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A DISFOCALITY FUNCTION FOR A NONLINEAR ORDINARY DIFFERENTIAL EQUATION

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

We will be concerned with the differential equation

$$y^{(n)} = f(x, y, \ldots, y^{(n-1)}) \tag{1}$$

where we will make some or all of the assumptions:

(A) f is continuous on $J \times \mathbf{R}^n$ (J a subinterval of the reals, $\mathbf{R}$).

(B) solutions of initial value problems (IVP's) are unique and exist on the whole interval J.

(C) if $\{y_n\}$ is a sequence of solutions which is uniformly bounded on a nondegenerate compact interval $[c, d] \subset J$, then there exists a subsequence $\{y_{n_k}\}$ such that each of the sequences $\{y_{n_k}^{(i)}\}$, $i = 0, \ldots, n-1$, converges uniformly on compact subintervals of J.

(D) $f_i(x, y, \ldots, y^{(n-1)}) = (\partial/\partial y^i)f(x, y, \ldots, y^{(n-1)})$, $i = 0, \ldots, n-1$ is continuous on $J \times \mathbf{R}^n$.

For information concerning the compactness condition (C) see [6] and the references given there.

We now introduce much of the same notation used by Muldowney [9]. Let $\tau = (t_1, \ldots, t_n)$. We say that $y(x)$ has n zeros at τ provided $y(t_i) = 0$, $1 \leqq i \leqq n$, and $y(t_i) = y'(t_i) = \cdots = y^{(m-1)}(t_i) = 0$ if a point t_i occurs m times in τ. A partition $(\tau_1; \ldots; \tau_\ell)$ of the ordered n-tuple $(t_1, \ldots, t_n)$ is obtained by inserting ℓ-1 semicolons in the expression. Let $m_i = |\tau_i|$ be the number of components of τ_i (so $\sum_{i=1}^{\ell} m_i = n$). We allow $m_i = 0$ (in which case we might think of τ_i as being a zero tuple or the empty set). We say that $(\tau_1; \ldots; \tau_\ell)$ is an increasing partition of $(t_1, \ldots, t_n)$ provided $t_1 \leqq t_2 \leqq \cdots \leqq t_n$ and if t is a component of τ_i and s is a component of τ_j with $i < j$ then either $t < s$ or $t = s$ and $i + m \leqq j$ where m is the multiplicity of t in τ_i.

We say that (1) is right $(m_1; \ldots; m_\ell)$-disfocal on J, $m_1 + \cdots + m_\ell = n$, $0 \leqq m_j \leqq n - j + 1$, provided there do not exist distinct solutions of

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(1) whose difference $u(x)$ satisfies $u^{(j-1)}(x)$ has m_j zeros at τ_j, $1 \leqq j \leqq \ell$ where $(\tau_1; \ldots; \tau_\ell)$ is any increasing sequence of n points in J with $m_j = |\tau_j|$, $1 \leqq j \leqq \ell$. Semicolons appear in $(m_1; \ldots; m_\ell)$ instead of comas to distinguish our concept from a similar but different concept used by Henderson [4], [5].

Let $\{n_j\}_{j=1}^{\ell}$ be a sequence of integers satisfying

$$n = n_1 > n_2 > \cdots > n_\ell \geqq 1. \tag{2}$$

Then let $\{m_j\}$ be a sequence of nonnegative integers such that

$$\begin{gathered} n = m_1 + \cdots + m_\ell, m_2 + \cdots + m_\ell \leqq n_2, \\ \cdots, m_{\ell-1} + m_\ell \leqq n_{\ell-1}, m_\ell \leqq n_\ell. \end{gathered} \tag{3}$$

We now define the disfocality function $\beta(t)$ as done by Muldowney [9]. Let $\{n_j\}_{j=1}^{\ell}$ be a sequence of integers satisfying (2), then define for $t \in J$

$$\begin{aligned} \beta(t) = \sup\{b > t\colon & L \text{ is right } (m_1; \ldots; m_\ell)\text{-} \\ & \text{disfocal on } [t, b] \text{ for all sequences} \\ & \{m_j\}_{j=1}^{\ell} \text{ satisfying (3)}\}. \end{aligned}$$

Examples. 1. If the sequence $\{n_j\}_{j=1}^{\ell}$ is the singleton sequence $\{n_1 = n\}$, then $\beta(t) = \eta_1(t)$ the first conjugate point function.

2. If the sequence $\{n_j\}_{j=1}^{\ell}$ is the sequence $\{n - k + 1\}_{k=1}^{n}$, then $\beta(t) = \nu_1(t)$ the first right focal point of t (see [8]).

