breadth, the Zarinski topology and the Frink ideal topology coincide. So it seems that in the situations in which the Frink ideal topology works well, it agrees with the Zariski topology. The next example is very similar to 2.5 but the differences are crucial.

EXAMPLE 2.6. Let $m = \{(x, y) \in R \times R : 0 < x \leq 1, 0 < y \leq 1\} \bigcup \{(0,0)\}$. Then on $M$ the Lawson topology (cf. [**11**]) is compact and Hausdorff and the meet operation is continuous. However, the join operation is not continuous. As with Example 2.5 the Zariski topology is the same as the topology that $M$ inherits from the plane. So it is Hausdorff and both operations are continuous. Very clearly it is not compact.

It is debatable whether the Zariski topology or the Lawson topology is a better topology for Example 2.6.

**3. The Zariski topology and essential extensions.** To continue our discussion about the qualities of the Zariski topology, in particular to discuss the manner in which it behaves relative to subspaces and quotients, we need to discuss the theory of essential extentions in some detail. In essence our work here will allow us to turn questions about distributive lattices into questions about Boolean algebras. An imbedding of distributive lattices $i : L \rightarrow M$ is essential if whenever $\varphi$ is a congruence on M such that $\varphi \bigcap (i(L) \times i(L)) \subseteq \Delta$, then $\varphi = \Delta$. When this situation holds, M is said to be an essential extension of $L$ via the imbedding $i : L \rightarrow M$. Frequently no mention is made of the specific imbedding involved. In [**4**], Banaschewski and Bruns showed that every distributive lattice has a unique maximal essential extension which turns out to be a complete Boolean algebra. We will need a realization of this maximal essential extension for our later work.

For a distributive lattice $L$, let $\Theta(L)$ be the (algebraic) lattice of all lattice congruences on $L$. It is a well known fact that $\Theta(L)$ is distributive. For $\varphi \in \Theta(L)$ define

$$\varphi^{\perp} = \sup\,\{\psi \in \Theta(L) : \psi \wedge \varphi = \Delta\}.$$

From a theorem of Glivenko (see also [**6,** p. 130]) the set $\Theta^*(L) = \{\varphi^{\perp} : \varphi \in \Theta(L)\}$ becomes a complete Boolean algebra with the order — but not the operations — which it inherits from $\Theta(L)$. For $a \in L$ we define congruences

$$\begin{aligned}\theta_a &:= \{(x,y) \in L \times L : x \vee a = y \vee a\}\\ \theta^a &:= \{(x,y) \in L \times L : x \wedge a = y \wedge a\}.\end{aligned}$$

One verifies almost immediately that $\theta_a$ is the complement of $\theta^a$ in the lattice $\Theta(L)$, therefore $\theta_a^{\perp} = \theta^a$ and $(\theta^a)^{\perp} = \theta_a$. It follows from results of Glivenko and Hashimoto [**17**] that $\Theta^*(L)$ is the maximal essential extension of $L$ via the imbedding $a \rightarrow \theta_a$ (see also [**14,** 1.6] for an explicit proof). The next results will be of importance (cf. [**14**]) in the rest of the paper.

PROPOSITION 3.1. *If $\theta$ is a congruence on the distributive lattice $L$ and if $\theta \in \Theta^*(L)$, then $(a,b) \in \theta$ if and only if $\theta_a \vee \theta = \theta_b \vee \theta$ in $\Theta^*(L)$.*

PROPOSITION 3.2. *Let $L \in \mathcal{D}, a, b \in L$ and let $\theta(a,b)$ denote the smallest lattice congruence identifying $a$ and $b$. If $a \leq b$, then a pair $(x,y) \in L \times L, x \leq y$, belongs to $\theta(a,b)$ if and only if*

$$x = (a \vee x) \wedge y$$
$$y = (b \vee x) \wedge y,$$

PROOF. From Gratzer's book [**16,** 2.9.3] we know that $(x,y) \in \theta(a,b)$ if and only if $b \vee x = b \vee y$ and $a \wedge x = a \wedge y$.

Clearly this implies that $(a \vee x) \wedge y = (a \wedge y) \vee x = (a \wedge x) \vee x = x$ and $(b \vee x) \wedge y = (b \vee y) \wedge y = y$.

Conversly, if $x = (a \vee x) \wedge y$ then $x \wedge a = (a \vee x) \wedge y \wedge a = (a \vee x) \wedge a \wedge y = a \wedge y$. Similarly, $y = (b \vee x) \wedge y$ implies $y \vee b = x \vee b$.

Implicitly contained in the proof of (3.2) is

PROPOSITION 3.3. *If $L$ is a distributive lattice and if $a, b \in L, a \leq b$, then*

(i) $\theta(a,b) = \theta_b \bigcap \theta^a$

(ii) $\theta(a,b)^{\perp} = \theta^b \vee \theta_a = \pi_{a,b}$ *where* $\pi_{a,b} := \{(x,y) : (a \vee x) \wedge b = (a \vee y) \wedge b\}$.

For an explicit proof of (3.3) as well as for more historical notes concerning essential extensions, please see [**14**].

Since every congruence $\theta$ on $L$ is the supremum of the congruences $\theta(a,b), (a,b) \in \theta$, de Morgan's law yields

$$\theta^{\perp} = \bigcap_{(a,b)\in\theta} \pi_{a,b}.$$

In fact, this characterizes congruences in $\theta^*(L)$.

