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# A BANG-BANG RESULT FOR AN UNDAMPED SECOND-ORDER EVOLUTION EQUATION OF SOBOLEV TYPE

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ABSTRACT. We determine bang-bang properties of an optimal control for an undamped second-order evolution equation of Sobolev type. The property is shown to depend on the point of control. However, smoothing of the Sobolev equation allows consideration of domains in $\mathbf{R}^p$ for $p = 1, 2$, or 3. A time optimal and a fixed-time problem are considered.

**1.** Let $\Omega$ be a nonempty open bounded subset of $\mathbf{R}^p$ with $p = 1, 2$, or 3 with smooth boundary $\Gamma$ and let $Q = \Omega \times (0, T)$ with lateral boundary denoted by $\Sigma = \Gamma \times (0, T)$. In this paper we study a control problem with an underlying equation given by the Sobolev equation

$$(1) \qquad \begin{aligned} (1 - \lambda\Delta)y_{tt} - \Delta y &= v(t)\delta(x - a) \text{ in } Q, \\ y(x, t) &= 0 \text{ on } \Sigma, \\ y(x, 0) = y_t(x, 0) &= 0 \text{ in } \Omega \end{aligned}$$

and with the optimization problem

$$(2) \qquad \begin{aligned} &\text{minimize} \int_\Omega (y(x, T; v) - z(x))^2 dx \\ &\text{subject to } v \in U_{\text{ad}} \end{aligned}$$

where

$$U_{\text{ad}} = \{v \in L^\infty(0, T) : \|v\|_{L^\infty(0,T)} \leqq 1\}.$$

We assume that $a \in \Omega$, $z \in L^2(\Omega)$, and $\lambda = 1$. For ease in our discussion we take $\Omega = (0, 1)$ and point out extensions to higher dimensions.

Problems such as (1) arise in the study of longitudinal vibrations in a beam, see Love [7], and [2, 3, 4, 9]. In these studies the presence of the

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$\lambda \Delta y_{tt}$ is a consequence of an inertial contribution of lateral motions in which the cross-sections are extended or contracted in their own planes. By retaining this term a better approximation of the velocity of wave-propagation in the beam is obtained [7]. The problem is also of interest mathematically, not only for itself, but also for the study of regularization of hyperbolic equations by artificial inertia [9].

We present here conditions characterizing the solution of (2). The results establish the bang-bang behavior of the optimal control of (1)–(2). This is of interest in comparison to the hyperbolic equation in which there is no general bang-bang result for an analogous problem [8]. Further, the bang-bang property is shown to hold for $\Omega$ in $\mathbf{R}^p$, $p = 1$, 2, or 3; however, we establish dependence of the bang-bang property upon the point $a$. Finally, we consider the time-optimal case to show that the bang-bang property holds as a necessary condition for optimality for this Sobolev problem.

**2. Existence and characterization.** The equation (1) may be rewritten as follows in $L^2(Q)$

$$\begin{aligned} y_{tt} - (1 - \Delta)^{-1}\Delta y &= v(t)g_a \text{ in } L^2(Q), \\ y|_\Sigma &= 0, \\ y(\cdot, 0) &= y_t(\cdot, 0) = 0 \end{aligned} \tag{3}$$

where $g_a \in L^2(\Omega)$ and is given by $(1 - \Delta)\, g_a = \delta_a$ in $\Omega$, $g_a|_\Gamma = 0$. Now $-\Delta(1 - \Delta)^{-1}$: $L^2(\Omega) \to L^2(\Omega)$. If $\varphi \in L^2(\Omega)$, we see that $\|\Delta(1 - \Delta)^{-1}\varphi\|_0 \leqq \|(1 - \Delta)^{-1}\varphi\|_2 \leqq c\|\varphi\|_0$. By considering the mapping $L^2(\Omega) \times L^2(\Omega)$ into $\mathbf{R}$

$$(\varphi, \psi) \to (\varphi, -\Delta(1 - \Delta)^{-1}\psi)_{L^2(\Omega)}$$

we see for each fixed $\varphi$ this determines a bounded linear functional on $L^2(\Omega)$. Hence, we may define a bounded linear operator $A$ on $L^2(\Omega)$ by

$$(A\varphi, \psi)_{L^2(\Omega)} = (\varphi, -\Delta(1 - \Delta)^{-1}\psi)_{L^2(\Omega)}.$$

For $\varphi$ in $H_0^1(\Omega) \cap H^2(\Omega)$ we see that $A\varphi = -(1 - \Delta)^{-1}\Delta\varphi$. Now on $H_0^1(\Omega) \cap H^2(\Omega)$, $-(1 - \Delta)^{-1}\Delta\varphi = -\Delta(1 - \Delta)^{-1}\varphi$. Thus, $A = A^*$ and is the extension of $-(1 - \Delta)^{-1}\Delta$ to $L^2(\Omega)$. Finally, the product $-(1 - \Delta)^{-1}\Delta$ is a positive operator. Hence, $A$ is a positive bounded self-adjoint operator on $L^2(\Omega)$. As such, there exists a positive bounded self-adjoint operator $B$ on $L^2(\Omega)$ with the property that $B^2 = A$.

