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MULTIPARAMETERED NONHOMOGENEOUS NONLINEAR EQUATIONS

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1. Introduction. Many problems in mathematical physics can be formulated as nonhomogeneous nonlinear equations involving one or more parameters. The purpose of this paper is to study the structure of the solution set of two classes of such operator equations where the nonlinear operator is compact.

The first class of operator equations is the single-parametered equation

$$u = \lambda F(\lambda, u) + z, \tag{1}$$

where λ is a real parameter, z is a fixed element of a Banach space, and the nonlinear operator F satisfies certain positivity conditions. For the special nonhomogeneous case with $z = 0$, but $F(\lambda, 0) \neq 0$, W. R. Derrick and H. J. Kuiper [2] have shown the existence of an unbounded continuum of positive solutions (λ, u). The first result extends their theorem to equation (1) and to an analogous multiparametered equation. Similar results for the homogeneous equation have been obtained by M. A. Krasnosel'skii [4], P. H. Rabinowitz [6], R. E. L. Turner [8] and others, in the single-parametered case; and by J. C. Alexander and J. A. Yorke [1] in the multiparametered case.

The second class of operator equations is given by the equation $u + F(u) = w$; where w is a fixed member of a Banach space. Assuming that $\|F(u)\| \leqq \alpha\|u\|$, $0 < \alpha < 1$, and that $I + F$ is locally one-to-one, C. Panchal has shown the existence of a solution [5]. Our second main result is a continuity theorem for this equation in the special case that $w = \sum_{i=1}^{n} \lambda_i z_i$, where λ_i is a real parameter and z_i is a fixed member of the Banach space for $1 \leqq i \leqq n$.

To illustrate possible applications of these results, we consider two problems in nonlinear elasticity. The first problem is the motion of an inelastic string with one endpoint free. Under the assumption that the string is acted on solely by forces of gravity and tension, this problem has been considered by I. I. Kolodner [3], using classical analysis, C. A. Stuart [6], using operator theory, and others. We consider the case that the string is also acted on by an external force and show the existence of

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a continuum of positive solutions of the corresponding boundary value problem.

The second problem is the motion of a simple pendulum acted on by electrostatic forces and an additional external force. D. W. Zachmann [9] has considered the problem and obtained local results. We show the existence of a unique unbounded branch of solutions.

2. Preliminaries.

2.1. Notation and definitions. Let $\mathbf{R}^{n+}$ denote the cone (see [4]) of all vectors in $\mathbf{R}^n$ with nonnegative components and C denote a cone in a real Banach space B. Then for $\lambda = (\lambda_i) \in \mathbf{R}^n$ with norm $|\lambda|$ and $u \in B$ with norm $\|u\|$, define $\mathscr{B} = \mathbf{R}^n \times B$ to be the Banach space with norm $\|(\lambda, u)\| = |\lambda| + \|u\|$ and with cone $\mathscr{C} = \mathbf{R}^{n+} \times C$. If $F = (F_i)$ represents a family of operators $F_i\colon \mathscr{D} \to B$, $1 \leqq i \leqq n$, with $\mathscr{D} \subseteqq \mathscr{B}$, then $\lambda \cdot F$ will denote the operator $\sum_{i=1}^{n} \lambda_i F_i\colon \mathscr{D} \to B$.

An operator is said to be compact if it transforms every bounded set into a (relatively) compact set.

A pair (λ_0, u_0) in $\mathscr{B}$ is called a solution of an operator equation $G(\lambda, u; z) = 0$, where $G(\cdot, \cdot; z)\colon \mathscr{B} \to B$ is some operator for z in B, if $G(\lambda_0, u_0; z) = 0$; it is called a positive solution if (λ_0, u_0) is also in $\mathscr{C}$.

A set is a continuum of solutions joining (λ_0, u_0) to ∞ if it is an unbounded, closed, connected set in $\mathscr{B}$ consisting of solutions and containing (λ_0, u_0) in $\mathscr{B}$.

If $u = u(\lambda)$ is a continuous function from a connected set in $\mathbf{R}^n$ into B, then $u(\lambda)$ is called a branch of solutions if $(\lambda, u(\lambda))$ is a solution for every λ in the domain of u. Note that if $u(\lambda)$ is a branch of solutions, then the closure of $\{(\lambda, u(\lambda))\}$ is a continuum of solutions.