PROPOSITION 3.4. *A congruence $\theta$ on the distributive lattice $L$ belongs to $\theta^*(L)$ if and only if $\theta = \bigcap_{i\in I} \pi_{a_i,b_i}$ for certain elements $a_i, b_i \in L, i \in I$.*

At this point we will begin to connect the idea essential extension with the Zariski topology by providing an alternative definition for the Zariski topology in terms of essential extensions.

THEOREM 3.5. *The map $a \to \theta_a : L \to \Theta^*(L)$ is topological imbedding when both $L$ and $\Theta^*(L)$ are equipped with the Zariski topology.*

PROOF. Since $\Theta^*(L)$ is a Boolean algebra, the Zariski topology on $\Theta^*(L)$ is the interval topology and this fact should be kept in mind throughout the proof. First of all, let $a, b \in L, a < b$, then

$$\begin{aligned}\{x \in L : \theta_x \leq \pi_{a.b}\} &= \{x \in L : \theta_x \leq \theta_a \vee \theta^b\} \\ &= \{x \in L : \theta_x \leq \theta_a \vee \theta_b^{\perp}\} \\ &= \{x \in L : \theta_x \wedge \theta_b \leq \theta_a\} \\ &= \{x \in L : \theta_{x \wedge b} \leq \theta_a\} \\ &= \{x \in L : x \wedge b \leq a\} = a/b.\end{aligned}$$

Let $e : L \to \Theta^*(L)$ denote the imbedding $a \mapsto \theta_a$. Then the equation $\{x \in L : \theta_x \pi_{a,b}\} = a/b$ may be reformulated as

$$a/b = e^{-1}(\downarrow \pi_{a,b}).$$

This last equation yields two things:

(1) The mapping $e$ is continuous. Indeed, if $\theta \in \Theta^*(L)$, then $\theta = \inf_{i \in I} \pi_{a_i,b_i}$ for certain $a_i, b_i \in L$ (see (2.6)). Hence $\downarrow \theta = \bigcap_{i \in I} \downarrow \pi_{a_i,b_i}$ and

$$e^{-1}(\downarrow \theta) = \bigcap_{i \in I} a_i/b_i$$

is closed. By symmetry $e^{-1}(\uparrow \theta)$ is closed, too. Since the sets of the form $\downarrow \theta$ and $\uparrow \theta$ form a subbase for the closed sets, $e$ is continuous.

(2) Every Zariski closed subset $A \subseteq L$ is of the form $A = e^{-1}(B)$, where $B \subseteq \Theta^*(L)$ is closed in the interval topology. This follows from (*) and its dual statement for the subbasic closed sets of the form $a/b$ and $b \backslash a$ and hence for every closed subset $A \subseteq L$

Theorem 3.5 turns out to be very useful in deriving properties for the Zariski topology. All we have to do is to verify the corresponding statement for complete Boolean algebras and then check whether the property is preserved under the formation of subspaces. A first application of this idea leads to the following result.

PROPOSITION 3.6. *Let $L$ be a distributive lattice with the Zariski topology. Then*

(i) *the translations $x \to - \wedge x : L \to L$ and $x \to - \vee x : L \to L$ are continuous for all $a \in L$;*

(ii) *$L$ has a subbase for the closed sets consisting of ideals and filters;*

(iii) *every point $x \in L$ has a neighborhood base consisting of order-convex open subsets of $L$.*

PROOF. All these properties are straightforward for Boolean algebras in the interval topology and obviously preserved under the informaiton of subspaces which at the same time are sublattices.

We now consider the question of whether the lattice operations $\wedge$ and $\vee$ are jointly continuous. Again, we need only turn to complete Boolean algebras to find a counterexample and use some ideas of E. Floyd (see [**8**]).

EXAMPLE 3.7. Let $B$ be a complete Boolean algebra. Then by 2.3 the Zariski topology on $B$ is the interval topology. Moreover, it is easy to see that the product topology on $B \times B$ is the interval topology. In fact this is true for every lattice $B$ with 0 and 1.

Next we take $B$ to be the Boolean algebra of regular open sets of the unit interval. For every natural number $n \in N$ let

$$U_n = \bigcup_{i=0}^{n-1} ]\frac{2i}{2n}, \frac{2i+1}{2n}[.$$

If $(U_{n_m})_m$ is any subsequence of the sequence $(U_n)_{n \in N}$, then $\bigcup_{m \in N} U_{n_m}$ is dense in $I$, the unit interval of $R$, and hence $\sup_{m \in N} U_{n_m} = I$ in $B$.

We now show that $I$ is a limit point of $(U_n)_{n \in N}$ in the interval topology. Indeed, a typical open neighborhood of $I$ is given by

$$\mathcal{A} = \{V \in B : V \not\subseteq W_i \text{ for } i = 1, ..., k\} \ ,$$

where the $W_i$ are certain fixed regular open sets. Now assume that $I$ is not a limit point of $(U_n)_{n \in N}$. Then there is a subsequence $(U_{n_m})_m$ of $(U_n)_n$ and an index $i_0 \in \{1, ..., k\}$ such that $U_{n_m} \subseteq W_{i_0}$ for all $m \in N$. We conclude that $I = \sup_{m \in N} U_{n_m} \subseteq U_{n_m}$, contradicting the fact that

$I \in \mathcal{A}$.