The concept of disfocality and right focal point given here disagrees with that first used in the calculus of variations [1]. This change was initiated by Nehari [10]. Now in most of the literature right focal point is defined as Nehari did.

Let $y(x)$ be a solution of (1) and assume that (D) holds, then the linear differential equation

$$z^{(n)} = \sum_{i=0}^{n-1} f_i(x, y(x), \ldots, y^{(n-1)}(x)) z^{(i)} \tag{4}$$

is called [2] the variational equation (V.E.) of (1) along $y(x)$. We will use $\beta(t)$ when referring to equation (1) and we will use $\beta(t; y(x))$ when referring to equation (4). In [11] the author proved that if (A)–(D) hold and $a \in J$ then

$$\eta_1(a) = \inf\{\eta_1(a; y(x))\colon y(x) \text{ is a solution of (1)}\}.$$

The natural question is does (A)–(D) and $a \in J$ imply

$$\beta(a) = \inf\{\beta(a; y(x))\colon y(x) \text{ is a solution of (1)}\}? \tag{5}$$

Shortly we will give an example to show that this result is not true. One of our main results (Theorem 3) is that with an additional assumption (5) does hold. Sometimes we will denote the right hand side of (5) by

$$\inf_{y(x)} \beta(a; y(x)).$$

We now develop our example where (5) does not hold. In this example we will see that $\beta(0) = 2$ but $\inf \beta(0; y(x)) = 1$. We start out by looking at a linear equation used by Muldowney [8]. Choose $\phi_1(x)$ and $\phi_2(x)$ so that

$$\begin{aligned}
&\phi_1(2) = 0,\ \phi_1'(x) < 0,\ x \in (-1, 3),\\
&\phi_2(0) = \phi_2'(1) = \phi_2'(2) = 0,\ \phi_2'(x) > 0,\ x \in (-1, 1) \cup (1, 2),\\
&\phi_2'(x) < 0,\ x \in (2, 3).
\end{aligned}$$

Define $a(x)$ and $b(x)$ on $J \equiv (-1, 2)$ by

$$W[\phi_1, \phi_2, y]/W[\phi_1, \phi_2] = y'' - a(x)y' - b(x)y$$

where this equation involves Wronskians.

Muldowney [9] gave the differential equation $y'' = a(x)y' + b(x)y$ as an example where $\beta(0) = 1$ but $\beta(0+) = 2$. In our example this differential equation will turn out to be the V.E. along the trivial solution.

Now consider the nonlinear differential equation

$$y'' = a(x)y' + b(x)y + \operatorname{Arctan}(y')^3 + \operatorname{Arctan} y^3. \tag{6}$$

In (2), let $n = n_1 = 2 > n_2 = 1$ and consider the corresponding $\beta(0)$ for (6) (which would be the same as the first right focal point $\nu_1(0)$). It can be shown that (6) satisfies (A)–(D) with $J = (-1, 3)$. Let $y(x)$ be a solution of (6), then the variational equation along $y(x)$ is

$$\begin{aligned}
z'' = a(x)z' + b(x)z &+ (3[y'(x)]^2/(1 + [y'(x)]^6)z'\\
&+ (3[y(x)]^2/(1 + [y(x)]^6))z.
\end{aligned} \tag{7}$$

Note $y(x) \equiv 0$ is a solution of (6) and $\beta(0; 0) = 1$. We claim that if $y(x)$ is not the trivial solution of (6), then $\beta(0; y(x)) \geqq 2$.

Assume that our claim is not true, that is, there is a nontrivial solution $y(x)$ of (1) and $\beta(0; y(x)) < 2$. This implies there are points $0 \leqq c < d < 2$ such that the solution $z(x; y(x))$ of the IVP (4), $z(c) = 0$, $z'(c) = 1$ satisfies $z'(d; y(x)) = 0$. Without loss of generality $z'(x; y(x)) > 0$ on $[c, d)$. It follows that $z(x; y(x)) > 0$ on $(c, d]$.