Having established the existence of a square root of $A$, we may represent the solution of (3) as

$$y(\cdot, t; v) = \left[\int_0^t \sin B(s - t)v(s)ds\right]B^{-1}g_a(\cdot). \tag{4}$$

From equation (4) the continuity of $t \to y(\cdot, t; v)$ is obvious as well as the continuity of $v \to y(\cdot, t; v)$ as a function of $L^2(0, T)$ into $L^2(\Omega)$. If $v_n \to v$ weakly in $L^2(0, T)$, then by introducing an adjoint equation

$$(5) \qquad \begin{aligned} (1 - \Delta)\psi_{tt} - \Delta\psi &= 0 \text{ in } Q, \\ \psi(\cdot, T) &= 0 \text{ in } \Omega, \\ \psi_t(\cdot, T) &= -(1 - \Delta)^{-1}\alpha \text{ in } \Omega, \\ \psi|_\Sigma &= 0 \end{aligned}$$

with $\alpha \in L^2(\Omega)$ we see that

$$(6) \qquad \int_\Omega y(\cdot, T; v_n)\alpha \, dx = \int_0^T v_n(t)\psi(a, t)dt.$$

Straightforward estimates [11] show that $\psi(a, \cdot)$ belongs to $L^2(0, T)$ so that in the limit, we have $\int_\Omega y(\cdot, T; v)\alpha \, dx = \int_0^T v(t)\psi(a, t)dt$. Hence, we see that $y(\cdot, T; v_n) \to y(\cdot, T; v)$ weakly in $L^2(\Omega)$ if $v_n \to v$ weakly in $L^2(0, T)$. We give these facts in the following.

PROPOSITION 1. *If $v \in L^2(0, T)$ then solution $y$ of* (1) *is given by equation* (4) *and the map $t \to y(\cdot, t; v)$ is a continuous map of* $[0, T]$ *into $L^2(\Omega)$. If $v_n \to v$ strongly (weakly) in $L^2(0, T)$ then $y(\cdot, T; v_n) \to y(\cdot, T; v)$ strongly (weakly) in $L^2(\Omega)$.*

We may now establish the existence of a solution to (2). To this end let $(u_n)$ be a sequence in $L^\infty(0, T)$ and $\|u_n\|_\infty \leqq 1$ with the property that

$$\|y(T; u_n) - z\|_{L^2(\Omega)} \to d = \inf_{v \in U_{\mathrm{ad}}} \|y(T; v) - z\|_{L^2(\Omega)}.$$

Since $\|u_n\|_{L^\infty(0,T)} \leqq 1$, there is a subsequence $(u_{n_i})$ such that $u_{n_i} \to u$ weak-star in $L^\infty(0, T)$. The limit $u$ also belongs to $U_{\mathrm{ad}}$. Now as $u_{n_i} \to u$ weak-star in $L^\infty(0, T)$ implies that $u_{n_i} \to u$ weakly in $L^2(0, T)$. We conclude from Proposition 1 that $y(T; u_{n_i}) \to y(T; u)$ weakly in $L^2(\Omega)$. Hence, we have

$$d \geqq \|y(T; u) - z\|_{L^2(\Omega)}.$$

Since $u \in U_{\mathrm{ad}}$, we conclude that $u$ is a solution of (2).

THEOREM 2. *There exists a solution $u$ to problem* (2).

By taking the variation of the control functional we may obtain a characterization of $u$ in terms of a variational inequality

$$(7) \qquad ((y(T; u) - z, y(\cdot, T; v) - y(\cdot, T; u))_{L^2(\Omega)} \geqq 0$$

for all $v \in U_{\mathrm{ad}}$.

If we introduce an adjoint equation similar to (5)

$$
(8) \qquad \begin{aligned} (1 - \Delta)p_{tt} - \Delta p &= 0 \text{ in } Q, \\ p(\cdot, T) &= 0 \text{ in } \Omega, \\ p_t(\cdot, T) &= (1 - \Delta)^{-1}(z - y(T; u)) \text{ in } \Omega, \\ y|_\Sigma &= 0, \end{aligned}
$$

the variational inequality (7) becomes

$$
(9) \qquad \int_0^T p(a, t)(v(t) - u(t))dt \geqq 0
$$

for all $v \in U_{\text{ad}}$.