The symbol $\partial\mathscr{O}$ represents the boundary of the set $\mathscr{O}$.

In the sequel we shall make use of the following two results.

THEOREM 2.2. [5, Th 3.4, pp. 34–36]. *Let $F\colon B \to B$ be a compact continuous operator. Suppose $I + F$ is locally one-to-one and $\|F(u)\| \leqq \alpha\|u\|$, where $0 < \alpha < 1$. Then for each w in B there exists exactly one u in B such that $u + F(u) = w$.*

The following lemma generalizes a result of Derrick and Kuiper [2, Th 2.5, pp. 181–182] to an analogous multiparametered equation.

LEMMA 2.3. *Let $F_i\colon \mathscr{C} \to C$, $1 \leqq i \leqq n$, be continuous operators (except possibly at $(0, 0)$) which are compact on $\mathscr{C} \cap \partial\mathscr{O}$ for every bounded open neighborhood $\mathscr{O}$ of $(0, 0)$. Then there exists a continuum of solutions of*

$$u = \lambda \cdot F(\lambda, u) \tag{2}$$

joining $(0, 0)$ to ∞, where $F = (F_i)$.

PROOF. Let $\mathbf{1}$ denote the vector (x_i) in $\mathbf{R}^n$ with $x_i = 1$, $1 \leqq i \leqq n$. Define $G: \mathscr{C} \to \mathscr{C}$ by

$$G(\lambda, u) = (\lambda + \mathbf{1}, (\lambda + \mathbf{1}) \cdot F(\lambda, u)).$$

Let $\mathcal{O}$ be a bounded open neighborhood of $(0, 0)$. Then G is continuous on $\mathscr{C} - (0, 0)$, compact on $\mathscr{C} \cap \partial\mathcal{O}$, and $\|G(\lambda, u)\| \geqq 1$ on $\mathscr{C} \cap \partial\mathcal{O}$. By Theorem 5.5 in [4], there exists (λ, u) in $\mathscr{C} \cap \partial\mathcal{O}$ such that for some $c > 0$, $G(\lambda, u) = c(\lambda, u)$. Since $c\lambda = \lambda + \mathbf{1}$, $c\lambda \cdot F(\lambda, u) = cu$. Thus, for every open neighborhood $\mathcal{O}$ of $(0, 0)$ there exists at least one positive solution (λ, u) of (2) on $\partial\mathcal{O}$. The existence of an unbounded continuum of positive solutions containing $(0, 0)$ now follows as in Theorem 2.5 in [2].

3. Continua of Solutions.

THEOREM 3.1. *Let z be a fixed element in B and let $\mathscr{C}^T$ denote the translate $\mathscr{C} + (0, z)$. Suppose $F = (F_i)$ represents the family of operators $F_i: \mathscr{C}^T \to C$, $1 \leqq i \leqq n$, that are continuous (except possibly at $(0, z)$) and compact on $\mathscr{C}^T \cap \partial\mathscr{S}$ for every bounded open neighborhood $\mathscr{S}$ of $(0, z)$. Then there exists a continuum of solutions of*

$$u = \lambda \cdot F(\lambda, u) + z \tag{3}$$

joining $(0, z)$ to ∞ in $\mathscr{C}^T$.

PROOF. Define a family of operators $F_z = (F_{zi})$ by

$$F_{zi}(\lambda, v) = F_i(\lambda, v + z). \tag{4}$$

Then $F_{zi}: \mathscr{C} \to C$, $1 \leqq i \leqq n$, are continuous (except possibly at $(0, 0)$) and compact on $\mathscr{C} \cap \partial\mathcal{O}$ for every bounded open neighborhood $\mathcal{O}$ of $(0, 0)$. Thus, by Lemma 2.3, there exists a continuum of solutions of $v = \lambda \cdot F_z(\lambda, v)$ joining $(0, 0)$ to ∞ in $\mathscr{C}$. Substitute $v = u - z$. The conclusion follows from (4).

When z is in C, it follows that $\mathscr{C}^T \subseteqq \mathscr{C}$; so we have the following corollary.

COROLLARY 3.2. *Suppose $F_i: \mathscr{C} \to C$, $1 \leqq i \leqq n$, are compact continuous with z in C fixed. Then there exists a continuum of positive solutions of* (3) *joining* $(0, z)$ *to* ∞.

