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A NOTE ON HIGHER DERIVATIONS AND ORDINARY POINTS OF CURVES

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ABSTRACT. In this note, we prove the following theorem: Let k be an algebraically closed field of arbitrary characteristic. Let C denote a reduced curve in $\mathbf{A}_k^n$ and let p be a point of C. Let R denote the local ring of C at p and let $\bar{R}$ denote the integral closure of R in its total quotient ring. Let $M_1, \ldots, M_h$ be the branches of C at p. Then p is an ordinary point of C if and only if the following two conditions are satisfied: (a) $\mathrm{Der}_k^q(R) \subseteqq \mathrm{Der}_k^q(\bar{R})$ for all $q \geqq 1$; (b) For t a common uniformizing parameter of C in R, there exists an x in the maximal ideal of R such that $\partial x/\partial t$ is a unit in $\bar{R}$, and $(\partial x/\partial t)$ Mod $M_i \neq (\partial x/\partial t)$ Mod_j for all $1 \leqq i < j \leqq h$.

Introduction. Throughout this paper, we shall let k denote an algebraically closed field of arbitrary characteristic. We shall let C denote a reduced curve in $\mathbf{A}_k^n$ (affine n-space over k). Let p denote a point of C. In this paper, we wish to characterize when p is an ordinary point of C in terms of the higher derivations on the local ring R of C at p. We shall first show that C is unramified at p precisely when every higher order k-derivation on R extends to the integral closure $\bar{R}$. To the best of my knowledge this result was first proven by T. Bloom in [2] for irreducible curves over the complex numbers $\mathbf{C}$. This result was later generalized to arbitrary fields of characteristic zero by J. Becker in [1]. In this paper, we present a purely algebraic argument which works in any characteristic.

In the last part of this paper, we present straightforward differential conditions which guarantee C has distinct tangents at p.

Preliminaries. In this section, we shall present the definitions and basic notation which will be used throughout the rest of this paper. We shall let C denote a reduced curve in $\mathbf{A}_k^n$. To be more specific; $C = \mathrm{Spec}\{k[X_1, \ldots, X_n]/\mathfrak{A}\}$ where $\mathfrak{A}$ is a radical ideal, unmixed of height

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$n - 1$ in $k[X_1, \ldots, X_n]$. Let $p \in C$. Without loss of generality, we can assume p is the origin in $\mathbf{A}_k^n$. Thus, $\mathfrak{A} \subseteqq (X_1, \ldots, X_n)$.

We shall let $R = \mathcal{O}_{C,p}$, the local ring of C at p. $\bar{R}$ will denote the integral closure of R in its total quotient ring $Q(R)$. We shall let $\mathfrak{m}$ denote the maximal ideal of R and $\{M_1, \ldots, M_h\}$ the maximal ideals of $\bar{R}$. We say C is unramified at p if $\mathfrak{m}\bar{R} = M_1 \cdots M_h$.

A q-th order, k-derivation δ of a k-algebra S into an S-module V is a linear map $\delta \in \mathrm{Hom}_k(S, V)$ such that for any $q + 1$ elements $x_0, \ldots, x_q \in S$ we have

$$(1) \quad \begin{aligned} &\delta(x_0 \cdots x_q) \\ &= \textstyle\sum_{s=1}^{q}(-1)^{s-1} \sum_{i_1<\cdots<i_s} x_{i_1} \cdots x_{i_s} \delta(x_0 \cdots \check{x}_{i_1} \cdots \check{x}_{i_s} \cdots x_q). \end{aligned}$$

The S-module of all q-th order, k-derivations of S into S will be denoted by $\mathrm{Der}_k^q(S)$. Any facts concerning these modules for which we do not give a specific reference can be found in [9].

We say an ideal $I \subseteqq S$ is differential under $\mathrm{Der}_k^q(S)$ if $\delta(I) \subseteqq I$ for all $\delta \in \mathrm{Der}_k^q(S)$. We shall need the following simple lemma in the next section.