Since $z(x; y(x))$ is a solution of the nonhomogeneous linear differential equation

$$z'' = a(x)z' + b(x)z + h(x)$$

where

$$h(x) = (3[y'(x)]^2/(1 + [y'(x)]^6)z'(x; y(x)) + (3[y(x)]^2/(1 + [y(x)]^6)z(x; y(x))$$

and if we let $z(x; 0)$ be the solution of the corresponding homogeneous equation satisfying the same initial conditions at c we have that

$$z(x; y(x)) = z(x; 0) + \int_c^x \mathfrak{K}(x, s)\, h(s)ds \tag{8}$$

where $\mathfrak{K}(x, s)$ is the Cauchy function for the D.E. $z'' = a(x)z' + b(x)z$. Since $y(x)$ is a nontrivial solution we have that $h(x) > 0$ on (c, d). Differentiating both sides of (8) and evaluating at d we get that

$$z'(d; y(x)) = z'(d, 0) + \int_c^d \mathfrak{K}_x(d, s)h(s)ds,$$

It follows from this that $z'(d; y(x)) > 0$ which is a contradiction.

Since $\beta(0; 0) = 1$ and $\beta(0; y(x)) \geqq 2$ for all nontrivial solutions $y(x)$ of (1) we have that $\inf_{y(x)} \beta(0; y(x)) = 1$.

But we now show that $\beta(0) = 2$. To this end we will first show that $\beta(0) \geqq 2$. To see this assume $\beta(0) < 2$. Then there are distinct solutions $y_1(x)$, $y_2(x)$ of (6) and points $0 \leqq c < d < 2$ such that $y_1(c) = y_2(c)$ and $y_1'(d) = y_2'(d)$. Without loss of generality we can assume that $y_2'(x) > y_1'(x)$ on $[c, d)$. It follows that $y_2(x) > y_1(x)$ on $(c, d]$.

Set $w(x) = y_2(x) - y_1(x)$, then $w(x)$ is a solution of

$$w'' = a(x)w' + b(x)w + k(x)$$

where

$$k(x) = (\text{Arctan}\,[y_2'(x)]^3 - \text{Arctan}\,[y_1'(x)]^3) + (\text{Arctan}\, y_2^3(x) - \text{Arctan}\, y_1^3(x)).$$

Since

$$w(c) = 0,$$
$$w'(c) = y_2'(c) - y_1'(c) \equiv \delta > 0,$$

we have that

$$w(x) = \delta z_c(x; 0) + \int_c^x \mathfrak{K}(x, s)\, k(s)ds \tag{9}$$

where $z_c(x; 0)$ is the solution of $z'' = a(x)z' + b(x)z$ with $z_c(c; 0) = 0$, $z_c'(c; 0) = 1$. By differentiating both sides of (9) and letting $x = d$ it is easy to see that $w'(d) > 0$ which is a contradiction. Hence $\beta(0) \geqq 2$.

To see that $\beta(0) = 2$ let $1 > \varepsilon > 0$ be given. For $h > 0$ sufficiently small we get that $h\phi_1 - \phi_2$ has an odd ordered zero in a small right hand neighborhood of 0 and $h\phi_1' - \phi_2'$ has an odd ordered zero in $(2, 2 + \varepsilon)$.

Now let $y_\delta(x)$ be the solution of (6) such that

$$y_\delta(0) = \delta[h\phi_1(0) - \phi_2(0)]$$
$$y_\delta'(0) = \delta[h\phi_1'(0) - \phi_2'(0)].$$

It follows from Theorem V-3.1, [2], that

$$\lim_{\delta\to 0} y_\delta^{(i)}(x)/\delta = h\phi^{(i)}(x) - \phi_2^{(i)}(x)$$

uniformly on compact subintervals of $(-1, 3)$ for $i = 0, 1$. Hence for $\delta \neq 0$, sufficiently small, $y_\delta(x)$ is a solution of (6) with a zero in a small right hand neighborhood of 0 and $y_\delta'(x)$ has a zero in $(2, 2 + \varepsilon)$. Hence $\beta(0) < 2 + \varepsilon$. It follows that $\beta(0) = 2$. Thus we have shown for the differential equation (6) that $2 = \beta(0) > \inf_{y(x)} \beta(0; y(x)) = 1$.

We will see in Theorem 3 that for (A)–(D) to imply (5) we will want to rule out the possibility of one of the variational equations having a solution like $\phi_2(x)$ in the above example. Before we state and prove our main result we have a preliminary lemma. As defined by Muldowney [9] we say that the linear differential equation

$$y^{(n)} = \sum_{i=0}^{n-1} p_i(x) y^{(i)}$$

has property I on an interval J_1 with respect to the sequence $\{n_j\}_{j=1}^{\ell}$ satisfying (2) provided there is a sequence of solutions $u_1, \ldots, u_n$ satisfying

$$W(u_1, \ldots, u_k) > 0, \quad k = 1, \ldots, n_1 = n,$$

for $j = 2, \ldots, \ell$-1 we have that

$$W(u_1^{(j-1)}, \ldots, u_k^{(j-1)}) > 0, \quad k = 1, \ldots, n_{j+1}$$
$$W(u_1^{(j-1)}, \ldots, u_k^{(j-1)}) \geqq 0, \quad k = n_{j+1} + 1, \ldots, n_j$$

and

$$W(u_1^{(\ell-1)}, \ldots, u_k^{(\ell-1)}) \geqq 0, \quad k = 1, \ldots, n_\ell$$

on J_1.

