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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF AN nTH ORDER DIFFERENTIAL EQUATION

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In this paper we give conditions which imply that the equation

$$u^{(n)} + f(t, u) = 0 \tag{1}$$

has a solution which behaves in a precisely specified way like a given polynomial of degree $< n$ as $t \to \infty$. We do not make the often imposed assumptions that f is continuous on $(0, \infty) \times (-\infty, \infty)$, or that it is majorized by a function which is nondecreasing in $|u|$. Moreover, our integral smallness conditions on f permit some of the improper integrals in question to converge conditionally.

Throughout the paper we write $f(t) = O(\psi(t))$ to indicate that $\overline{\lim}_{t\to\infty}|f(t)/\psi(t)| < \infty$, and $f(t) = o(\psi(t))$ to indicate that $\lim_{t\to\infty} f(t)/\psi(t) = 0$.

The following is our main theorem.

THEOREM 1. *Suppose k is an integer in $\{0, 1, \ldots, n-1\}$ and ϕ is positive, continuous, and nonincreasing on $[\bar{T}, \infty)$ for some $\bar{T} \geqq 0$; moreover, if $k \neq 0$, suppose there is a number γ such that*

$$\gamma < 1 \text{ and } t^{\gamma}\phi(t) \text{ is nondecreasing on } [\bar{T}, \infty). \tag{2}$$

Let p be a given polynomial of degree $<n$, and suppose there are constants $M > 0$ and $T_0 \geqq \bar{T}$ such that f is continuous on the set

$$\Omega = \{(t, u) \mid t \geqq T_0, |u - p(t)| \leqq M\phi(t)t^k\}, \tag{3}$$

and

$$|f(t, u_1) - f(t, u_2)| \leqq g(t)|u_1 - u_2| \text{ if } (t, u_i) \in \Omega, i = 1, 2, \tag{4}$$

where $g \in C[T_0, \infty)$,

$$\int^{\infty} s^{n-1} g(s)\, \phi(s)\, ds < \infty, \tag{5}$$

and

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$$(6)\qquad \overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}\int_t^\infty s^{n-1}g(s)\phi(s)\,ds=\rho_1<(n-k-1)!\prod_{j=1}^{k}(j-\gamma).$$

Suppose also that

$$(7)\qquad \int^\infty s^{n-k-1}f(s,p(s))\,ds$$

converges – perhaps conditionally – and that

$$(8)\qquad \overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}\left|\int_t^\infty s^{n-k-1}f(s,p(s))\,ds\right|=\rho_2<\infty,$$

where

$$(9)\qquad \rho_2+\rho_1 M<M(n-k-1)!\prod_{j=1}^{k}(j-\gamma).$$

Then (1) *has a solution* u_0 *which is defined for* t *sufficiently large and satisfies*

$$(10)\qquad u_0^{(r)}(t)=p^{(r)}(t)+O(\phi(t)t^{k-r}),\ 0\leqq r\leqq n-1;$$

moreover, if $\rho_1=\rho_2=0$, *then* (10) *can be replaced by*

$$(11)\qquad u_0^{(r)}(t)=p^{(r)}(t)+o(\phi(t)t^{k-r}),\ 0\leqq r\leqq n-1.$$

PROOF. For $t_0\geqq T_0$, let $H(t_0)$ be the Banach space of continuous functions h on $[t_0,\infty)$ such that $h(t)=O(\phi(t)t^k)$, with norm

$$(12)\qquad \|h\|=\sup_{t\geqq t_0}\{t^{-k}(\phi(t))^{-1}|h(t)|\}.$$

Let $H_M(t_0)=\{h\in H(t_0)|\ \|h\|\leqq M\}$. Define $\hat h=Th$ *by*

$$(13)\qquad \hat h(t)=\int_t^\infty\frac{(t-s)^{n-1}}{(n-1)!}f(s,p(s)+h(s))\,ds \text{ if } k=0,$$

or by

$$(14)\qquad \hat h(t)=\int_{t_0}^t\frac{(t-\lambda)^{k-1}}{(k-1)!}\,d\lambda\int_\lambda^\infty\frac{(\lambda-s)^{n-k-1}}{(n-k-1)!}f(s,p(s)+h(s))\,ds$$
$$\text{if } k=1,2,\ldots,n-1.$$

We will show that T is a contraction mapping of $H_M(t_0)$ into itself if t_0 is sufficiently large. It will then follow that there is a function h_0 in $H_M(t_0)$ such that $h_0=Th_0$, and we will show that the function $u_0=p+h_0$ has the stated properties.

