ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 3, Summer 1984

IRREDUCIBLE REPRESENTATIONS OF INSEPARABLE C*-ALGEBRAS

C.J.K. BATTY

ABSTRACT. Irreducible representations of a C^*-algebra are shown to induce irreducible representations on certain arbitrarily large separable subalgebras. Many structural properties of C^*-algebras can also be reduced to separable subalgebras.

1. Introduction. There are several problems concerning C^*-algebras which were solved some time ago assuming separability conditions, but which are outstanding in inseparable cases. For example, Dixmier **[3]** showed in 1960 that prime ideals of separable C^*-algebras are primitive, but it is unknown whether this is valid in inseparable cases. Meanwhile, Glimm **[6]** showed that the following conditions on a separable C^*-algebra A are equivalent:

(i) A is postliminal,
(ii) A is smooth,
(iii) A has no (factorial) representations of type III,
(iv) A has no (factorial) representations of type II,
(v) Irreducible representations of A with equal kernels are unitarily equivalent.

Subsequently, Sakai **[7, 8]** established the equivalence of conditions (i), (ii) and (iii) even for inseparable C^*-algebras, but it remains unknown whether they are implied by (iv) or by (v) (see **[2, 12]** for some partial results about (iv)).

It was already implicit in Glimm's argument that any antiliminal C^*-algebra A contains a separable C^*-subalgebra B which is not postiliminal. Passing to an antiliminal quotient, the same result follows whenever A is not postliminal. This paper introduces a technique for reducing some properties of this type to the corresponding properties of the separable subalgebras. For example, it will be shown that (prime) antiliminal C^*-algebras contain arbitrarily large separable (prime) antiliminal subalgebras.

Throughout the paper, suffixes i, j, m, n, r, etc. take on (non-negative) integer values.

Received by the editors on July 8, 1983, and in revised form on October 16, 1983.

2. Irreducible representations. Firstly we consider separable reductions of irreducible and factorial representations. The following result contains the essential features of the reduction.

THEOREM 1. *Let A be a C^*-algebra, B_0 be a separable C^*-subalgebra of A, and, for $r \geqq 1$, let π_r be an irreducible (respectively, factorial) representation of A on a Hilbert space $\mathscr{H}_r$ and $\mathscr{K}_r$ be a separable subspace of $\mathscr{H}_r$. There exists a separable C^*-subalgebra B of A containing B_0 such that, for each r, π_r induces an irreducible (respectively, factorial) representation of B on $[\pi_r(B)\mathscr{K}_r]$.*

PROOF. Suppose first that each π_r is irreducible. We shall construct inductively an increasing sequence $\{B_n : n \geqq 0\}$ of separable C^*-subalgebras of A such that, for any $r \geqq 1$, $\varepsilon > 0$ and unit vectors η and η' in $\mathscr{K}_{rn} = [\pi_r(B_n)\mathscr{K}_r]$, there exists b in B_{n+1} with $\|b\| \leqq 1$ and $\|\pi_r(b)\eta - \eta'\| < \varepsilon$. The algebra B_0 is given in the statement of the theorem.

Suppose that B_n has been constructed. Let $\{\xi_{jrn} : j \geqq 1\}$ be a dense sequence in the unit sphere of $\mathscr{K}_{rn}$. For each pair (i, j), there exists a_{ijrn} in A such that $\pi_r(a_{ijrn})\xi_{jrn} = \xi_{irn}$ and $\|a_{ijrn}\| = 1$. Let B_{n+1} be the C^*-subalgebra of B generated by B_n and $\{a_{ijrn} : i, j, r \geqq 1\}$. Then B_{n+1} has the properties required in the inductive construction.

Having constructed the algebras B_n, let $B = (\bigcup_{n\geqq 0} B_n)^-$. Then B has the properties required in the theorem.

The proof of the factorial case is quite similar to the irreducible case above. If π_r is factorial, the C^*-algebra $\tilde{A}_r$ generated by $\pi_r(A)$ and $\pi_r(A)'$ is irreducible on $\mathscr{H}_r$. One constructs inductively increasing sequences $\{B_n : n \geqq 0\}$ of separable C^*-subalgebras of A and $\{C_{rn} : n \geqq 0\}$ of separable unital C^*-subalgebras of $\pi_r(A)'$ with the property that (for each r) if $\tilde{A}_{rn}$ is the C^*-subalgebra of $\tilde{A}_r$ generated by $\pi_r(B_n)$ and C_{rn}, and $\mathscr{K}_{rn} = [\tilde{A}_{rn}\mathscr{K}_r]$, then, for any $\varepsilon > 0$ and unit vectors η and η' in $\mathscr{K}_{rn}$, there exists y in $\tilde{A}_{rn}$ with $\|y\| \leqq 1$ and $\|y\eta - \eta'\| < \varepsilon$. Then one takes $B = (\bigcup_{n\geqq 0} B_n)^-$, $C_r = (\bigcup_{n\geqq 0} C_{rn})^-$ and $\tilde{B}_r$ to be the C^*-subalgebra of $\tilde{A}_r$ generated by $\pi_r(B)$ and C_r. The restriction to $[\tilde{B}_r\mathscr{K}_r]$ of the identity representation of $\tilde{B}_r$ on $\mathscr{H}_r$ is irreducible. Since $\pi_r(B)$ and C_r commute, it follows easily that π_r induces a factorial representation of B on $[\pi_r(B)\mathscr{K}_r]$ (see [**9**, Lemma IV.6.31]).