LEMMA 1. *Assume* (A)–(D) *hold and that for each solution* $y(x)$ *of* (1) *the* V.E. (4) *has property I on J, then* (1) *is right* $(m_1; \ldots; m_\ell)$ – *disfocal on J for all sequences* $\{m_j\}_{j=1}^{\ell}$ *satisfying* (3).

PROOF. By Theorem 1, [9] we get that for each solution $y(x)$ of (1) the corresponding V.E. (4) is right $(m_1; \ldots; m_\ell)$ – disfocal on J for all sequences $\{m_j\}_{j=1}^{\ell}$ satisfying (3). We will show that this implies the conclusion of Lemma 1.

Assume not. Then there is a sequence $\{m_j\}_{j=1}^{\ell}$ satisfying (3) such that

(1) is not right $(m_1; \ldots ; m_\ell)$ – disfocal on J. Order the allowable such sequences $\{m_j\}_{j=1}^{\ell}$ by the lexicographic ordering. Without loss of generality $\{m_j\}_{j=1}^{\ell}$ is the maximum sequence satisfying (3) such that (1) is not right $(m_1; \ldots ; m_\ell)$ – disfocal on J. It follows that there are distinct solutions $y_1(x)$, $y_2(x)$ of (1) and an increasing partition $(\tau_1; \ldots ; \tau_\ell)$ of n points $(t_1, \ldots, t_n)$ in J such that $m_i = |\tau_i|$, $1 \leqq i \leqq \ell$, $y_1^{(i-1)} - y_2^{(i-1)}$ has m_i zeros at τ_i, $1 \leqq i \leqq \ell$. By Theorem 1, **[11]**, $(n; 0; \ldots ; 0) > (m_1; \ldots ; m_\ell)$. We break the remainder of this proof into the two cases of $m_1 = n - 1$ and $m_1 < n - 1$.

Assume $m_1 = n - 1$. Then there is a k, $2 \leqq k \leqq \ell$, such that $m_k = 1$ and $m_j = 0$, $2 \leqq j \leqq \ell, j \neq k$. In this case $\tau_1 = (t_1, \ldots, t_{n-1})$, $\tau_k = (t_n)$, and $\tau_i = \phi$ (or zero tuple) for $2 \leqq i \leqq \ell$, $i \neq k$ and $y_1 - y_2$ has $n - 1$ zeros at τ_1 and $y_1^{(k-1)} - y_2^{(k-1)}$ has a zero at t_n. Let $u(x, s)$ be the solution of (1) such that $u(x, s) - y_1(x)$ has $n - 1$ zeros at τ_1 and $u^{(\alpha)}(t_{n-1}, s) = s$ where α is the number of times t_{n-1} occurs in τ_1 (If $\alpha = n - 1$, then the existence and uniqueness of $u(x, s)$ is guaranteed by (A), (B), otherwise the existence and uniqueness of $y(x, s)$ is guaranteed by the disconjugacy of (1) on J).

Now set $K = \{s$: there is such a solution $y(x, s)$ of (1)$\}$. If $t_1 = \cdots = t_{n-1}$ (the IVP case) then K is the real line. If $t_1 < t_{n-1}$, then the disconjugacy and (A)–(D) gives that K is the real line (see **[3]**).

Let s_1, s_2 be the distinct real numbers $s_1 = y_1^{(\alpha)}(t_{n-1})$ and $s_2 = y_2^{(\alpha)}(t_{n-1})$. If $t_1 = \cdots = t_{n-1}$ (the IVP case) the following steps follow from Theorem V-3.1, **[1]**; while if $t_1 < t_{n-1}$ the following steps follow from Theorem 8 **[12]**.