We assume henceforth that $h\in H_M(t_0)$, with $t_0\geqq T_0$.

We must first show that the improper integral in (13) or (14) converges. To this end, we first show that the integral

$$(15)\qquad I(t;h)=\int_t^\infty s^{n-k-1}f(s,p(s)+h(s))\,ds$$

converges. By assumption, this is true for the integral

$$I(t; 0) = \int_t^\infty s^{n-k-1} f(s, p(s))\, ds. \tag{16}$$

If $h_1, h_2 \in H_M(t_0)$, then the integral

$$\int_t^\infty s^{n-k-1} [f(s, p(s) + h_1(s)) - f(s, p(s) + h_2(s))]\, ds$$

converges absolutely, by (5) and Weierstrass's test, since

$$\begin{aligned} s^{n-k-1} &|f(s, p(s) + h_1(s)) - f(s, p(s) + h_2(s))| \\ &\leqq s^{n-k-1} g(s) |h_1(s) - h_2(s)| && \text{(see (4))} \\ &\leqq \|h_1 - h_2\| s^{n-1} g(s)\phi(s) && \text{(see (12))} \end{aligned}$$

if $s \geqq t \geqq t_0$. From this, (16), and the convergence of (7), we can conclude that $I(t; h)$ converges, that

$$|I(t; h_1) - I(t; h_2)| \leqq \|h_1 - h_2\| \int_t^\infty s^{n-1} g(s)\phi(s)\, ds, \tag{17}$$

and that

$$\begin{aligned} |I(t; h)| &\leqq |I(t; 0)| + |I(t; h) - I(t; 0)| \\ &\leqq |I(t; 0)| + M \int_t^\infty s^{n-1} g(s)\phi(s)\, ds, \end{aligned} \tag{18}$$

where the last inequality follows from (17) with $h_1 = h$ and $h_2 = 0$, and the assumption that $\|h\| \leqq M$.

Since $I(t;h)$ converges, so do integrals of the form $\int_t^\infty s^j f(s, p(s) + h(s))\, ds$, $0 \leqq j \leqq n - k - 1$, by Dirichlet's test. Therefore, the improper integral in (13) or (14) converges. Thus, $\hat{h}$ is continuous, and, in fact, n times differentiable on $[t_0, \infty)$.

Now we estimate $\hat{h}^{(r)}(t)$, $0 \leqq r \leqq n - 1$. First we observe from (18) that

$$|I(t; h)| \leqq \sigma(t)\phi(t), \tag{19}$$

where

$$\sigma(t) = \sup_{\tau \geqq t} \left\{ (\phi(\tau))^{-1} \left[M \int_\tau^\infty s^{n-1} g(s)\phi(s)\, ds + \left| \int_\tau^\infty s^{n-k-1} f(s, p(s)\, ds \right| \right\}. \tag{20}$$

(Note that ϕ is well defined, because of (6) and (8).) We will first show that

$$|\hat{h}^{(r)}(t)| \leqq 2\sigma(t)\phi(t) t^{k-r}/(n-r-1)!, \quad k \leqq r \leqq n - 1. \tag{21}$$

Differentiating (13) or (14) yields

$$\hat{h}^{(r)}(t) = \int_t^\infty \frac{(t-s)^{n-r-1}}{(n-r-1)!} f(s, p(s) + h(s))\, ds, \quad k \leqq r \leqq n - 1, \tag{22}$$

which can be rewritten as

$$(23)\qquad \hat{h}^{(r)}(t) = -\frac{1}{(n-r-1)!}\int_t^\infty \left(\frac{t}{s}-1\right)^{n-r-1} s^{k-r} I'(s;\,h)\,ds.$$

(See (15).) If $k \leqq r < n-1$, integrating this by parts yields

$$(24)\quad \hat{h}^{(r)}(t) = \frac{1}{(n-r-1)!}\int_t^\infty I(s;h)\frac{d}{ds}\left[\left(\frac{t}{s}-1\right)^{n-r-1} s^{k-r}\right]ds,\ k \leqq r \leqq n-2.$$

But

$$(25)\qquad \left|\frac{d}{ds}\left[\left(\frac{t}{s}-1\right)^{n-r-1} s^{k-r}\right]\right| \leqq (r-k)s^{k-r-1} + t^{k-r}\frac{d}{ds}\left(1-\frac{t}{s}\right)^{n-r-1}$$

if $s \geqq t$ and $r > k$. This, (19), (24) and the monotonicity of $\sigma\phi$ imply (21) for $k \leqq r \leqq n-2$. Setting $r = n-1$ in (23) and integrating by parts yields

$$(26)\qquad \hat{h}^{(n-1)}(t) = t^{k-n+1} I(t;\,h) + (k-n+1)\int_t^\infty s^{k-n} I(s;\,h)\,ds,$$

and therefore (19) also implies (21) with $r = n-1$.