COROLLARY 2. *Let A be a C^*-algebra, B_0 be a separable C^*-subalgebra of A and $\mathscr{P}$ be a countable set of pure states of A. There is a separable C^*-subalgebra B of A containing B_0 such that $\phi|_B$ is a pure state of B for each ϕ in $\mathscr{P}$.*

In general, the algebra B constructed in Theorem 1 or Corollary 2 depends not only on B_0 but also on the representations π_r or on the collection $\mathscr{P}$ of pure states. Teleman [**10**, Theorem 1] has proved a more sophisticated version of Corollary 2 for maximal measures in Choquet theory. Corollary 2 is merely Teleman's result for atomic measures on the pure states of A.

3. Structural properties. Using the basic technique of Theorem 1, it is now possible to reduce many structural properties of C^*-algebras to the separable subalgebras.

PROPOSITION 3. *Let A be a C^*-algebra, and suppose that every separable C^*-subalgebra of A is liminal. Then A is liminal.*

PROOF. If A is not liminal, there is an irreducible representation π on A on $\mathscr{H}$ and a self-adjoint element a of A such that $\pi(a)$ is not compact. By spectral theory, there is a separable infinite-dimensional subspace $\mathscr{K}$ of $\mathscr{H}$ and a real number $\delta > 0$ such that

$$\|\pi(a)\xi\| \geqq \delta\|\xi\| \qquad (\xi \in \mathscr{K}).$$

By Theorem 1, there is a separable C^*-subalgebra B of A containing a such that π induces an irreducible representation ρ of B on $[\pi(B)\mathscr{K}]$. Since $\|\rho(a)\xi\| \geqq \delta\|\xi\|$ $(\xi \in \mathscr{K})$, $\rho(a)$ is not compact. Thus B is not liminal.

Glimm's arguments **[6]** established that an antiliminal C^*-algebra contains a separable C^*-subalgebra which is not postliminal. The next result shows that the subalgebra can be arranged to be antiliminal and arbitrarily large (subject to being separable.)

PROPOSITION 4. *Let B_0 be a separable C^*-subalgebra of an antiliminal C^*-algebra A. There is a separable, antiliminal C^*-subalgebra of A containing B_0.*

PROOF. For a representation π of a C^*-algebra on a Hilbert space $\mathscr{H}$, let $\dot{\pi}$ denote the composition of π with the quotient map into the Calkin algebra over $\mathscr{H}$. Since A is antiliminal, the direct sum of the representations $\dot{\pi}$ over all irreducible representations π of A is faithful, and hence isometric, on A.

We shall construct inductively an increasing sequence $\{B_n: n \geqq 0\}$ of separable C^*-subalgebras of A and irreducible representations π_{rm} of A on $\mathscr{H}_{rm}$ and separable subspaces $\mathscr{K}_{rm}$ of $\mathscr{H}_{rm}$ $(r, m \geqq 1)$ such that for $r \geqq 1$ and $1 \leqq m \leqq n$, π_{rm} induces an irreducible representation ρ_{rmn} of B_n on $\mathscr{K}_{rmn} = [\pi_{rm}(B_n)\mathscr{K}_{rm}]$, and, for b in B_{n-1},

$$\sup_r \|\dot{\pi}_{rnn}(b)\| \geqq \frac{1}{2}\|b\|.$$

The algebra B_0 is given in the proposition.