The following result is a continuity theorem for a special case of the nonhomogeneous equation $u + F(u) = w$.

THEOREM 3.3. *Let $F: B \to B$ be a compact continuous operator such that $I + F$ is locally one-to-one. Suppose*

$$\|F(u)\| \leqq \alpha\|u\|, \ 0 < \alpha < 1. \tag{5}$$

Then for each linearly independent set $\{z_i\}_{i=1}^n$ *in* B, *there exists a unique branch* $u(\lambda)$ *of solutions of*

$$(6) \qquad u + F(u) = \lambda \cdot z,$$

such that $u(\lambda)$ *is one-to-one on* $\mathbf{R}^n$, $u(0) = 0$, *and* $u(\lambda) \to \infty$ as $|\lambda| \to \infty$.

PROOF. Let $\{z_i\}_{i=1}^n$ be a fixed linearly independent set in B. By Theorem 2.2 and linear independence, for every λ in $\mathbf{R}^n$ there exists a unique u in B satisfying (6). Let S be the set $(I + F)^{-1}(\{\lambda \cdot z: \lambda \in \mathbf{R}^n\})$. By continuity S is closed.

Suppose (λ, u) satisfies (6). From (5) and (6),

$$(7) \qquad (1 - \alpha)\|u\| \leqq \textstyle\sum_{i=1}^n |\lambda_i| \|z_i\| \leqq k(1 + \alpha)\|u\|$$

for some k independent of λ. Thus, a subset of S is bounded if and only if the corresponding set in $\mathbf{R}^n$ is bounded.

We now show that closed and bounded subsets of S are compact. Let $\{u^k\}$ be a bounded sequence in S. The corresponding set $\{\lambda^k\}$ is bounded from (7) and, thus, has a subsequence converging to λ in $\mathbf{R}^n$. Since F is compact, a subsequence of $\{F(u^k)\}$ converges to w in B. Re-indexing, we see that

$$u^k = \lambda^k \cdot z - F(u^k) \to \lambda \cdot z - w.$$

Since S is closed, $\lambda \cdot z - w$ is in S.

Let $A_r = \{u \in S: \|u\| \leqq r\}$. Since A_r is a closed and bounded subset of S, A_r is compact. For every λ in $\mathbf{R}^n$ such that $\|\lambda \cdot z\| \leqq (1 - \alpha)r$, (7) implies that u is in A_r. Therefore, the set $(I + F)(A_r)$ contains the finite dimensional ball $X = \{\lambda \cdot z: \|\lambda \cdot z\| \leqq (1 - \alpha)r\}$.

The mapping $I + F: A_r \to (I + F)(A_r)$ is one-to-one by the uniqueness of solutions of (6). Since $I + F$ is also an onto and continuous mapping on a compact metric space A_r, $(I + F)^{-1}$ is continuous. Thus, $(I + F)^{-1}(X)$ determines a component in A_r containing 0; that is, by continuity there exists a unique branch $u(\lambda)$, for $\|\lambda \cdot z\| \leqq (1 - \alpha)r$, such that $u(0) = 0$. Since r was arbitrary, the branch $u(\lambda)$ exists for all λ, is one-to-one by (6), and $u(\lambda) \to \infty$ as $|\lambda| \to \infty$ by (7).

From (7) obvious upper and lower bounds for the branch of solutions of (6) can be obtained.

If $\{z_i\}_{i=1}^n$ is linearly dependent, then extract the maximal independent subset and apply Theorem 3.3 for the obvious analog.

4. Applications.

4.1. Consider the motion of an inelastic flexible string of length L of uniform cross-section with one endpoint free. Assume that the string is acted on by some external force in addition to the forces of gravity and tension.

4.1.1. Formulation of the nonlinear boundary value problem. We assume that the fixed endpoint is at the origin of a rectangular coordinate system and that gravity acts in the direction of the positive z-axis. Let s be the arclength measured from the free endpoint, $\mathbf{x}(s, t) = (x(s, t), y(s, t), z(s, t))$ be the position vector and $T(s, t)$ be the tension in the string at the point s at time t, and ρ be the mass per unit length. Then the equations of motion can be written

$$(\rho \mathbf{x}_t)_t = \rho \mathbf{g} + (T\mathbf{x}_s)_s + \mathbf{f}, \tag{8}$$