Notice that one term drops out of (25) if $r = k$, or out of (26) if $n-1 = k$. This means that if $r = k$, then (21) can be replaced by the sharper inequality

$$(27)\qquad |\hat{h}^{(k)}(t)| \leqq \sigma(t)\phi(t)/(n-k-1)!.$$

We will use this to obtain bounds on $\hat{h}^{(r)}(t)$, $0 \leqq r \leqq k-1$. Differentiating (14) and using (22) with $r = k$ yields

$$\hat{h}^{(r)}(t) = \int_{t_0}^t \frac{(t-\lambda)^{k-r-1}}{(k-r-1)!}\hat{h}^{(k)}(\lambda)\,d\lambda,\ 0 \leqq r \leqq k-1.$$

From this and (27),

$$(28)\qquad |\hat{h}^{(r)}(t)| \leqq \frac{1}{(k-r-1)!(n-k-1)!}\int_{t_0}^t (t-\lambda)^{k-r-1}\sigma(\lambda)\phi(\lambda)\,d\lambda.$$

Since σ is nonincreasing and $t^\gamma\phi(t)$ is nondecreasing, this means that

$$|\hat{h}^{(r)}(t)| \leqq \frac{\sigma(t_0)\phi(t)t^\gamma}{(k-r-1)!(n-k-1)!}\int_0^t (t-\lambda)^{k-r-1}\lambda^{-\gamma}\,d\lambda.$$

(Recall that $t_0 > 0$ and $\gamma < 1$.) Now repeated integration by parts yields

$$(29)\qquad |\hat{h}^{(r)}(t)| \leqq \frac{\sigma(t_0)\phi(t)t^{k-r}}{(n-k-1)!\prod_{j=1}^{k-r}(j-\gamma)},\ 0 \leqq r \leqq k-1.$$

In particular, setting $r = 0$ and recalling (12) shows that

$$\|\hat{h}\| \leqq \frac{\sigma(t_0)}{(n-k-1)!\prod_{j=1}^{k}(j-\gamma)}. \tag{30}$$

With ρ_1 and ρ_2 as in (6) and (8), choose ρ_1' and ρ_2' so that

$$\rho_1 < \rho_1' < (n-k-1)!\prod_{j=1}^{k}(j-\gamma), \tag{31}$$

$$\rho_2 < \rho_2', \tag{32}$$

and

$$\rho_2' + \rho_1' M \leqq M(n-k-1)!\prod_{j=1}^{k}(j-\gamma), \tag{33}$$

where the latter is possible because of (9). Now choose t_0 so that

$$\int_t^\infty s^{n-1}\, g(s)\phi(s)\, ds \leqq \rho_1'\phi(t),\ t \geqq t_0, \tag{34}$$

which is possible because of (6) and (31), and so that

$$\sigma(t_0) \leqq \rho_2' + \rho_1' M, \tag{35}$$

which is possible because of (6), (8), (9), (20), (31), and (32). Now (30), (33), and (35) imply that $\|\hat{h}\| \leqq M$; thus, T maps $H_M(t_0)$ into itself. Moreover, if $h_1, h_2 \in H_M(t_0)$, we can infer from (17) that $|I(t; h_1) - I(t; h_2)| \leqq \sigma_1(t)\phi(t)$, with

$$\sigma_1(t) = \|h_1 - h_2\| \sup_{\tau \geqq t}\left\{(\phi(\tau))^{-1}\int_\tau^\infty s^{n-1} g(s)\phi(s)\, ds\right\}. \tag{36}$$

Now we can write $\hat{h}_1 - \hat{h}_2$ as an integral by means of (13) or (14) and repeat the argument that led from (19) to (30), to conclude that

$$\|\hat{h}_1 - \hat{h}_2\| \leqq \sigma_1(t_0)/(n-k-1)!\prod_{j=1}^{k}(j-\gamma). \tag{37}$$