$$\begin{aligned} 0 &= y_2^{(k-1)}(t_n) - y_1^{(k-1)}(t_n) \\ &= u^{(k-1)}(t_n, s_2) - u_1^{(k-1)}(t_n, s_1) \\ &= (s_2 - s_1)\frac{\partial}{\partial s} u^{(k-1)}(t_n, \bar{s}) \\ &= (s_2 - s_1) z^{(k-1)}(t_n; u(x, \bar{s})), \end{aligned}$$

where $\bar{s}$ is between s_1 and s_2 and $z(x; u(x, \bar{s}))$ is the solution of the V.E. (4) along $u(x, \bar{s})$ such that $z(x; u(x, \bar{s}))$ has $n - 1$ zeros at τ_1 and $z^{(\alpha)}(t_{n-1}) = 1$. But from above we get that $z^{(k-1)}(t_n; u(x, \bar{s})) = 0$. This contradicts the right $(m_1; \ldots ; m_\ell)$ – disfocality of (4) along $u(x, \bar{s})$ (here $m_1 = n - 1, m_k = 1, m_j = 0, 2 \leqq j \leqq \ell, j \neq k$).

It remains to consider the case $m_1 < n - 1$. Pick ℓ_0 so that $m_{\ell_0} = |\tau_{\ell_0}| \geqq 1$ and $m_j = 0, j = \ell_0 + 1, \ldots, \ell$.

In this case let $u(x, s)$ be the solution of (1) such that $u^{(j-1)}(x, s) - y_1^{(j-1)}(x)$ has m_j zeros at $\tau_j, j = 1, \ldots, \ell_0 - 1$,

$$u^{(\ell_0-1)}(x, s) - y_1^{(\ell_0-1)}(x) \text{ has } m_{\ell_0} - 1 \text{ zeros at } \bar{\tau}_{\ell_0},$$

where $\bar{\tau}_{\ell_0}$ is τ_{ℓ_0} with its last component (which is t_n) removed, and

$$u^{(\alpha)}(t_{m_1}, s) = s$$

where α is the number of times t_{m_1} occurs in τ_1. The maximality of the sequence $\{m_j\}_{j=1}^{\ell}$ guarantees the uniqueness of $u(x, s)$ (i.e., we use here the fact that (1) is right $(m_1 + 1; m_2; \cdots ; m_{\ell_0} - 1; 0; \cdots; 0)$ – disfocal of J). In regard to the existence of $u(x, s)$ for various values of s set $k = \{s$: there is such a solution $u(x, s)\}$. Since $y_1^{(\alpha)}(t_{m_1}) \in K$, K is not the empty set. By an application of the Brouwer invariance of domain theorem, as in [3], or like Theorem 1, [13], K is an open set. Define $\bar{s}_i = y_i^{(\alpha)}(t_{m_1})$, $i = 1, 2$, and assume, without loss of generality, that $\bar{s}_2 > \bar{s}_1$. The claim is that $[\bar{s}_1, \bar{s}_2] \subset K$ (the same argument can be used to show that K is a connected set, but we use in this proof only that $[\bar{s}_1, \bar{s}_2] \subset K$). Assume not and define $s = \sup\{s \in A: [\bar{s}_1, s] \subset A\}$. Since A is open, $\bar{s}_1 < \delta < \bar{s}_2$ and $\delta \notin A$. To see that this leads to a contradiction assume $\{s_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of numbers in $[\bar{s}_1, \delta]$ with $\lim_{n\to\infty} s_n = \delta$. Consider the sequence $\{u_n(x) = u(x, s_n)\}_{n=1}^{\infty}$ of solutions of (1). If for some $\varepsilon > 0$, $\{u_n(x)\}_{n=1}^{\infty}$ is uniformly bounded on $[t_{m_1}, t_{m_1} + \varepsilon]$, then by (C) there is a subsequence $\{u_{n_k}(x)\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} u_{n_k}^{(i)}(x) = y^{(i)}(x)$$

uniformly on compact subsets of J, where $y(x)$ is a solution of (1) and $i = 0, 1, \ldots, n-1$. But it then would follow that