where $\mathbf{g} = (0, 0, g)$ is the acceleration due to gravity vector and $\mathbf{f}$ is an external forcing vector. In addition, since the string is inelastic, we have the constraint

$$|\mathbf{x}_s|^2 = 1. \tag{9}$$

The boundary conditions are

$$\mathbf{x}(L, t) = 0 \text{ and } T(0, t) = 0. \tag{10}$$

We assume that the external force $\mathbf{f}$ has the form

$$\mathbf{f}(s, t) = (f(s) \cos \omega t, f(s) \sin \omega t, 0), \tag{11}$$

where ω is a constant and f is continuously differentiable. Since the string is fixed at $s = L$, we have

$$f(L) = 0. \tag{12}$$

We seek only those motions in which the string rotates with angular velocity ω about the z-axis. Assuming that such motion exists, we have

$$\mathbf{x}(s, t) = (x(s) \cos \omega t, x(s) \sin \omega t, z(s)) \tag{13}$$

and

$$T(s, t) = T(s). \tag{14}$$

Substitute (11)–(14) into (8)–(10) to obtain

$$-\rho\omega^2 x(s) = (T(s)\,x'(s))' + f(s). \tag{15}$$

$$0 = \rho g + (T(s)z'(s))', \tag{16}$$

$$x'(s)^2 + z'(s)^2 = 1, \tag{17}$$

and

$$T(0) = 0, \; x(L) = z(L) = 0. \tag{18}$$

Applying the technique of Kolodner [3, p. 396], we reduce these three ordinary differential equations to a single second order equation

$$-u''(s^*) = \lambda u(s^*)(u(s^*)^2 + s^{*2})^{-1/2} + L^* f^{*\prime}(s^*) \tag{19}$$

for the variable $u(s^*) = T(s)x'(s)/L\rho g$, where $s^* = s/L$, $f^*(s^*) = f(s)/L$, $L^* = L/\rho g$, and $\lambda = \omega^2 L/g$. From (15)–(18) and this change of variables, the boundary conditions become

$$u(0) = u'(1) = 0. \tag{20}$$

We note that in Kolodner's formulation, T, x, and z are expressed in terms of s and u so that solving (19)–(20) for u does solve (15)–(18).

4.1.2. Formulation of the operator equation. Invert the operator $-d^2/ds^2$ with boundary conditions (20) to obtain the integral equation

$$u(s) = \lambda \int_0^1 k(s, t)u(t)(u(t)^2 + t^2)^{-1/2}\, dt + \int_0^1 k(s, t)Lf'(t)dt \tag{21}$$

where the Green's function is given by

$$k(s, t) = \begin{cases} s, & s \leqq t. \\ t, & s \geqq t. \end{cases}$$

Let B denote the Banach space $\{u \in C^1[0, 1]: u(0) = 0\}$ with norm $\| \ \|_1$. Define K: $C[0, 1] \to B$ by $Ku(s) = \int_0^1 k(s, t)u(t)dt$ and N: $B \to C[0, 1]$ by

$$N(u)(s) = \begin{cases} s^{-1}u(s)/[s^{-2}u(s)^2 + 1]^{1/2}, & 0 < s \leqq 1. \\ u'(0)/[u'(0)^2 + 1]^{1/2}, & s = 0. \end{cases}$$

Note that $\lim_{s \to 0} u(s)/s = \lim_{s \to 0} u'(s) = u'(0)$. Thus, N is bounded and continuous. Since K is a compact linear operator, the operator $F = K \circ N$: $B \to B$ is compact continuous. Let C be the cone of nonnegative functions in B. Then by definition, F: $C \to C$. Thus, (21) becomes the compact nonhomogeneous equation

$$u = \lambda F(u) + z, \tag{22}$$

where $z(s) = \int_0^1 k(s, t)Lf'(t)dt$. We emphasize that equations (22) and (2) are not of the same form due to the fact that the nonhomogeneity z is independent of λ; thus, the Derrick and Kuiper result [2] does not apply.

4.1.3. Existence of a continuum. Since all the hypotheses of Corollary 3.2 are satisfied, there exists a continuum of positive solutions in $\mathscr{C}$ $(n = 1)$ of (22) joining $(0, z)$ to ∞. Since u satisfies (21), direct calculation shows that $u'(1) = 0$. Also, since u is in $C^1[0, 1]$ and in the range of the integral operator $K \circ N + z$, u is twice-continuously differentiable. Thus, the continuum is one of positive solutions of (19)–(20).