Given $n \in N$, we let

$$V_n = \bigcup_{i=0}^{n-1} ]\frac{2i+1}{2n}, \frac{2i+2}{2n}[.$$

Then $V_n$ is the complement of $U_n$ in the Boolean algebra of regular open subsets of $I$. The same reasoning as before shows that the sequence $V_n$ converges to $I$. Now assume that the mapping $\wedge : B \times B \to B$ is continuous. Then $I = I \bigcap I$ would be a limit point of the sequence $(U_n \wedge V_n)_{n \in N}$, i.e., a limit point of the constant sequence $(\phi)_{n \in N}$, contradicting the fact that $\downarrow \phi = \{\phi\}$ is closed.

**4. Topological suprema and topological infima.** For compatible topologies directed sets converge to their suprema and infima whenever they exist. Thus no further refinement is necessary. For topologies such as the Zariski topology for which this is no longer true, we need further specification.

DEFINITION 4.1. Suppose that $D$ is an upward directed subset of the distributive lattice $L$. If sup $D$ exists and if $D$ converges to sup $D$ in the Zariski topology, then sup $D$ is said to be a topological supremum of $D$ relative to the Zariski topology. Topology infima relative to the Zariski topology are defined dually.

Since we will only be discussing the Zariski topology in this section, we shall suppress the phrase "relative to the Zariski topology" and simply use the terms "topological infima" and "topological suprema."

Before we come to the various characterizations of topological suprema and infima, let us make two remarks.

PROPOSITION 4.2.

(i) *Let $L$ be a distributive lattice and let $D$ be an upward directed subset of $L$ such that supD exists. If the net formed from $D$ has a cluster point $x$, then $x = \sup D$ and $D$ converges to $x$. Especially, in this case $D$ has a unique limit point.*

(ii) *If $L$ is a complete relatively pseudocomplemented lattice, then every upward directed set has a topological supremum and every downward directed set has a topological infimum.*

PROOF.

(i) Let $d_\infty = \sup D$. For every given $d_\alpha \in D$ the net $(d)_{d\in D}$ is eventually in the closed set $[d_\alpha, d_\infty]$ and hence every cluster point belongs to $[d_\alpha, d_\infty]$. Since $\bigcap_{d_\alpha \in D}[d_\alpha, d_\infty] = \{d_\infty\}$, it follows that $d_\infty$ is the only possible cluster point of $D$.

(ii) If $L$ is a double Heyting algebra, then the Zariski topology is the interval topology. Then since $L$ is also complete it follows that the interval topology is compact. Finally our result follows from the fact that the interval topology is compatible. It is possible that a directed set has a supremum without having a cluster point. To illustrate this fact, take $L$ and $D$ as in Example 2.5. Then $D$ has no cluster point in $L$, however $\sup D = (1,1)$ exists in $L$. This is also an example for a directed set $D$ which has a supremum yet this supremum is not topological.

The idea behind part (i) of Proposition 4.2 will appear again in the proof of the next theorem.

THEOREM 4.3. *Let $L$ be a distributive lattice and let $D$ be an upward directed subset of $L$. Then the following conditions are equivalent:*

(a) *The net $(d)_{d\in D}$ converges in the Zariski topology.*

(b) *The set of $D$ admits a topological supremum.*

(c) $\sup D$ *exists in $L$ and the imbedding $a \to \theta_a : L \to \Theta^*(L)$ preserves* $\sup D$.

(d) *If $\theta = \sup\{\theta_d : d \in D\}$ in $\Theta^*(L)$, then there is $d_\infty$ in $L$ such that $\theta = \theta_{d_\infty}$.*

(e) $\sup D$ *exists in $L$ and for every $a \in L$, $a \wedge \sup D = \sup\{a \wedge d : d \in D\}$.*

*Especially, if $L$ is complete, then $L$ is a complete Heyting algebra if and only if all suprema are topological.*

PROOF. The equivalence of (c) and (d) is evident.

(a) $\Rightarrow$ (b). Let $d_\infty = \lim_{d\in D} d$. We have to show that $d_\infty = \sup D$. First we will verify that $d_\infty$ is an upper bound of $D$, hereby establishing the fact that the set of upper bounds of $D$ is nonempty. Indeed, given $d_0 \in D$ the net $(d)_{d\in D}$ is eventually in the closed set $\uparrow d_0$, hence $d_\infty \in \uparrow d_0$ i.e., $d_0 \le d_\infty$. Since $d_0 \in D$ was arbitrary, $d_\infty$ is an upper bound for $D$.

Now let $a \in L$ be any upper bound of $D$. Then the whole net $(d)_{d\in D}$

is contained in the closed set $\downarrow a$ and therefore $d_\infty \leq a$. This shows that $d_\infty$ is the least upper bound of $D$.

(b) $\Rightarrow$ (c). By definition, $\sup D$ exists and$\sup D = \lim D$. Let $d_\infty = \sup D$. Since the imbedding $a \mapsto \theta_a : L \to \Theta^*(L)$ is topological, we obtain $\theta_{d_\infty} = \lim_{d\in D} \theta_d$. Now we use the proof of $(a) \mapsto (b)$ again in order to show that $\lim_{d\in D} \theta_d = \sup_{d\in D} \theta_d$. We conclude $\theta_{d_\infty} = \sup_{d\in D} \theta_d$.