$$\delta = \lim_{n\to\infty} s_n = y^{(\alpha)}(t_{m_1})$$

is in K, which is a contradiction. Hence for any $\varepsilon > 0$, $\{u_n(x)\}_{n=1}^{\infty}$ can not be uniformly bounded on $[t_{m_1}, t_{m_1} + \varepsilon]$. It follows from this and the boundary conditions at t_{m_1} that $\{u_n^{(\alpha)}(x)\}_{n=1}^{\infty}$ is not uniformly bounded on $[t_{m_1}, t_{m_1} + \varepsilon]$ for any $\varepsilon > 0$. From this we get that for sufficiently large n, $u_n^{(\alpha)}(x)$ either touches $y_1^{(\alpha)}(x)$ or $y_2^{(\alpha)}(x)$ in a right hand neighborhood of t_{m_1}. To be specific we will assume $u_n^{(\alpha)}(x)$ first touches $y_2^{(\alpha)}(x)$ in a right hand neighborhood of t_{m_1} for infinitely many values of n. By taking subsequences and relabeling we can assume that for each n, $u_n^{(\alpha)}(x)$ touches $y_2^{(\alpha)}(x)$ at a first point, say z_n, to the right of t_{m_1} where $\{z_n\}_{n=1}^{\infty}$ is a decreasing sequence with limit t_{m_1}. Now

$$y_1^{(\alpha)}(x) < u_n^{(\alpha)}(x) < y_2^{(\alpha)}(x) \text{ on } [t_{m_1}, z_n)$$

and the boundary conditions at t_{m_1} imply that

$$y_1^{(j)}(x) \leqq u_n^{(j)}(x) \leqq y_2^{(j)}(x) \text{ on } [t_{m_1}, z_n)$$

for $0 \leqq j \leqq \alpha$. It follows that

$$\lim_{n\to\infty} u_n^{(j)}(z_n) = y_2^{(j)}(t_{m_1})$$

for $j = 0, \ldots, \alpha$. Recall that u_n and y_2 satisfied several common boundary conditions. It follows from the right $(m_1 + 1; m_2; \ldots; m_{\ell_0} - 1; 0; \ldots; 0)$ – disfocality (this is an application of the Brouwer invariance of domain theorem for this type of problem similar to Theorem 1, [13)] that

$$\lim_{n\to\infty} u^{(j)}(x, s_n) = y_2^{(j)}(x)$$

uniformly on compact subsets of J, $j = 0, \ldots, n-1$. In particular,

$$\lim_{n\to\infty} u^{(\alpha)}(t_{m_1}, s_n) = y_2^{(\alpha)}(t_{m_1}),$$

which contradicts

$$\lim_{n\to\infty} u^{(\alpha)}(t_{m_1}, s_n) = \lim_{n\to\infty} s_n = \delta.$$

Hence $[\bar{s}_1, \bar{s}_2] = [y_1^{(\alpha)}(t_{m_1}), y_2^{(\alpha)}(t_{m_1})] \subset A$.

Now using a theorem like Theorem 8, [12] or Theorem 3, [13] but for the right $(m_1 + 1; m_2; \ldots; m_{\ell_0} - 1; 0; \ldots; 0)$ – boundary value problem we get that (where β is the number of times t_n occurs in τ_{ℓ_0})

$$0 = y_2^{(\ell_0+\beta-2)}(t_n) - y_1^{(\ell_0+\beta-2)}(t_n),$$
$$0 = u^{(\ell_0+\beta-2)}(t_n, \bar{s}_2) - u^{(\ell_0+\beta-2)}(t_n, \bar{s}_1),$$
$$0 = (\bar{s}_2 - \bar{s}_1) \frac{\partial}{\partial s} u^{(\ell_0+\beta-2)}(t_n, \bar{s}),$$
$$0 = (\bar{s}_2 - \bar{s}_1)\, z^{(\ell_0+\beta-2)}(t_n; u(x, \bar{s})),$$

where $\bar{s}$ is between $\bar{s}_1$ and $\bar{s}_2$ and $z(x; u(x, \bar{s}))$ is the solution of the V.E. along $u(x, \bar{s})$ such that $z^{(i-1)}(x; u(x, \bar{s}))$ has m_i zeros at τ_i, $i = 1, \ldots, \ell_0 - 1$, $m_{\ell_0} - 1$ zero at τ_{ℓ_0}, and

$$z^{(\alpha)}(t_{m_1}; u(x, \bar{s})) = 1.$$

But from above we get that

$$z^{(\ell_0+\beta-2)}(t_n; y(x, \bar{s})) = 0.$$

This contradicts the right $(m_1; \ldots; m_\ell)$-disfocality of the V.E. along $u(x, \bar{s})$.

From the proof of Lemma 1 we get the following very useful result. This result reduces the disfocality of the nonlinear differential equation (1) to that of the linear differential equations (4). Using known uniqueness implies existence results one also gets the existence of solutions in various special cases (see Theorem 3, [5] and various results in [7]). An important application is that if f satisfies a uniform Lipschitz condition with respect to $y, y', \ldots, y^{(n-1)}$ then the compactness condition is satisfied and $\inf \beta(a; y(x))$ in this result can be replaced by a lower bound depending on the Lipschitz coefficients for certain boundary value problems (see [7]).