We now consider the implications of the inequality (9). Suppose that $|u(t)| < 1$ for $t$ in a set $E \subset [0, 1]$ with meas $E > 0$. If $p$ is the solution of (9), then $p(a, t)$ is a continuous function of $t$. Hence, there is a number $M > 0$ such that $|p(a, t)| \leqq M$ on $[0, T]$. Set

$$
E_n = \{t \in E: 1 - |u(t)| \geqq 1/n\}.
$$

By assumption then there is $k$ such that meas $E_k \neq 0$. Define the function $v$ on $[0, t]$ by

$$
v(t) = \begin{cases} u(t) \pm \delta_k p(a, t) & \text{on } E_k, \\ u(t) & \text{on } [0, T] - E_k \end{cases}
$$

with the number $\delta_k \in (0, 1/kM)$ such that

$$
\begin{aligned} |v(t)| &= |u(t) \pm \delta_k p(a, t)| \\ &\leqq |u(t)| + \delta_k M \\ &\leqq 1 - (1/k) + \delta_k M. \end{aligned}
$$

Hence, we see that $|v(t)| \leqq 1$ on $E_k$ and $v \in U_{\text{ad}}$. From the inequality (9) it follows now that

$$
(10) \qquad \int_{E_k} p^2(a, t)dt = 0
$$

and so $p(a, t) = 0$ a.e. in $E_k$. Since $E$ is the countable union of such sets, we conclude that

$$
(11) \qquad p(a, t) = 0 \text{ a.e. in } E.
$$

Hence, we have the following.

THEOREM 3. *If $u$ is a solution of* (1), (2) *and if $p$ is the solution of* (8), *then*

$$
(12) \qquad (|u(t)| - 1)p(a, t) = 0
$$

*holds* a.e. *in* $[0, T]$.

**3. A bang-bang result.** We begin by studying the implications of the equation

$$p(a, t) = 0 \tag{13}$$

on a set $E \subset [0, T]$ with meas $E > 0$. For ease we consider the case for $\Omega = [0, 1]$. Set $z - y(T; u) = \sum_{k=1}^{\infty} \zeta_k \sin k\pi x$ and we give the Fourier series solution of (8)

$$p(x, t) = -\sum_{k=1}^{\infty} \frac{\zeta_k}{\sqrt{\mu_k}(1 + k^2\pi^2)} \sin[\sqrt{\mu_k}(T - t)]\sin(k\pi x) \tag{14}$$

where $\mu_k = k^2\pi^2/(1 + k^2\pi^2)$. Note that this series converges uniformly on $[0, 1] \times [0, T]$. Setting $\tau = T - t$ condition (13) gives

$$0 = \sum_{k=1}^{\infty} \frac{\zeta_k}{\sqrt{\mu_k}(1 + k^2\pi^2)} \sin(\sqrt{\mu_k}\, \tau)\sin(k\pi a) \tag{15}$$

for $\tau \in F = T - E$. We consider now the complex-valued function given by

$$f(z) = \sum_{k=1}^{\infty} \frac{\zeta_k}{\sqrt{\mu_k}(1 + k^2\pi^2)} \sin(\sqrt{\mu_k}\, z)\sin(k\pi a). \tag{16}$$

Since the numbers $\mu_k = k^2\pi^2/(1 + k^2\pi^2)$ are bounded, partial sums of the series in (16) converge uniformly on compact sets in the complex plane $\mathbf{C}$. Hence, the function $f(z)$ is an analytic complex-valued function with the property that $f(z) = 0$ for $z \in F \subset [0, T]$. Since the set $F$ has positive measure, it has a cluster point. Thus, by the identity theorem, we conclude that, in fact, $f(z) \equiv 0$. Therefore, equation (15) holds for all $\tau \in (-\infty, \infty)$, that is,

$$p(a, t) \equiv 0 \tag{17}$$

for $t \in (-\infty, \infty)$.

Calculating the Laplace transform of $p$, we use (15) and integrate inside the sum since the series converges uniformly on $[0, \infty)$ to obtain

$$\hat{p}(a, s) = \sum_{k=1}^{\infty} (\zeta_k/\sqrt{\mu_k}(1 + k^2\pi^2))\ (1/(s^2 + \mu_k))\sin(k\pi a). \tag{18}$$

Equation (17), however, implies $\hat{p}(a, s) = 0$. Now considering function

$$g(z) = \sum_{k=1}^{\infty} \frac{\zeta_k}{\sqrt{\mu_k}(1 + k^2\pi^2)} \frac{1}{(z^2 + \mu_k)} \sin(k\pi a),$$

we see that $g$ is a meromorphic function with poles at $\pm i\sqrt{\mu_k}$ and has the property that $g(z) = 0$ for $z$ real and positive. This implies, however, that the residues of $g$ are zero. Hence, we conclude that

$$\zeta_k \sin(k\pi a) = 0 \tag{19}$$

for each $k$. It is from equation (19) that we may deduce properties of $\zeta_k$. If $a$ is an irrational number, then equation (19) implies $\zeta_k = 0$ for each $k$. Hence, we may conclude that $p(x, t) \equiv 0$ so that $y(T; u) = z$ in $L^2(\Omega)$, cf. [10].