LEMMA. *Let S be any noetherian k-algebra and I a radical ideal in S. If $\delta \in \mathrm{Der}_k^q(S)$, and $\delta(I) \subseteqq I$, then $\delta(\mathscr{P}) \subseteqq \mathscr{P}$ for any associated prime $\mathscr{P}$ of I.*

PROOF. Let $\mathscr{P}_1, \ldots, \mathscr{P}_s$ denote the associated primes of I. Since I is a radical ideal, each $\mathscr{P}_i$ is an isolated prime of I. In particular, $\mathscr{P}_i \not\supset \bigcap_{j \neq i} \mathscr{P}_j$. Let $x \in \mathscr{P}_i$ and choose $y \in \{\bigcap_{j\neq i} \mathscr{P}_j\} - \mathscr{P}_i$. Then for all $n \geqq 1$, $xy^n \in I$. Applying equation (1) to $\delta(xy^q)$, we see $y^q\delta(x) \in \mathscr{P}_i$. Thus, $\delta(x) \in \mathscr{P}_i$, and $\mathscr{P}_i$ is differential under δ.

We note that I being a radical ideal of S is essential for the validity of the Lemma. For example, if $S = k[X]$, and $\delta \in \mathrm{Der}_k^2(k[X])$ is defined by $\delta(X) = 1$, and $\delta(X^2) = X^2$, then $I = (X^2)$ is differential under δ whereas $\sqrt{I} = (X)$ is not.

Finally, we recall the definition of p being an ordinary point of C. The maximal ideals $M_1, \ldots, M_h$ in $\bar{R}$ are called the branches of C at p (See [10]). If each branch M_i is linear (i.e., $\mathfrak{m}\bar{R} = M_1 \ldots M_h$), then the canonical map π_i: $\mathfrak{m}/\mathfrak{m}^2 \to M_i/M_i^2$ induces an injection π_i^*: $\mathrm{Hom}_k(M_i/M_i^2, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$. We identify $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ with the tangent space $\mathscr{T}_{C,p}$ of C at p. For each $i = 1, \ldots, h$, let $\mathscr{T}_i = \mathrm{Im}\ \pi_i^*$. Then $\mathscr{T}_i$ is a one dimensional (i.e., a line) subspace of $\mathscr{T}_{C,p}$ called the tangent to M_i. We say p is an ordinary point of C if C is unramified at p, and the tangents $\mathscr{T}_1, \ldots, \mathscr{T}_h$ are all distinct.

Main results. It is well known that $\mathrm{Der}_k^q(R) \otimes_R Q(R) \cong \mathrm{Der}_k^q(Q(R))$. Thus, any q-th order, k-derivation δ on R can be viewed as a q-th order derivation on the total quotient ring $Q(R)$. It then makes sense to enquire when $\delta(\bar{R}) \subseteq \bar{R}$. If $\delta(\bar{R}) \subseteq \bar{R}$, we shall say δ extends to $\bar{R}$. If every q-th order, k-derivation on R extends to $\bar{R}$, we shall write $\mathrm{Der}_k^q(R) \subseteq \mathrm{Der}_k^q(\bar{R})$. In [**11**], A. Seidenberg showed that $\mathrm{Der}_k^1(R) \subseteq \mathrm{Der}_k^1(\bar{R})$ whenever the characteristic of k is zero. In the same paper, Seidenberg gave an example of a first order k-derivation on R which did not extend to $\bar{R}$ when the characteristic of k was not zero. In [**3**], the author exhibited an example in characteristic zero, where $\mathrm{Der}_k^2(R) \nsubseteq \mathrm{Der}_k^2(\bar{R})$. In both examples, the curve C failed to be unramified at p. In studying these examples, one is naturally led to our first theorem.

THEOREM 1. *Let C denote a reduced curve in $\mathbf{A}_k^n$ and let p be a point of C. Let R denote the local ring of C at p and $\bar{R}$ the integral closure of R in its total quotient ring. Then C is unramified at p if and only if* $\mathrm{Der}_k^q(R) \subseteq \mathrm{Der}_k^q(\bar{R})$ *for all* $q \geqq 1$.