(c) $\Rightarrow$ (e). Let $D \subseteq L$ be a directed subset and assume that $\sup D$ exists and that $\theta_{\sup_{d\in D} \theta_d}$. Further, let $a \in L$. We now use the meet–continuity of $\Theta^*(L)$ in order to calculate

$$\begin{aligned}\theta_{a\wedge \sup D} &= \theta_a \wedge \theta_{\sup D}\\ &= \theta_a \wedge \sup_{d\in D} \theta_d\\ &= \sup_{d\in D} \theta_{a\wedge d}.\end{aligned}$$

Let $u$ be an upper bound of $\{a \wedge d : d \in D\}$. Then $\theta_u \geq \theta_{a\wedge d}$ for all $d \in D$ and hence $\theta_u \geq \sup_{d\in D} \theta_{a\wedge d} = \theta_{a\wedge \sup D}$. We conclude that $u \geq a \wedge \sup D$. This shows that in fact $a \wedge \sup D$ is the smallest upper bound of $\{a \wedge d : d \in D\}$. (Note that $a \wedge \sup D$ always bounds $\{a \wedge d : d \in D\}$ from above!)

(e) $\Rightarrow$ (c). Assume we are given a directed subset $D \subseteq L$ such that $\sup D$ exists and such that we have $a \wedge \sup D = \sup_{d\in D} a \wedge d$ for every $d \in D$. Let $d_\infty = \sup D$. We wish to show that $\theta_{d_\infty} = \sup_{d\in D} \theta_d$. Now note that $\theta_{d_\infty} \leq \sup_{d\in D}\theta_d$ is equivalent to

$$\theta_{d_\infty}^\perp \supseteq \bigcap_{d\in D} \theta_d^\perp$$

i.e.

$$\theta^{d_\infty} \supseteq \bigcap_{d\in D} \theta^d.$$

We have

$$\begin{aligned}(x,y) \in \theta^{d_\infty} &\text{ iff } x \wedge d_\infty = y \wedge d_\infty,\\ &\text{ iff } x \wedge \sup D = y \wedge \sup D,\\ &\text{ iff } \sup_{d\in D} x \wedge d = \sup_{d\in D} y \wedge d.\end{aligned}$$

If we let $(x,y) \in \bigcap_{d\in D} \theta^d$ then by definition of $\theta^d$ we always have $x \wedge d = y \wedge d$ for all $d \in D$. Clearly this implies $\sup_{d\in D} x \wedge d =$

$\sup_{d\in D} y \wedge d$, i.e., $(x,y) \in \theta^{d_\infty}$, establishing the inculsion $\bigcap_{d\in D} \theta^d \subseteq \theta^{d_\infty}$ and therefore $\theta_{d_\infty} \leq \sup_{d\in D} \theta_d$. Since the other inequality always holds, we have verified (c).

(c) $\Rightarrow$ (a). Let us assume (c). Recalling 4.2 (ii) we observe that $(\theta_{\mathrm{d}})_{d\in D}$ converges to $\sup_{d\in D} \theta_d = \theta_{\sup D}$. Since the imbedding $a \to \theta_a$ is topological, the net $(d)_{d\in D}$ converges to $\sup D$.

We now have enough information to turn our attention to closures of ideals and filters in the Zariski topology.

PROPOSITION 4.4. *Let $J$ be an ideal in the distributive lattice $L$. Then cl(J), the closure of $J$ in the Zariski topology, is again an ideal. Moreover, $x \in cl(J)$ if and only if $x$ is the topological supremum of $\downarrow x \bigcap J$.*

PROOF. Again, we consider the imbedding $a \to \theta_a : L \to \Theta^*(L)$. Let $\theta = \mathrm{a} \in \mathrm{J}_{\theta_a}$. Then $\downarrow \theta$ is a closed ideal in $\Theta^*(L)$ and hence $Y = \{x : \theta_x \leq \theta\}$ is a Zariski closed ideal of $L$. It suffices to show that every $x \in Y$ is the topological supremum of $\downarrow x \bigcap J$. Now $\theta_x \leq \theta$ and the meet–continuity of $\Theta^*(L)$ imply $\theta_x = \theta_x \wedge \theta = \sup_{a\in J} \theta_x \wedge \theta_a = \sup_{a\in J} \theta_{x\wedge a}$. Hence $x = \sup(\downarrow x \bigcap J)$ and the imbedding $a \mapsto \theta_a$ preserves $\sup(\downarrow x \bigcap J)$. Now use 4.3 to show that $x$ is the topological supremum of $\downarrow x \bigcap J$.

THEOREM 4.5. *Let $f : L \to M$ be homomorphism; then $f$ is continuous relative to the Zariski topologies defined on both $L$ and $M$ if and only if $f$ preserves topological suprema and topological infima.*

PROOF. If $f$ is continuous relative to the Zariski topologies of $L$ and $M$, then from (4.3) it follows that $f$ preserves topological infima and topological suprema.