THEOREM 2. *Assume* (A)–(D) *hold and* $a \in J$, *then* $\beta(a) \geqq \inf \beta(a; y(x))$ *where the infimum is taken over all solutions of* (1).

PROOF. In the first two sentences of Lemma 1 we pointed out that the fact that for each solution $y(x)$ of (1) the V.E. is right $(m_1; \ldots; m_\ell)$-disfocal for all sequences $\{m_j\}_{j=1}^{\ell}$ satisfying (3) is what we used to prove the conclusion of Lemma 1.

THEOREM 3. *Assume* (A)–(D) *hold, then either there is a solution* $y(x)$ *of* (1) *such that the* V.E. (4) *has a nontrivial solution* $z(x)$ *satisfying the* $n + 1$ *boundary conditions*

$$z^{(j-1)}(a) = 0, \quad j = 1, \ldots, n - k,$$

$$z^{(j-1)}(b) = 0, \quad j = n - k + 1, \ldots, n + 1,$$

$a = b$ $(b = \beta(a; y(x)))$ *for some* k *satisfying* $n_{n-k+2} < k \leqq n_{n-k+1}$ *where* $1 \leqq n - k + 1 \leqq \ell$ $(n_{\ell+1} \equiv 0)$ *or* $\beta(a) = \inf \beta(a; y(x))$ *where the infimum is taken over all solutions* $y(x)$ *of* (1).

PROOF. From Theorem 2 we have that $\beta(a) \geqq \inf \beta(a; y(x))$. Assume $\beta(a) > \inf \beta(a; y(x))$. To complete the proof of this theorem we will show that this assumption leads to the existence of a solution $z(x)$ as described in the statement of this theorem.

First of all our assumption implies there is a solution $y(x)$ of (1) such that $\beta(a) > \beta(a; y(x))$. Set $b = \beta(a; y(x))$, then by Theorem 3, [9], there is an $h \in \{1, \ldots, \ell\}$ and a k satisfying $n_{h+1} < k \leqq n_h$ such that there is a nontrivial solution $z(x; y(x))$ of (4) satisfying

$$z^{(j-1)}(a) = 0, \quad j = 1, \ldots, n - k, \tag{10}$$

$$z^{(j-1)} = 0, \quad j = h, \ldots, h + k - 1. \tag{11}$$

A close look at the proof of Theorem 3, [9] shows that $z(x; y(x))$ is the essentially unique solution of (4) satisfying (1), (11) that $z^{(n-k)}(a; y(x)) \neq 0$, $h + k - 1 \leqq n$, and if $h + k - 1 < n$, then $z^{(h+k-1)}(b) \neq 0$. If $h + k - 1 = n$, then either $z^{(h+k-1)}(b; y(x)) = z^{(n)}(b; y(x))$ equals zero or is not zero. If $h + k - 1 = n$ and $z^{(n)}(b; y(x)) = 0$, then $z(x) \equiv z(x; y(x))$ satisfies the properties of $z(x)$ in the statement of the theorem and we are done. This leaves only the case where $z^{(h+k-1)}(b) \neq 0$ (whether $h + k - 1 < n$ or $h + k - 1 = n$). We now show that this leads to a contradiction.

For each $j = 1, \ldots, n$, let $z\,(x; y(x))$ be the solution of (4) satisfying

$$z_j^{(i-1)}(a;\, y(x)) = \delta_{ij}, \quad i = j, \ldots, n.$$

By the essential uniqueness of $z(x;\, y(x))$ mentioned above, we can assume without loss of generality (just multiply $z(x;\, y(x))$ by the appropriate nonzero constant) that for $k = 1$ $z(x;\, y(x)) = z_n(x;\, y(x))$ and for $k > 1$

$$z(x;\, y(x)) = \begin{vmatrix} z_{n-k+1}(x;\, y(x)) & \cdots & z_n(x;\, y(x)) \\ z_{n-k+1}^{(h-1)}(b;\, y(x)) & \cdots & z_n^{(h-1)}(b;\, y(x)) \\ \cdots & & \cdots \\ z_{n-k+1}^{(h+k-3)}(b;\, y(x)) & \cdots & z_n^{(h+k-3)}(b;\, y(x)) \end{vmatrix}$$

(here one uses the fact that the $(h + k - 2) - nd$ derivative of the right hand side of this last equation is zero at b by Theorem 3, **[9]**).