Therefore, (31), (34), (36), and (37) imply that $\|\hat{h}_1 - \hat{h}_2\| \leqq \theta\|h_1 - h_2\|$, where $\theta < 1$; that is, T is a contraction mapping of $H_M(t_0)$ into itself. Consequently, there is an h_0 in $H_M(t_0)$ such that $h_0 = Th_0$; that is, either (13) or (14)—whichever is appropriate—holds with $\hat{h} = h = h_0$. Differentiating either of these equations (with $\hat{h} = h = h_0$) n times shows that the function $u_0 = p + h_0$ satisfies (1); moreover, (21) and (29) with $\hat{h} = h_0$ imply that $h_0^{(r)}(t) = O(\phi(t)t^{k-r})$, $0 \leqq r \leqq k-1$, and this implies (10).

Now suppose $\rho_1 = \rho_2 = 0$. Then

$$\lim_{t\to\infty} \sigma(t) = 0 \tag{38}$$

(see (6), (8), and (20)), and therefore (21) with $\hat{h} = h_0$ implies that

(39) $$h_0^{(r)}(t) = o(\phi(t)t^{k-r})$$

for $k \leqq r \leqq n-1$. If $0 \leqq r \leqq k-1$, then (28) with $\hat{h} = h_0$ and (2) imply that

(40) $$|h_0^{(r)}(t)| \leqq \frac{t^{k-r-1+\gamma}\phi(t)}{(k-r-1)!(n-k-1)!} \int_{t_0}^{t} \sigma(\lambda)\lambda^{-\gamma}\, d\lambda.$$

But

$$\int_{t_0}^{t} \sigma(\lambda)\,\lambda^{-\gamma}\, d\lambda \leqq \int_{t_0}^{t} \sigma(\lambda)\,\lambda^{-\gamma}\, d\lambda + \sigma(t_1)\frac{t^{1-\gamma} - t_1^{1-\gamma}}{1-\gamma}$$

if $t \geqq t_1 \geqq t_0$. This and (40) imply that

$$\overline{\lim_{t\to\infty}}\, t^{-k+r}(\phi(t))^{-1}\, |h_0^{(r)}(t)| \leqq \sigma(t_1)/(k-r-1)!(n-k-1)!(1-\gamma).$$

Since this holds for all $t_1 \geqq t_0$, (38) implies (39) for $0 \leqq r \leqq k-1$. Hence, u_0 satisfies (11). This completes the proof of Theorem 1.

REMARK 1. If (5) is replaced by $\int^\infty s^{n-1} g(s)\, ds < \infty$, then (6) holds for any nonincreasing ϕ, with $\rho_1 = 0$. Also, if $\lim_{t\to\infty} \phi(t) \neq 0$, then we may as well assume that $\phi(t) \equiv 1$. Then obviously $\rho_1 = \rho_2 = 0$, and so (11) holds with $\phi(t) = 1$.

REMARK 2. In some instances we may take M arbitrarily large. Then (9) is no restriction.

REMARK 3. We believe it to be particularly significant that one of our integral smallness conditions on f in (1) does not require absolute convergence. There are relatively few instances in the literature where possibly conditional convergence is permitted in integral smallness conditions. For examples, see **[2]**, [3], [**4**, p. 379], [**6**], [7], [**8**], [**9**], and [**10**].

EXAMPLE 1. Suppose P, $F \in C(0, \infty)$ and the integrals

(41) $$\int^\infty t^{2n-2}\, P(t)\, dt \quad\text{and}\quad \int^\infty t^{n-1}\, F(t)\, dt$$

converge—perhaps conditionally—while

(42) $$\int^\infty t^{n-1}\, |P(t)|\, dt < \infty.$$

Let p be any polynomial of degree $< n$. Then Theorem 1 implies that the equation

(43) $$u^{(n)} + P(t)u = F(t)$$

has a solution u_0 such that

(44) $$u_0^{(r)}(t) = p^{(r)}(t) + o(t^{-r}),\ 0 \leqq r \leqq n - 1.$$

To see this, take $k = 0$, $\phi(t) = 1$, and

(45) $$f(t, u) = P(t)u - F(t).$$

From Dirichlet's test and the convergence of the first integral in (41), the integrals $\int^{\infty} t^{n+k-1} P(t)dt$, $k = 0, 1, \ldots, n - 1$, all converge, and, therefore, so does (7), with $\rho_2 = 0$ in (8). Obviously f in (45) is continuous on any set of the form (3) with $T_0 > 0$, and (4) and (6) (the latter with $\rho_1 = 0$) hold with $g(t) = |P(t)|$. (See (42).) Thus, the stated conclusion follows. To obtain the same conclusion from standard theorems, it would be necessary to assume that the integrals in (41) converge absolutely. (Note the order of approximation in (44), and recall that p is an arbitrary polynomial of degree $< n$.) By a more delicate argument which makes specific use of the linearity of (43), the same conclusion can be obtained without assuming (42); that is, no assumption requiring absolute convergence is needed. This is a special case of work on linear equations which will appear elsewhere.