Conversely, suppose that $f$ preserves topological suprema and topological infima. By symmetry we need to verify that $f^{-1}(a/b)$ is closed in $L$ for every pair $a, b \in M$. Note that $a/b$ and hence $f^{-1}(a/b)$ are ideals. Assume that $x \in \mathrm{cl}(f^{-1}(a/b))$. From 4.4 we know that x is the topological supremum of $\downarrow x \bigcap f^{-1}(a/b)$. Thus $f(x)$ is the topological supremum of $f(\downarrow x \bigcap f^{-1}(a/b))$. Since $f(\downarrow x \bigcap f^{-1}(a/b)$ which shows that $f^{-1}(a/b)$ is closed.

*Subspaces.* The Zariski topology works very well for subspaces. As a point of comparison let us review a bit of the subspace situation for the interval topology. As an example we will consider the unit interval I. Both $I$ and $I \times I$ are completely distributive, so their usual topology is the interval topology. The sublattice $L = \{x \in I : x = 1 \text{ or } x < \frac{1}{2}\}$ of $I$ is compact and connected in its own interval topology, but it has neither of these properties relative to the topology it inherits from $I$. The sublattice $M = \{(x, y) \in I \times I : x \wedge y \neq 0 \text{ and } x \vee y \neq 1\}$ of $I \times I$ fails even more poorly. As an ordered set, $M$ is isomorphic to $R \times R$. Thus as we already know, $M$ is not a Hausdorff space in its own interval topology. As a subspace of the Hausdorff space $I \times I$, it is Hausdorff in the topology it inherits from $I \times I$. Hence, for subspaces the interval topology can be coarser or finer than the restriction of the interval defined on the superspace. This is not the case for the Zariski topology.

LEMMA 5.1. *If $L$ is a sublattice of the distributive lattice $M$, then the Zariski topology defined on $L$ is contained in the restriction of the Zariski topology defined on $M$ to $L$.*

PROOF. One need only note that for elements $a, b \in L$ with $a \leq b, \{s \in L : x \vee a \geq b\} = \{x \in M : x \vee a \geq b\} \bigcap L$.

For open sublattices of completely distributive lattices, we are able to show the reverse containment to that established in 5.1.

PROPOSITION 5.2. *Let $M$ be a connected completely distributive lattice with the interval topology and let $L$ be an open sublattice of $M$. Then the topology that $L$ inherits as a subspace of $M$ is equivalent to the Zariski topology defined on $L$.*

PROOF. From Lemma 5.1 we see that the Zariski topology on $L$ is finer than the topology it inherits from the Zariski topology on $M$. We need to verify that for every point $a \in M$, the set $\uparrow a \bigcap L$ and $\downarrow a \bigcap L$ are closed in the Zariski topology of $L$. Suppose that $a \in M$ is given. We need only check that $\downarrow a \bigcap L$ is closed; the other case then follows from duality. Let $x \in L / \downarrow a$. Then $a \wedge x \neq x$ and the interval $[a \wedge x, x] \subseteq M$ is a nondegenerate connected subset of $M$ (it is the continuous image of the connected space $M$ under the map $y \mapsto (x \wedge y) \vee (a \wedge x) : M \to [a \wedge x, x]$). Since $x \in L$ and $L$ is an open subset of $M$, $L$ must intersect the interval $[a \wedge x, x]$ in more than the single point $x$; otherwise $x$ would be isolated in $[a \wedge x, x]$.

Let $y \in L \bigcap [a \wedge x, x]$, $y \neq x$. Then for every $b \in M$, $b \leq a$ implies $b \wedge x \leq a \wedge x \leq y$, i.e., $\downarrow a \subseteq \{b \in M : x \wedge b \leq y\}$ and therefore $\downarrow a \bigcap L \subseteq \{b \in L : x \wedge b \leq y\} = y/x$. Since $y < x$ we conclude that $x \notin y/x$ and hence $\downarrow a \bigcap L$ is an intersection of Zariski closed sets.

It remains an open problem whether (5.2) is still true for arbitrary lattices $M$ which are connected in their Zariski topology. The next example shows that the connectedness of $M$ in Theorem 5.2 is not superfluous.

EXAMPLE 5.3. Let $M = \{(x, y) \in I \times I : (x, y) \leq (\frac{1}{3}, \frac{1}{3})$ or $(x, y) \geq (\frac{2}{3}, \frac{2}{3})\}$ Then $L = \{(x, y) \in I \times I : (x, y) \leq (\frac{1}{3}, \frac{1}{3})$ or $x > \frac{2}{3}, y > \frac{2}{3}\}$ is an open sublattice of $M$. However, the Zariski topology on $L$ is strictly coarser than the topology from the interval topology on $M$ (which, by the way, is the topology induced from the usual topology on the square).