We remark that the application of such an abstract result as Corollary 3.2 is only the beginning in the analysis of a physical problem. The next step is to determine some properties of this continuum; for instance, does the continuum exist for all $\lambda > 0$ or does it approach ∞ as λ approaches some finite value? In the following application, we are able to answer such questions.

4.2. As an application of our second main result, we consider the motion of a simple pendulum under hypotheses proposed by D. W. Zachmann [9].

4.2.1. Formulation of the nonlinear boundary value problem. Suppose the pendulum has unit length and has a bob of unit mass which carries a unit of positive charge. Assume that the pendulum is supported at the origin and is free to swing in the xz plane. Let $u = u(t)$ measure the deflection from the positive z-axis. Assume that there are two wires of infinite length fixed on the lines $x = \pm d$, $d > 1$, and that the wires carry c units of positive charge per unit length. In addition, assume there is an external force whose tangential component is

$$f(t) = \lambda_1 \sin t + \lambda_2 \sin 2t.$$

We seek periodic solutions. Thus, the boundary value problem is

$$-u'' = g \sin u + (2c \sin 2u)/(d^2 - \sin^2 u) + f \tag{23}$$

$$u(0) = u(\pi) = 0. \tag{24}$$

See [9, p. 899] for further details.

4.2.2. Formulation of the operator equation. Let B denote the Banach space $\{u \in C'[0, \pi]: u(0) = u(\pi) = 0\}$ with the sup norm. Define K: $C[0, \pi] \to B$ by $Ku(s) = \int_0^\pi k(s, t)u(t)\, dt$, where the Green's function $k(s, t) = (s + t)/2 - |s - t|/2 - st/\pi$, $0 \leqq s, t \leqq \pi$, and N: $B \to C[0, \pi]$ by

$$N(u)(s) = g \sin u(s) + (2c \sin 2u(s))/(d^2 - \sin^2 u(s)).$$

Since $d > 1 \geqq \sin u(s)$ for all s, N is bounded and continuous. Since K is a compact linear operator, the operator $F = K \circ N$: $B \to B$ is compact continuous. Thus, inverting the operator $-d^2/ds^2$ with boundary conditions (24), we obtain the corresponding nonhomogeneous operator equation (6), where

$$z_m(s) = \int_0^\pi k(s, t) \sin mt\, dt, \; m = 1, 2.$$

4.2.3. Existence of a branch. We now show that the operator F satisfies the hypotheses of Theorem 3.3. First, note that the Frechet derivative of F at w is given by $F'_w(u) = \int_0^\pi k(s, t)N'(w(t))u(t)dt$, where $N'(u) = g \cos u + (4c \cos 2u)/(d^2 - \sin^2 u) + (2c \sin^2 2u)/(d^2 - \sin^2 u)^2$.

If

$$\|F'_w\| = \pi(g + 2c(2d^2 - 1)/(d^2 - 1)^2) < 1, \tag{25}$$

$I + F'_w$ is invertible. Thus, by a lemma of C. Panchal [5, p. 38], $I + F$ is locally one-to-one. Second,

$$\|F(u)\| = \|K\circ N(u)\| \leqq \pi(g + 2c/(d^2 - 1))\|u\|.$$

If

$$\alpha = \pi(g + 2c/(d^2 - 1)) < 1, \tag{26}$$

inequality (5) is satisfied. (Note that $g < 1/\pi$).

Therefore, if the physical parameters g, c, and d satisfy inequalities (25) and (26), by Theorem 3.3 there exists a unique branch of solutions $u = u(\lambda)$: $\mathbf{R}^2 \to B$ of (6) such that $u(\lambda)$ is one-to-one on $\mathbf{R}^2$, $u(0) = 0$, and $u(\lambda) \to \infty$ as $|\lambda| \to \infty$. Since for each λ, $u(\lambda)$ is in $C'[0, \pi]$ and in the range of the operator $-K\circ N + \lambda\cdot z$, $u(\lambda)$ is a twice-continuously differentiable function. Thus, $u(\lambda)$ is a solution of (23)–(24) for each λ. Obviously, the same conclusion follows if $f(t) = \sum_{m=1}^{n} \lambda_m \sin mt$.

We remark that it is possible to reformulate (23)–(24) as a multiparametered operator equation of form (3) and apply Corollary 3.2.

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