THEOREM 4. *If $a \in (0, 1)$ is an irrational number, then either*

$$y(\cdot, T; u) = z(\cdot) \text{ in } L^2(\Omega) \tag{20}$$

*or*

$$|u(t)| = 1 \text{ a.e. in } [0, T] \tag{21}$$

*in which case* $u(t) = -\operatorname{sgn}(p(a, t))$ a.e. $(0, T)$.

REMARK 5. An analogous result holds if $\Omega$ is a rectangle $[0, l_1] \times [0, l_2]$ or a parallelopiped $[0, l_1] \times [0, l_2] \times [0, l_3]$ where the lengths of the sides are independent with respect to the integers. This assures that the eigenvalues have multiplicity one. In this case, if a has coordinates that are irrational multiples of the lengths of the sides, then the result holds.

REMARK 6. For the case of the wave equation in place of (1) we may define a solution by transposition. Further, Theorems 2 and 3 hold as well. In this case $\mu_k = k^2\pi^2$ and equation (14) becomes

$$p(x, t) = -\sum_{k=1}^{\infty} (\zeta_k/k\pi)\sin[k\pi(T-t)]\sin(k\pi x) \tag{22}$$

which converges in $L^2(Q)$. However, this series cannot be extended analytically to conclude $p(a, t) \equiv 0$ since here $\sqrt{\mu_k} = k\pi$ is no longer bounded. Indeed, apparently a bang-bang principle does not hold for hyperbolic equations [8].

**4. Time optimal control.** Consider the equations

$$\begin{aligned} (1 - \Delta)y_{tt} - \Delta y &= v(t)\delta(x - a) \text{ in } \Omega \times (0, \infty) \\ y(x, t) &= 0 \text{ on } \Gamma \times (0, \infty) \\ y(x, 0) &= y_t(x, 0) = 0 \text{ in } \Omega \end{aligned} \tag{23}$$

with problem

$$\begin{aligned} &\text{Find } T_0 = \min\{T\colon \text{there exists } |v(t)| \leqq 1 \text{ with} \\ &\text{the property } \|y(T; v) - z\|_{L^2(\Omega)} \leqq \rho\}. \end{aligned} \tag{24}$$

Here we assume that a solution $u$ exists such that

$$\|y(T_0, u) - z\|_{L^2(\Omega)} \leqq \rho,$$

and $T_0$ is the least such $T$ for which this is true. We show that as a consequence of the proof of Theorem 4 that $|u(t)| = 1$ for almost all $t \in$

$[0, T_0]$. As a corollary of the continuity of the map $t \to y(\cdot, t; v)$ from $[0, \infty)$ into $L^2(\Omega)$ given in Proposition 1, we immediately have the following, see also [**5**, **6**].

PROPOSITION 7. *If $u$ and $T_0$ provide a solution of problem* (24), *then*

$$\|y(\cdot, T_0; u) - z(\cdot)\|_{L^2(\Omega)} = \rho. \tag{25}$$

PROOF. If $\|y(T_0, u) - z\|_{L^2(\Omega)} < \rho$, then by the continuity of $t \to y(t; u)$ there exists a $\delta > 0$ such that if $T$ satisfies $|T - T_0| < \delta$ then $\|y(T; u) - z\|_{L^2(\Omega)} < \rho$. But then $\|y(T_0 - \delta/2, u) - z\|_{L^2(\Omega)} \leqq \rho$ contradicting the minimality of $T_0$.

We also have the result.

PROPOSITION 8. *The set of attainability at time $T$, $\{y(\cdot, T: v): |v(t)| \leqq 1\}$, is closed and convex in $L^2(\Omega)$.*

By Proposition 8 it now follows that a necessary condition for $u$ to be a solution of (24) is that

$$(z - y(T_0, u), y(T_0, v) - y(T_0, u))_{L^2(\Omega)} \leqq 0 \tag{26}$$

for all $|v(t)| \leqq 1$ almost everywhere in $[0, T_0]$. This corresponds to the inequality in (7). The same proof now follows as in section 3 to obtain a result similar to Theorem 4. In the time optimal case, however, as a consequence of Proposition 7, we have $y(T_0, u) \neq z$. Hence, we must have $|u(t)| = 1$ a.e. in $[0, T_0]$.

THEOREM 9. *Let $T_0$ and $u$ be a solution of a problem* (24) *and let $a \in (0, 1)$ be irrational. Then $|u(t)| = 1$ almost everywhere in $[0, T_0]$ and $u(t) = -\operatorname{sgn}(p(a, t))$.*

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