In order to verify this statement, note that $\{(x, y) \in I \times I : x > \frac{2}{3}, y > \frac{2}{3}\}$ is relatively closed in $L$. It is not too complicated to see that every Zariski closed subset of $L$ containing $\{(x, y) \in I \times I : x > \frac{2}{3}, y > \frac{2}{3}\}$ also contains the point $(\frac{1}{3}, \frac{1}{3}) \in L$. Hence the Zariski topology is strictly coarser than the relative topology

Lemma 5.1, when combined with Theorem 4.5, immediately yields

PROPOSITION 5.4. *Let $i : L \to M$ be an imbedding of distributive lattices. Then the following conditions are equivalent:*

(a) *$i$ is continuous when both $L$ and $M$ are equipped with the Zariski topology.*

(b) *$i$ is a topology imbedding when both $L$ and $M$ are equipped with the Zariski topology.*

(c) *$i$ preserves topological suprema and topological infima.*

Essential extensions always respect topological suprema and topological infima. Indeed if $M$ is an essential extension of $L$, then (up to isomorphism) $M$ is a sublattice of $\Theta^*(L)$, the maximal essential extension of $L$. From 4.3 we know that the imbedding $L \to \Theta^*(L)$ preserves topological suprema and topological infima. It is now straightforward to check that these suprema and infima are also preserved when mapping $L$ into the smaller space $M$. Hence from either the preceding remarks or from (3.5) directly we obtain

COROLLARY 5.5. Let $i : L \to M$ be an essential imbedding of distributive lattices. Then it also is a topological imbedding when $L$ and $M$ are both given the Zariski topology.

With 5.5 we have established a firm bond between the Zariski topology and essential extensions.

EXAMPLE 5.6. Let $L = \{(x, y) \in I \times I : (x, y) \leq (\frac{1}{2}, \frac{1}{2}) \text{ or } \frac{1}{2} < x \wedge y\}$. Then it is easy to see that $I \times I$ is an essential extension of $L$. Hence the Zariski topology on $L$ is the same as the topology that $L$ inherits as a subspace of $I \times I$.

The connection between the Zariski topology and essential extension is so strong that it seems to be plausible (at least to the authors) that dense imbeddings give rise to essential extensions. However, the following example refutes this conjecture.

EXAMPLE 5.7. Let $C$ denote the first uncountable cardinal number. Then $[0, 1]^c$ is separable and hence contains a dense countable sublattice $L$. But the inclusion map $i : L \hookrightarrow [0, 1]^c$ cannot be an essential extension since otherwise $[0, 1]^c$ would be metrizable by [13].

One naturally expects the Zariski topology to be invariant under the formation of closed sublattices. This is not the case.

EXAMPLE 5.8. Let $M := \{(r, s) \in I \times I : r, s < \frac{1}{2}\}$. Then $L := \{(r, s) \in M : r - s = 0 \text{ or } \frac{1}{2} < r, s\}$ is a closed sublattice of $L$. In the relative topology on $M$ the point $(0, 0)$ is isolated. However, $\{(r, s) \in L : \frac{1}{2} < r, s\}$ is a filter in $L$ which has $(0, 0)$ as topological infimum.

**6. Remarks on quotient maps.** In this section we record some observations on topological quotient maps between distributive lattices with their Zariski topologies. If the lattices in question are infinitely distributive (and hence carry the interval topology), the situation is completely understood.

PROPOSITION 6.1. *Let $L$ and $M$ be infinite distributive and let $f : L \to M$ be a surjective lattice homomorphism. Then the following are equivalent:*

(i) *$f$ preserves arbitrary infima and arbitrary suprema.*

(ii) $f$ *is continuous for the Zariski topologies.*
(iii) $f$ *is continuous and closed for the Zariski topologies.*
(iv) $f$ *is a topological quotient map for the Zariski topologies.*

PROOF. With the exception of the implication of (i) $\Rightarrow$ (iii), all those statements are straightforward. To verify that (i) implies (iii), assume that $A \subset L$ is closed and let $B = f(A)$. We have to show that $B$ is closed. Let $\mathcal{G}$ be an ultrafilter on $B$ and let $y$ be a limit point of $\mathcal{G}$. We have to show that $y \in B$. Pick an ultrafilter $\mathcal{F}$ on $A$ such that $f(\mathcal{F}) = g$. Since $A$ is closed, all limit points of $\mathcal{F}$ belong to $A$. It follows that $[\liminf \mathcal{F}, \limsup \mathcal{F}] \subset A$. Since $f$ preserves arbitrary suprema and arbitrary infima, and since $f$ is a surjective lattice homomorphism and hence preserves intervals, we conclude that $f[\liminf \mathcal{F}, \limsup \mathcal{F}] = [f \liminf \mathcal{F}, \liminf \mathcal{F}[\limsup f\mathcal{F}] = [\liminf \mathcal{G}, \limsup \mathcal{G}]$. Now since $y$ was a limit point of $\mathcal{G}$, we conclude that $y \in [\liminf \mathcal{G}, \liminf \mathcal{G}] \subset f(A) = B$.

If the range and domain of a lattice homomorphism $f : L \to M$ are no longer infinitely distributive, the situation is not so clear. In the remainder of this section, we will give an example of meet–continuous complete distributive lattice $L$ and two points $a, b, \in L$ such that the canonical quotient map $f : L \to L/\theta(a,b)$ is continuous but not a quotient map for the Zariski topologies.