PROOF. We can assume the point p is the origin in $\mathbf{A}_k^n$ without any loss of generality. If p is a simple point of C, then $R = \bar{R}$, and the result is trivial. Hence, we assume p is a singular point of C. Then $R \neq \bar{R}$.

Let $\hat{R}$ and $\hat{\bar{R}}$ denote the $\mathfrak{m}$-adic completions of R and $\bar{R}$ respectively. Since R is a reduced, excellent local ring, the completion $\hat{R}$ is reduced, and the integral closure of $\hat{R}$ in $Q(\hat{R})$ is just $\hat{\bar{R}}$. The following facts are well known:

$$\mathrm{Der}_k^q(R) \otimes_R \hat{R} \cong \mathrm{Der}_k^q(\hat{R}) \text{ for all } q \geqq 1, \tag{2}$$

$$\mathrm{Der}_k^q(\bar{R}) \otimes_{\bar{R}} \hat{\bar{R}} \cong \mathrm{Der}_k^q(\hat{\bar{R}}) \text{ for all } q \geqq 1, \tag{3}$$

and

$$Q(\bar{R}) \cap \hat{\bar{R}} = \bar{R} \tag{4}$$

A proof of equation (2) can be found in [**6**; Prop. 1]; equation (3) in [**7**; Lemma 2]; and (4) in [**8**; (18.4)]. It easily follows from equations (2) through (4) that we can assume without loss of generality that R is complete.

If $q_1, \ldots, q_h$ denote the minimal primes of R, then $\bar{R}$ has the following form:

$$\bar{R} = V_1 \oplus \cdots \oplus V_h \tag{5}$$

In equation (5), V_i is the integral closure of R/q_i in $Q(R/q_i)$. Each V_i is a complete, discrete, rank one, valuation ring containing a copy k of its residue class field. The reader is referred to [**4**; pp. 119–122] for the proofs of the statements made above.

Now let us assume C is unramified at p. Then $\mathfrak{m}\bar{R} = M_1 \cdots M_h$. It follows that $\mathfrak{m}V_i$ is the maximal ideal of V_i. Thus, $\mathfrak{m}(R/q_i)$ contains a uniformizing parameter of V_i. Since both rings are complete, we conclude $R/q_i = V_i$. So, $\bar{R} = \bigoplus_{i=1}^h R/q_i$. Now if $\delta \in \mathrm{Der}_k^q(R)$, then $\delta(0) \subseteqq (0)$. Hence, by the Lemma, $\delta(q_i) \subseteqq q_i$ for $i = 1, \ldots, h$. Thus, δ induces a q-th order k-derivation on each summand R/q_i of $\bar{R}$. It readily follows that $\delta(\bar{R}) \subseteqq \bar{R}$, and, thus, $\mathrm{Der}_k^q(R) \subseteqq \mathrm{Der}_k^q(\bar{R})$.

Conversely, suppose $\mathrm{Der}_k^q(R) \subseteqq \mathrm{Der}_k^q(\bar{R})$ for all $q \geqq 1$. We wish to argue $\mathfrak{m}\bar{R} = M_1 \ldots M_h$. The proof given here is a modification of an argument due to Y. Ishibashi in [7].

We have $\mathfrak{m}\bar{R} = M_1^{s_1} \cdots M_h^{s_h}$ with $s_i \geqq 1$. Suppose $s_i > 1$ for some i. We can assume $s_1 = u > 1$. We shall then construct a $\delta \in \mathrm{Der}_k^q(R)$ for a suitable $q \gg 1$ such that $\delta(\bar{R}) \not\subset \bar{R}$.