EXAMPLE 2. In [5], Tong presented a theorem concerning the asymptotic behavior of solutions of (1) with $n = 2$, and applied it to show that, for some nonzero values of a_1, the equation

(46) $$u'' + t^{-4}u^2 \cos u = 0$$

has a solution u_0 which is asymptotic to

(47) $$p(t) = a_0 + a_1 t.$$

Tong did not specify the errors $u_0 - p$ and $u_0' - p'$. Theorem 1 provides more precise information on this problem; namely, if $|a_1| < 1$ and a_0 is arbitrary, then (46) has a solution u_0, defined for sufficiently large t, such that

$$u_0(t) = a_0 + a_1 t + O(t^{-1}),\ u_0'(t) = a_1 + O(t^{-2}).$$

To see this, we observe that here $f(t, u) = t^{-4}u^2 \cos u$ and

(48) $$f_u(t, u) = t^{-4}\,[2u \cos u - u^2 \sin u].$$

It is easy to show that, with p as in (47),

$$\overline{\lim_{t\to\infty}}\, t \left| \int_t^{\infty} sf(s, p(s))\, ds \right| = |a_1|;$$

hence, (8) holds, with

(49) $$\phi(t) = t^{-1},\ n = 2,\ k = 0,\ \rho_2 = |a_1|.$$

Moreover, f is continuous on Ω in (3) for any $M > 0$, $T_0 > 0$, and, by (48) and the mean value theorem, (4) holds, with

$$g(t) = t^{-4}[2(|a_0| + |a_1|\, t + M/t) + (|a_0| + |a_1| t + M/t)^2].$$

Therefore,

$$\overline{\lim_{t\to\infty}}\, t \int_t^\infty g(s)\, ds = a_1^2 = \rho_1,$$

so (6) holds with ϕ, n, and k as in (49), if $|a_1| < 1$. Since we may choose M arbitrarily, (9) is no restriction (Remark 2), so the hypotheses of Theorem 1 are satisfied, and the conclusion follows.

It can be shown from Theorem 1 that if a_0 is arbitrary, then (46) has a solution u_0 such that

$$u_0(t) = a_0 + O(t^{-2}),\ u_0'(t) = O(t^{-3}).$$

Many theorems dealing with the asymptotic behavior of solutions of (1) require that $|f(t, u)| \leqq w(t, |u|)$, where $w(t, r)$ is nondecreasing in r for each t. Since most of the large body of results concerning the equation

$$u^{(n)} + P(t)u^\alpha = F(t) \tag{50}$$

(for examples, see **[1]**, **[11]**. and **[12]**) are based on this approach, they necessarily require that $\alpha > 0$, and that the integral smallness conditions on P and F be stated entirely in terms of absolute convergence. The following corollary shows how Theorem 1 permits these conditions to be relaxed. (For other results along these lines for (49), see **[10]**.)

COROLLARY 1. *Suppose P, $F \in C(0, \infty)$ and k and ϕ satisfy the assumptions of Theorem* 1. *In addition, assume that*

$$\lim_{t\to\infty} \phi(t) = 0, \tag{51}$$

$$\int_t^\infty s^{n-k-1} F(s)\, ds = O(\phi(t)), \tag{52}$$

$$\int_t^\infty s^{n-1+k(\alpha-1)} P(s)\, ds = O(\phi(t)) \tag{53}$$

(*where the two integrals may converge conditionally*), *and*

$$\int_t^\infty s^{n-1+k(\alpha-1)}\, |P(s)|\phi(s)\, ds = O(\phi(t)), \tag{54}$$

where α is an arbitrary nonzero real number. Let $p(t) = at^k$, where a is a given positive number. Then (50) *has a solution u_0 which satisfies* (10), *provided $a^{\alpha-1}$ is sufficiently small.*