EXAMPLE 6.2. Let $[0,1]$ be the real unit interval with its usual order and topology. We consider the following sublattice $L \subset [0,1]^2$:

$$L = \{(x,y) \in [0,1]^2 : y < 1 \text{ or } x = y = 1\}.$$

Let $a = (\frac{1}{2}, 0)$and $b = (1,0)$. Then $L/\theta(a,b)$ is isomorphic to $\{(x,y) \in [0,1]^2 : (x \leq \frac{1}{2} \text{ and } y < 1) \text{ or } (x = \frac{1}{2} \text{ and } y = 1)\}$. Under this identification, the canonical map $f : L \to L/\theta(a,b)$ is the restriction of the map$(x,y) \to (\min\{s, \frac{1}{2}\}, y) : [0,1]^2 \to L$. Moreover, $[0,1]^2$ is an essential extension of $L$ and $[0, \frac{1}{2}] \times [0,1]$ is an essential extension of $L/\theta(a,b)$. Hence the Zariski topologies on $L$ and $L/\theta(a,b)$ are the natural topologies inherited from the Euclidean plane. This makes $f$ a continuous map. However, $f$ is not a quotient map: Consider the set $B = \{(x - \frac{1}{2}, x) \in [0,1] \times [0,1] : \frac{1}{2} \leq x < 1\} \subset L/\theta(a,b)$. Then $f^{-1}(B) = B$ is closed in $L$, but $B$ is not closed in $L/\theta(a,b)$, since $(\frac{1}{2}, 1)$ belongs to the closure of $B$ in $L/\theta(a,b)$.

**7. Hausdorffness conditions.** In this section we will try to answer the question: Under which conditions is a lattice $L$ Hausdorff in its Zariski topology. A first solution of this problem was given by R. Ball in [**3**]. We shall restate his result here and give a different, more direct proof. Let us start with some preliminary results.

PROPOSITION 7.1. *Let $M$ be a distributive complete lattice which is meet–and–join continuous, let $L$ be a sublattice of $M$ and let $A$ be a closed subset of $M$ in the interval topology. If $L \subseteq A$, then $A$ also contains the smallest complete sublattice generated by $L$.*

PROOF. Let us introduce some new notations which we also will use later on.

If $B \subseteq M$, then

$$B^+ = \{\sup C : C \subseteq B, C \text{ up–directed}\}.$$
$$B^- = \{\inf C : C \subseteq B, C \text{ down–directed}\}.$$

Since directed suprema and infima are limits in the intercal topology, we obtain $L^+ \subseteq A, L^- \subseteq A$, where $L$ and $A$ are as specified in the proposition. Moreover, since $M$ is meet–continuous and join–continuous, $L^+$ and $L^-$ are sublattices. We obtain $L^+ \subseteq A, L^{+-+} \subseteq A$, etc. Since the smallest complete sublattice $L'$ generated by $L$ is obtained by applying the operators $+$ and $-$ recursively, an easy transfinite induction shows $L \subseteq A$.

PROPOSITION 7.2. *Let $L$ be a distributive complete lattice which is meet–and-join–continuous. If any two comparable elements $x, y \in L$ can be separated in the interval topology, then $L$ is Hausdorff in the interval topology.*

PROOF. We have to show that every ultrafilter $\hat{u}$ on $L$ has a unique limit point. Since limits are given by lim inf's and lim sup's, it suffices to show that $\liminf \hat{u} = \limsup \hat{u}$ for every ultrafilter $\hat{u}$. Assume not. Then there is an ultrafilter $\hat{u}$ such that lim inf $\hat{u} \not\leq$ lim sup $\hat{u}$. Choose disjoint open neighborhoods $U$ of lim inf $\hat{u}$ and $V$ lim sup $\hat{u}$ and lim sup $\hat{u}$ are limit points of $\hat{u}$, we conclude that $U, V \in \hat{u}$, contradicting $U \bigcap V = 0$.

If $L$ is a distributive lattice, then $L^* \subseteq \Theta^*(L)$ denotes the smallest complete sublattice of $\Theta^*(L)$ containing $\{\theta_a : a \in L\}$. It is clear that $L^*$ is an essential extension of $L$.

THEOREM 7.3 (BALL, 1983). *Let $L$ be a distributive lattice. Then the following conditions are equivalent:*
(i) *$L$ is Hausdorff in the Zariski topology.*
(ii) *$L^*$ is completely distributive.*
(iii) *$L$ admits an essential extension which is completely distributive.*

PROOF. (i) $\Rightarrow$ (ii). Let $\theta, \psi \in L^*, \theta \not\leq \psi$. Using (1.1), we have to find closed subsets $A, B \in L^*$ such that
(i) $A \bigcup B = L^*$,
(ii) $\theta \notin A, \psi \notin B$; in this case $L^* \backslash A$ and $L^* \backslash B$ are disjoint open neighborhoods of $\theta$ and $\psi$ respectively.