Let $\mathfrak{f}$ denote the conductor of R in $\bar{R}$. If $J = \bigcap_{i=1}^h M_i$, then there exists an integer $n \gg 1$ such that $J^n \subseteqq \mathfrak{f} \subseteqq \mathfrak{m}$. $\bar{R}$ is a principal ideal ring, and, consequently, $J = t\bar{R}$ for some $t \in \bar{R}$. Let us write $t = (t_1, \ldots, t_h) \in \bigoplus_{i=1}^h V_i = \bar{R}$. Then $V_i = k[[t_i]]$, and $t^n \in \mathfrak{f}$.

Since $t^{n+2} R$ is a primary ideal for $\mathfrak{m}$ in R, $R/t^{n+2} R$ is a finite dimensional k-vector space. Let $\{y_1, \ldots, y_s\}$ be elements of $\mathfrak{m}$ whose residues modulo $t^{n+2} R$ form a k-basis of $\mathfrak{m}/t^{n+2}R$. For any $x \in \bar{R}$, let us denote the i-th component of x (in V_i) as x_i. Then $x = (x_1, \ldots, x_h)$. Let $\nu_i\colon Q(V_i) \to \mathbf{Z} \cup \{\infty\}$ denote the canonical valuation given by $\nu_i(z) = \mathrm{ord}_{t_i}(z)$. Since $\mathfrak{m}V_1 = t_1^u V_1$ with $u > 1$, we see $\nu_1(y_{j1}) \geqq 2$ for $j = 1, \ldots, s$. Using the fact that $V_1 = k[[t_1]]$ and taking k-linear combinations of $y_1, \ldots, y_s$ if need be, we can assume, without loss of generality, that $2 \leqq \nu_1(y_{11}) < \nu_1(y_{21}) < \cdots < \nu_1(y_{s1})$.

Now let $D = \{D_0 = 1, D_1, \ldots\}$ be the k-derivation of infinite rank on V_1 given by the following equations:

$$D_i(t_1^\beta) = \begin{cases} \binom{\beta}{i} t_1^{\beta-i} & \text{if } \beta \geqq i \\ 0 & \text{if } \beta < i \end{cases} \tag{6}$$

For the definition and existence of such a derivation of infinite rank, we refer the reader to [5; *IV* (pt 4), 16.11.2]. Using equation (6), one easily checks that if $f \in V_1$, and $\nu_1(f) = \alpha$, then the following relations are satisfied:

$$\nu_1(D_i(f)) \geqq \alpha - i \quad \text{for} \quad i < \alpha \tag{7}$$

and $\nu_1(D_\alpha(f)) = 0$. Let us set $\nu_1(y_{j1}) = n_j$ for $j = 1, \ldots, s$. Then $2 \leqq n_1 < n_2 < \cdots < n_s$.

For any $a_1, \ldots, a_s \in V_1$, we can consider the differential operator Δ defined by the following equation:

$$\Delta = \frac{1}{t_1} D_1 + \textstyle\sum_{i=1}^s a_i t_1^{n_i-2} D_{n_i}. \tag{8}$$

Since D is a higher derivation of infinite rank, each D_i is an i-th order, k-derivation on V_1 ([**9**; Prop, 5]). Thus, using [**9**; Prop. 4], we can view Δ as an $(n_s - 2)$-order, k-derivation on $Q(V_1) = k((t_1))$. We claim there exists a choice of constants $a_1, \ldots, a_s \in V_1$ such that $\Delta(y_{j1}) = 0$ for all $j = 1, \ldots, s$. To see this, we consider the following system of linear equations in unknowns $x_1, \ldots, x_s$:

$$(9) \qquad \textstyle\sum_{i=1}^{s} x_i\, t_1^{n_i-2}\, D_{n_i}(y_{j1}) = -\dfrac{1}{t_1} D_1(y_{j1}), j = 1, \ldots, s.$$

We regard the equations in (9) as s-equations in s-unknowns with coefficients in the field $k((t_1))$. If we solve the equations in (9) using Cramer's rule and use equation (7) to check the ν_1-value of our answers, we see we get a solution $a_1, \ldots, a_s$ in V_1.