Suppose that, for some $n \geqq 1$, B_m, π_{rm} and $\mathscr{K}_{rm}$ have been constructed for all $r \geqq 1$ and $1 \leqq m < n$. Let $\{b_{rn}: r \geqq 1\}$ be a countable dense subset of B_{n-1}. Let π_{rn} be an irreducible representation of A on $\mathscr{H}_{rn}$ such that $\|\dot{\pi}_{rn}(b_{rn})\| \geqq \frac{3}{4}\|b_{rn}\|$. By spectral theory, there is a separable infinite-dimensional subspace $\mathscr{K}_{rn}$ of $\mathscr{H}_{rn}$ such that

$$\|\pi_{rn}(b_{rn})\xi\| \geqq \frac{1}{2}\|b_{rn}\|\,\|\xi\| \qquad (\xi \in \mathscr{K}_{rn}).$$

By Proposition 1, there is a separable C^*-subalgebra B_n of A containing B_{n-1} such that, for $r \geqq 1$ and $1 \leqq m \leqq n$, π_{rm} induces an irreducible representation ρ_{rmn} of B_n on $\mathscr{K}_{rmn}$. Furthermore

$$\|\dot{\rho}_{rnn}(b_{rn})\| \geqq \frac{1}{2}\|b_{rn}\|.$$

It follows from the density of $\{b_{rn}: r \geqq 1\}$ in B_{n-1} that

$$\sup_r \|\dot{\rho}_{rnn}(b)\| \geqq \frac{1}{2}\|b\| \qquad (b \in B_{n-1}).$$

This completes the inductive construction.

Now let $B = (\bigcup_{n \geqq 0} B_n)^-$. Then π_{rm} induces an irreducible representation ρ_{rm} of B on $[\pi_{rm}(B)\mathscr{K}_{rm}]$. For each b in B_{m-1}, $\|\dot{\rho}_{rm}(b)\| \geqq \frac{1}{2}\|b\|$ for some r. Thus for each non-zero b in B, $\rho_{rm}(b)$ is not compact for some irreducible representation ρ_{rm} of B. Thus B is antiliminal.

Next we show that if the algebra A in Proposition 4 is prime as well as antiliminal, then the subalgebra can also be arranged to be prime.

PROPOSITION 5. *Let B_0 be a separable C^*-subalgebra of a prime, antiliminal C^*-algebra A. There is a separable, prime (hence primitive), antiliminal C^*-subalgebra of A containing B_0.*

PROOF. If A has no unit, the C^*-algebra obtained by adjoining one is prime and antiliminal. Thus we may assume that A has a unit. (In fact, the results from **[4, 5, 11]** which will be quoted are valid in non-unital C^*-algebras, so this observation is unnecessary.) Increasing B_0 if necessary, we may assume that B_0 contains the unit of A and is at least two-dimensional. Let $S(A)$ be the set of all states of A, and $P(A)$ be the set of all pure states of A. It was shown by Glimm **[5]** (see **[4,** 11.2.4]) that $P(A)$ is weak* dense in $S(A)$.

We shall construct inductively increasing sequences $\{B_n: n \geqq 0\}$ of separable C^*-subalgebras of A and $\{\mathscr{P}_n: n \geqq 1\}$ of countable subsets of $P(A)$ such that $\{\phi|_{B_{n-1}}: \phi \in \mathscr{P}_n\}$ is weak* dense in $S(B_{n-1})$ and, for each ϕ in $\mathscr{P}_n$, $\phi|_{B_n} \in P(B_n)$. The algebra B_0 is given in the proposition.

Suppose that, for some $n \geqq 0$, B_n and $\mathscr{P}_n$ have been constructed (take $\mathscr{P}_0$ to be empty). Let $\{\psi_{rn}: r \geqq 1\}$ be a countable weak* dense set in the weak* separable space $S(B_n)$, and let $\{b_{in}: i \geqq 1\}$ be a countable dense set in B_n. For $k, m, r \geqq 1$, there exists ϕ_{krmn} in $P(A)$ with

$$|\phi_{krmn}(b_{in}) - \psi_{rn}(b_{in})| < \frac{1}{m} \qquad (1 \leqq i \leqq k).$$

Let

$$\mathscr{P}_{n+1} = \mathscr{P}_n \cup \{\phi_{krmn}: k, r, m \geqq 1\}.$$

By Corollary 2, there is a separable C^*-subalgebra B_{n+1} of A containing B_n such that $\phi|_{B_{n+1}} \in P(B_{n+1})$ for each ϕ in $\mathscr{P}_{n+1}$. This completes the inductive construction.

Now let $B = (\bigcup_{n\geqq 0} B_n)^-$ and $\mathscr{P} = \bigcup_{n\geqq 1} \mathscr{P}_n$. Then B is a separable C^*-subalgebra of A containing B_0, $\{\phi|_B: \phi \in \mathscr{P}\}$ is weak* dense in $S(B)$, and, for each ϕ in $\mathscr{P}$, $\phi|_B \in P(B)$. Thus $P(B)$ is weak* dense in $S(B)$. It follows that B is prime and antiliminal [**11**, Theorem 2].

Next we prove the corresponding version of Propositions 4 and 5 for prime C^*-algebras.