First of all, $L^*$ is an essential extension of $\{\theta_a : a \subset L\}$. Hence there are elements $a, b \in L$, $a \not\leq b$, such that $\theta_a$ and $\theta_b$ are identified by the smallest congruence collapsing $\theta$ and $\psi$. As we already noticed in the proof of 5.7, this means

$$\theta_a = (\theta_a \vee \theta) \wedge \theta_b, \quad \theta_b = (\theta_a \vee \psi) \wedge \theta_b.$$

Since that Zariski topology on $\{\theta_a : a \in L\}$ is Hausdorff and since the Zariski topology is the relative topology of the interval topology on $\Theta^*(L)$ by 3.5, we can find open neighborhoods $U \subseteq \Theta^*(L)$ of $\theta_a$ and $V \subseteq \Theta^*(L)$ of $\theta_b$ such that $U \bigcap V \bigcap \{\theta_a : a \in L\} = 0$. We conclude that $\{\theta_a : a \in L\} \subseteq (\Theta^*(L) \backslash U) \bigcup (\Theta^*(L) \backslash V)$. Since the latter set is closed in the interval topology, we obtain $L^* \subseteq (\Theta^*(L) \backslash U) \bigcup (\Theta^*(L) \backslash V)$ by (4.1). Let

$$A' := L^* \bigcap (\Theta^*(L) \backslash U),$$
$$B' := L^* \bigcap (\Theta^*(L) \backslash V).$$

Then, $\Theta^*(L)$ also being an essential extension of the larger sublattice $L^*$, the sets $A'$ and $B'$ are closed in the Zariski topology of $L^*$ by (5.5). Note that $L^*$ as a complete sublattice of $\Theta(L)$ is infinitely distributive, hence the Zariski topology and the interval topology agree on $L^*$. Now let

$$f : L^* \to L^*; \quad \Gamma \to (\theta_a \vee \Gamma) \wedge \theta_b.$$

Then $f$ is continuous by 3.6, $f(\theta) = \theta_a \in B' \backslash A'$ and $f(\psi) \in A' \backslash B'$. Let

$$A := f^{-1}(A');$$
$$B := f^{-1}(B').$$

Then $A$ and $B$ are closed subsets having the required properties.

(ii) $\Rightarrow$ (iii) is trivial, since $L^*$ is an essential extension of $L$. (iii) $\Rightarrow$ (i) follows from (1.1) and (5.5).

COROLLARY 7.4. *Let $L$ be a distributive lattice and assume that $L$ is Hausdorff in the Zariski topology. Then $L$ is a completely regular topological lattice in the Zariski topology.*

PROOF. This follows immediately from the fact that $L$ is algebraically and topologically a (dense) sublattice of $L^*$.

Thus, we can say that lattices which are Hausdorff in the Zariski topology are those topological lattices $L$ which allow a compactification $L^*$. It is not hard to see that $L^* = L^{+-} = L^{-+}$. If we start with $L$ being the open square, then $L^*$ is the closed square. It seems to be not without interest to study the categorical properties of the imbedding $L \to L^*$; we shall return to this question in a later paper.
The characterizations given in (7.3) are external in the sense that they use extensions of $L$. It would also be desirable to have an internal characterization.

DEFINITION 7.5. A distributive lattice $L$ is called *reductive* if for every pair of comparable elements $x, y \in L$, say $x \not\leq y$, there are elements $a, b \in L$ such that $x \leq a \not\leq b \leq y$ and such that the interval $[a, b]$ is a chain.

Reductive lattices were introduced and studied in [14]. There are several other algebraic properties leading to reductivity – one is the property that every element be the infimum of finitely many primes; a second is that the lattice does not contain a copy of $2^\omega$ as a complete sublattice.

PROPOSITION 7.6. *A distributive lattice $L$ is reductive if and only if $L$ admits an essential extension which is a product of (complete) chains*

PROOF. This was shown in [**14**] without the adjective "complete." Hence (7.6) would follow from the following two observations:

(i) The Dedekind–MacNeille completion of a chain is an essential extension.

(Indeed, let $C$ be a chain and let $C'$ be its Dedekind–MacNeille complete. If $x, y \in C'$, $x < y$, then there are elements $a, b \in C$ such that $x \leq a < b \leq y$. Hence every congruence relation on $C'$ which is not the smallest congruence $\Delta$ has a non-trivial restriction to $C$.)

(ii) If $L_i \subset M_i$ are essential extensions for every $i \in I$, where $I$ is an arbitrary index set, then $\Pi_{i\in I} M_i$ is an essential extension of $\Pi_{i\in I} L_i$.

(Let $\theta$ be a non-trivial congruence relation on $\prod_{i\in I} M_i$ and take elements $f, g \in \prod_{i\in I} M_i$ such that $f < g$ and $(f, g) \in \theta$. Pick an index $i \in I$ such that $f_i < g_i$. Since $M_i$ is an essential extension of $L_i$ there are elements $x, y \in L_i$ such that $x < y$ and such that $(x, y)$ belongs to the smallest congruence relation $\theta(f_i, g_i)$ on $M_i$ identifying $f_i$ and $g_i$. From (3.2) we conclude that

$$x = (f_i \vee x) \wedge y,$$
$$y = (g_i \vee x) \wedge y.$$

For every $j \in I\backslash\{i\}$ let $a_j \in L_j$ be an arbitrary element. Define elements $s, r \in \prod_{i\in I} L_i$ by

$$s_j = r_j = a_j \text{ for } j \neq i,$$
$$s_i = x,$$
$$r_i = y.$$

It follows that $s = (f \vee s) \wedge r$ and $r = (g \vee s) \wedge r$. By (3.2), $(s, r)$ belongs to the smallest congruence containing $(f, g)$. Especially, $(r, s) \in \theta$.)

Since a product of complete chains is completely distributive, we obtain

COROLLARY 7.7. *If $L$ is reductive, then $L$ is Hausdorff in the Zariski topology.*

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