Now let $a_1, \ldots, a_s \in V_1$ be a solution to the equations in (9) and define $\Delta \in \mathrm{Der}_k^{n_s-2}(k((t_1)))$ from these $a_1, \ldots, a_s$ via equation (8). Set $q = n_s - 2$. Since $Q(\bar{R}) = \bigoplus_{i=1}^{h} k((t_i))$, one easily checks that $\mathrm{Der}_k^q(Q(\bar{R})) = \bigoplus_{i=1}^{h} \mathrm{Der}_k^q(k((t_i)))$. Thus $\delta = (\Delta, 0, \ldots, 0)$ is a well defined q-th order, k-derivation on $Q(\bar{R})$. We claim that $\delta(R) \subseteqq R$, but $\delta(\bar{R}) \nsubseteq \bar{R}$. To see this, let $f \in R$. Then $f = \alpha_0 + \alpha_1 y_1 + \cdots + \alpha_s y_s + t^{n+2}\gamma$. Here $\alpha_0, \ldots, \alpha_s \in k$, and γ is some element in R. Since Δ vanishes on $\alpha_0, y_{11}, \ldots, y_{s1}$, we have $\delta(f) = (\Delta(t_1^{n+2}\gamma_1), 0, \ldots, 0)$. Again, using equations (7) and (8), one easily checks that $\nu_1(\Delta(t_1^{n+2}\gamma_1)) \geqq n$. Thus, $\Delta(t_1^{n+2}\gamma_1) = t_1^n\sigma_1$ for some element $\sigma_1 \in V_1$. Therefore, $\delta(f) = t^n(\sigma_1, 0, \ldots, 0) \in R$ since $t^n \in \mathfrak{f}$. We have now shown $\delta(R) \subseteqq R$. Since $\delta(t) = (\Delta(t_1), 0, \ldots, 0) = (1/t_1, 0, \ldots, 0) \notin \bar{R}$, we see δ does not extend to $\bar{R}$. This is impossible, and, hence, completes the proof of the theorem.

Let us assume C is unramified at p. Since k is infinite, there exists a sufficiently general k-linear combination of the basis elements of $\mathfrak{m}$, say $t \in \mathfrak{m}$, such that $\mathfrak{m}\bar{R} = t\bar{R}$. Then in the complete case, $\bar{R} = K[[t]]$ where $K = \bar{R}/J$. Thus, for any $x \in \bar{R}$, $\partial x/\partial t$ is well defined. Now if $x \in \mathfrak{m}$, the value of $(\partial x/\partial t)$ Modulo M_i is the same as the value of x/t at the corresponding point $M_i \cap R[t^{-1}\mathfrak{m}]$ of $\mathrm{Spec}(R[t^{-1}\mathfrak{m}])$. So, for x a sufficiently general k-linear combination of a basis for $\mathfrak{m}$, the number of distinct values of $\partial x/\partial t$ at the closed points of $\mathrm{Spec}(\bar{R})$ will be precisely the number of maximal ideals of $R[t^{-1}\mathfrak{m}]$, i.e., the number of tangents at p. Thus, we get the following corollary to the theorem.

THEOREM 2. *Let k be an algebraically closed field of arbitrary characteristic. Let C denote a reduced curve in $\mathbf{A}_k^n$ and let p be a point on C. Let R denote the local ring of C at p, and let $\bar{R}$ denote the integral closure of R in its total quotient ring. Let $M_1, \ldots, M_h$ be the branches of C at p. Then p is an ordinary point of C if and only if the following two conditions are satisfied*:

(a) $\mathrm{Der}_k^q(R) \subseteqq \mathrm{Der}_k^q(\bar{R})$ *for all* $q \geqq 1$.

(b) *For* t *a common uniformizing parameter of* C *in* R, *there exists an* x *in the maximal ideal of* R *such that* $\partial x/\partial t$ *is a unit in* $\bar{R}$, *and* $(\partial x/\partial t) \bmod M_i \neq (\partial x/\partial t) \bmod M_j$ *for all* $1 \leqq i < j \leqq h$.

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