PROOF. Here $f(t, u) = P(t)u^\alpha - F(t)$, so (52) and (53) imply (8), and f is continuous and differentiable with respect to u, with

$$f_u(t, u) = \alpha P(t)u^{\alpha-1}, \tag{55}$$

on any subset of $(0, \infty) \times (0, \infty)$. If M, $T_0 > 0$ are such that

$$M\phi(T_0) \leqq a/2 \tag{56}$$

and $(t, u) \in \Omega$ as in (3), then $t \geqq T_0$ and

$$(a/2)t^k \leqq u \leqq (3a/2)t^k. \tag{57}$$

Therefore, f is continuous on Ω and (55), (57), and the mean value theorem imply (4), with $g(t) = C\,a^{\alpha-1}\,t^{k(\alpha-1)}|P(t)|$, where C is a positive constant. With this g, (54) implies that

$$\overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}\int_t^\infty s^{n-1}\,g(s)\phi(s)\,ds = C_1 a^{\alpha-1}, \tag{58}$$

where C_1 is a positive constant; hence, (6) holds if $a^{\alpha-1}$ is sufficiently small. Now choose M to satisfy (9), and then choose T_0 to satisfy (56). (This is possible, by (50).) This verifies the hypotheses of Theorem 1, and so the conclusion follows.

EXAMPLE 3. The equation

$$u^{(n)} + (t^{-n-k(\alpha-1)}[1+\log t]^{-1}\sin t)u^\alpha = t^{-n+k+1}[1+\log t]^{-1}\cos t, \tag{59}$$

with k in $\{0, 1, \ldots, n-1\}$ and $\alpha \neq 0$, satisfies the assumptions of Corollary 1, with $\phi(t) = (\log t)^{-1}$. Therefore, if $A > 0$ and $A^{\alpha-1}$ is sufficiently small, then (59) has a solution u_0 such that

$$u_0^{(r)}(t) = \begin{cases} A[1 + O(1/\log t)]\, t^{k-r}/(k-r)!, & 0 \leqq r \leqq k, \\ O(t^{k-r}/\log t), & k+1 \leqq r \leqq n-1. \end{cases}$$

REMARK 4. If (54) is replaced by the stronger condition $\int^\infty s^{n-1+k(\alpha-1)}|P(s)|\,ds < \infty$, then $C_1 = 0$ in (58), and it is not necessary to assume that $a^{\alpha-1}$ is small. Also, if α is a rational number with odd denominator, so that a^α has a real value for any $a \neq 0$, then the conclusion of Corollary 1 holds for any nonzero a such that $|a|^{\alpha-1}$ is sufficiently small. If $a < 0$ in this case, the proof is the same as that given above, except that a must be replaced by $|a|$ throughout, and u must be replaced by $|u|$ in (57).

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REFERENCES

1. T.G. Hallam, *Asymptotic behavior of the solutions of an n-th order nonhomogeneous ordinary differential equation*, Trans. Amer. Math. Soc., **122** (1966), 177–194.

2. ———, *Asymptotic integration of second order differential equations with integrable coefficients*, SIAM J. Appl. Math., **19** (1970), 430–439.

3. ———, *Asymptotic integration of a nonhomogeneous differential equation with integrable coefficients*, Czechoslovak Math. J., **21** (1971), 661–671.

4. P. Hartman, *Ordinary Differential Equations*, John Wiley, New York, 1964.

5. J. Tong, *The asymptotic behavior of a class of nonlinear differential equations of second order*, Proc. Amer. Math. Soc., **84** (1982), 235–236.

6. W. F. Trench, *Asymptotic behavior of solutions of perturbed disconjugate equations*, J. Diff. Eq., **11** (1972), 661–671.

7. ———, *Asymptotic integration of linear differential equations subject to integral smallness conditions involving ordinary convergence*, SIAM J. Math. Anal., **7** (1976), 213–221.

8. ———, *Asymptotic theory of perturbed general disconjugate equations*, Hiroshima Math. J., **12** (1982), 43–58.

9. ———, *Systems of differential equations subject to mild integral conditions*, Proc. Amer. Math. Soc., **87** (1983), 263–270.

10. ———, *Asymptotic integration of* $y^{(n)} + P(t)y^{\gamma} = f(t)$ *under mild integral smallness conditions*, Funk. Eckvac. (to appear).

11. P. Waltman, *On the asymptotic behavior of solutions of a nonlinear equation*, Proc. Amer. Math. Soc., **15** (1964), 918–923.

12. ———, *On the asymptotic behavior of solutions of an nth order equation*, Monatsh. Math., **69** (1965), 427–430.

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