PROPOSITION 6. *Let B_0 be a separable C^*-subalgebra of a prime C^*-algebra A. There is a separable, prime (hence primitive) C^*-subalgebra of A containing B_0.*

PROOF. Let f: $[0, \infty) \to \mathbf{R}$ be any fixed continuous function which vanishes near 0, with $f(1) \neq 0$. Note that if B is a C^*-subalgebra of A, I is a (closed two-sided) ideal of B, and a_r is a sequence of positive elements of B converging to a limit a in I, then $\pi_1(f(a_r)) = f(\pi_1(a_r)) = 0$ for large r, where $\pi_1: A \to A/I$ is the quotient map. Thus $f(a_r) \in I$ for large r.

We shall construct inductively an increasing sequence $\{B_n: n \geqq 0\}$ of separable C^*-subalgebras of A such that, for any non-zero ideals I and J of B_n, $IB_{n+1}J \neq (0)$. The algebra B_0 is given in the proposition.

Suppose that B_n has been constructed, and let $\{b_{jn}: j \geqq 1\}$ be a countable dense sequence in the positive part of the unit sphere of B_n. Since A is prime and $f(b_{in})$ and $f(b_{jn})$ are non-zero, for each pair (i, j) there exists a_{ijn} in A such that $f(b_{in})a_{ijn}f(b_{jn}) \neq 0$. Let B_{n+1} be the C^*-subalgebra of A generated by B_n and $\{a_{ijn}: i, j \geqq 1\}$.

Suppose I and J are non-zero ideals of B_n. It follows from the first paragraph of the proof that $f(b_{in}) \in I$ and $f(b_{jn}) \in J$ for some (i, j). Now, $0 \neq f(b_{in})a_{ijn}f(b_{jn}) \in IB_{n+1}J$. This shows that B_{n+1} has the required properties.

Now let $B = (\bigcup_{n\geqq 0} B_n)^-$. Let I and J be non-zero ideals of B, $I_n = I \cap B_n$ and $J_n = J \cap B_n$. It follows from the first paragraph of the proof that I_n and J_n are non-zero for large n. By construction, $(0) \neq I_nB_{n+1}J_n \subset IBJ$. Thus B is prime.

Proposition 6 may alternatively be proved by a method analogous to the proof of Proposition 5 given above, but replacing the results of [**5**] and [11] by Archbold's result [2] that A is prime if and only if the factorial states are weak* dense in $S(A)$, and using the factorial version of Theorem 1.

Proposition 5 can be deduced from Propositions 4 and 6 by constructing separable, prime C^*-subalgebras B_n and separable, antiliminal C^*-subalgebras B_n' with $B_n \subset B_n' \subset B_{n+1}$, and then taking $B = (\bigcup_{n \geqq 0} B_n)^-$.

Finally, we give the corresponding version of the above results for simple C^*-algebras.

PROPOSITION 7. Let B_0 be a separable C^*-subalgebra of a simple C^*-algebra A. There is a separable, simple C^*-subalgebra B of A containing B_0.

PROOF. We shall construct inductively an increasing sequence $\{B_n: n \geqq 0\}$ of separable C^*-subalgebras of A such that, for each non-zero (closed two-sided) ideal I of B_n, the ideal of B_{n+1} generated by I contains B_n. The algebra B_0 is given in the proposition. Let f be as in the proof of Proposition 6.

Suppose B_n has been constructed, and let $\{b_{jn}: j \geqq 1\}$ be a countable dense set in the positive part of the unit sphere of B_n. Since A is simple and $f(b_{in})$ is non-zero, for each $i, j, m \geqq 1$, there exist x_{ijmnr} and y_{ijmnr} in A $(1 \leqq r \leqq k_{ijmn})$ with

$$\left\| \sum_{r=1}^{k_{ijmn}} x_{ijmnr} f(b_{in}) y_{ijmnr} - b_{jn} \right\| < \frac{1}{m}.$$

Let B_{n+1} be the C^*-subalgebra of A generated by B_n and $\{x_{ijmnr}, y_{ijmnr}: 1 \leqq r \leqq k_{ijmn}; i, j, m \geqq 1\}$. If I is a non-zero ideal of B_n, then $f(b_{in}) \in I$ for some i, so b_{jn} lies in the ideal of B_{n+1} generated by I. Thus B_{n+1} has the required properties.

Now let $B = (\bigcup_{n \geqq 0} B_n)^-$. Let I be a non-zero ideal of B, and $I_n = I \cap B_n$. For large n, I_n is non-zero, so B_n is contained in the ideal of B generated by I_n. Thus $I = B$, so B is simple.

ADDED NOTE. S. Wright has kindly pointed out that Propositions 6 and 7 above already appeared and been applied in work of B. Blackadar, and of G.A. Elliott and L. Zsido, respectively. Professor Wright also informs the author that Proposition 4 has been obtained independently by A.J. Lazar (unpublished).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF EDINBURGH, KING'S BUILDINGS, EDINBURGH EH9 3JZ, SCOTLAND