We will only complete the proof for the more complicated case where $k > 1$. For $\varepsilon > 0$, sufficiently small, define

$$z_\varepsilon(x;\, y(x)) = \begin{vmatrix} z_{n-k+1}(x;\, y(x)) & \cdots & z_n(x;\, y(x)) \\ z_{n-k+1}^{(h-1)}(b-\varepsilon;\, y(x)) & \cdots & z_n^{(h-1)}(b-\varepsilon;\, y(x)) \\ \cdots & & \cdots \\ z_{n-k+1}^{(h+k-3)}(b-\varepsilon;\, y(x)) & \cdots & z_n^{(h+k-3)}(b-\varepsilon;\, y(x)) \end{vmatrix}.$$

Note that $z_\varepsilon(x;\, y(x))$ is a solution of the V.E. (4) and

$$z_\varepsilon^{(j-1)}(a;\, y(x)) = 0, \qquad j = 1, \ldots, n-k,$$
$$z_\varepsilon^{(j-1)}(b-\varepsilon;\, y(x)) = 0, \quad j = h, \ldots, h+k-2.$$

Furthermore

$$\lim_{\varepsilon \to 0^+} z_\varepsilon^{(i)}(x;\, y(x)) = z^{(i)}(x;\, y(x))$$

uniformly on compact subsets of J, $i = 0, \ldots, n-1$.

Since $b = \beta(a;\ y(x)) > b - \varepsilon$, $z_\varepsilon^{(a+k-2)}(b - \varepsilon;\ y(x)) \neq 0$. Hence $z^{(h-1)}(x;\, y(x))$ has a zero at b of order exactly one more than the order of the zero of $z_\varepsilon^{(h-1)}(x;\ y(x))$ at $b - \varepsilon$. It follows from this and the uniform convergence that there is an $\varepsilon_0 > 0$ such that $z_{\varepsilon_0}^{(h-1)}(x;\, y(x))$ has an odd ordered zero at, say, x_0 where $b - \varepsilon_0 < x_0 < \beta(a)$.

Now since (1) is right $(m_1;\, \ldots;\, m_h)$-disfocal on $[a, x_0]$ with $m_1 = n - k$, $m_2 = \cdots = m_{h-1} = 0$, $m_h = k$ (this is like Theorem 1, **[13]**) we have for all $\delta > 0$, sufficiently small, that the BVP (1)

$$y_\delta^{(j-1)}(a) = y^{(j-1)}(a), \qquad j = 1, \ldots, n-k,$$
$$y_\delta^{(j-1)}(b-\varepsilon_0) = y^{(j-1)}(b-\varepsilon_0), \quad j = h, \ldots, h+k-2,$$
$$y_\delta^{(h+k-2)}(b-\varepsilon_0) = \delta z_{\varepsilon_0}^{(h+k-2)}(b-\varepsilon_0;\, y(x)) + y^{(h+k-2)}(b-\varepsilon_0)$$

has a solution $y_\delta(x)$. It follows from a result similar to Theorem 3, **[9]** for the type of boundary value problem considered here that

$$\lim_{\delta\to 0^+} \frac{y_\delta^{(j-1)}(x) - y^{(j-1)}(x)}{\delta} = z_{\varepsilon_0}^{(j-1)\varepsilon}(x; y(x))$$

uniformly on compact subsets of J, $j = 1, \ldots, n$. Since $z_{\varepsilon_0}^{(h-1)}(x; y(x))$ has an odd ordered zero at x_0 it follows that there is a $\delta_0 > 0$ such that

$$(y_{\delta_0}^{(h-1)}(x) - y^{(h-1)}(x))/\delta_0$$

has a zero (near x_0, which is between $b - \varepsilon_0$ and $\beta(a)$). But this implies that $y_{\delta_0} - y$ has a zero of order $n - k$ at a followed by k zeros of $y_{\delta_0}^{(h-1)} - y^{(h-1)}$ in $(a, \beta(a))$, $n_{h+1} < k \leqq n_h$, which contradicts the definition of $\beta(a)$. This completes the proof of this theorem.

One would like to know when one could rule out the existence of solutions like $z(x)$ in Theorem 3. For an example we refer to a result of Nehari. Nehari proved **[10]** that no nontrivial solution $y(x)$ of $Ly + p(x)y = 0$, where $p(x)$ is of one sign on an interval I and $Ly = 0$ is disconjugate on I, satisfies y has k zeros at c and the kth quasi derivative of y has $n - k + 1$ zeros at $d > c$, $c, d \in I$.

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