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SOME PROPERTIES OF RELATIVE PRINCIPAL COFIBRATIONS

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ABSTRACT. This paper studies the basic homotopy properties of relative principal cofibrations including a special suspension needed to extend the short exact sequences for relative mapping cones and attaching maps to a long exact sequence. Useful in classifying relative extensions.

Introduction. §1 contains the definition of a relative principal cofibration, an exact sequence of sets of homotopy classes and a group action useful in enumerating homotopy classes of extensions of a given map. This one sequence can be applied to all types of homotopy classes of extensions, from those relative to the domain of the map being extended to those relative to the base point. From this sequence and group actions, we can recover the result of Barcus-Barratt [1] as a special case. The results of §1 are dual to those of McClendon [5] and hence the proofs will be omitted. Complete details are given in Kruse [3].

A suspension operation $\bar{\Sigma}$ is defined in §2 and the homotopy equivalence between the relative mapping cone of $\bar{\Sigma} f$ and the suspension of the mapping cone of f is proven. Also, the exactness of the sequence

$$\cdots \leftarrow \bigoplus_i \pi_{r(i)+k}(Z) \xleftarrow{(\bar{\Sigma}^k f)^*} \pi_{n+k}(Z) \xleftarrow{i} [K(\bar{\Sigma}^k f), Z]^D$$
$$\leftarrow \bigoplus_i \pi_{r(i)+k+1}(Z) \xleftarrow{(\bar{\Sigma}^{k+1} f)^*} \pi_{n+k+1}(Z)$$

can be obtained as a special case of Corollary 2.10, where

$$f = \langle f_i \rangle : S^{r(1)} \vee S^{r(2)} \vee \cdots \vee S^{r(w)} \to D \vee S^n,$$

$\pi = \langle \mathrm{id}, * \rangle : D \vee S^n \to D$ and $\pi \circ f_i \sim 0$ for each i. $\bar{\Sigma}^k f = \langle \bar{\Sigma}^k f_i \rangle$ where $\bar{\Sigma}^k f_i$ is the k-fold one-sided suspension of f_i: $S^{r(i)} \to D \vee S^n$ as defined in (2.3). $K(\bar{\Sigma}^k f)$ is the ordinary mapping cone of $\bar{\Sigma}^k f$ and $f^*(\alpha)$ means $f^*\langle \check{z}, \alpha \rangle$ where $\check{z} : D \to Z$ is a fixed map. This sequence plays a central role in [4].

NOTATION. Top$(C \to D)$ will denote the category, whose objects are triples $(X, \check{x}, \hat{x})$ where $\check{x}: C \to X$, $\hat{x}: X \to D$ are continuous and $\hat{x}\check{x} = u$:

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$C \to D$ is a fixed map. The morphisms of $\mathrm{Top}(C \to D)$ are continuous functions f: $(X, \check{x}, \hat{x}) \to (Y, \check{y}, \hat{y})$ such that $\check{y} = f\check{x}$ and $\hat{y}f = \hat{x}$. Unless we wish to emphasize specific structure maps $\check{x}$ and $\hat{x}$, the object $(X, \check{x}, \hat{x})$ will be denoted by X. If $C = D$ and $u = \mathrm{id}$, then $\mathrm{Top}(C \to D) = \mathrm{Top}(D = D)$ will be denoted by Top D. The homotopy classes of maps from X to Y in $\mathrm{Top}(C \to D)$ will be denoted by $[X, Y]^C_D$.

The cone construction in Top C is defined by

$$K_C X = \frac{X \times I}{R}$$

where R is the equivalence relation generated by

a) $(\check{x}(c), t) \sim (x(\check{c}), t')$ $\quad c \in C,\ t, t' \in I$, and

b) $(x, 0) \sim (x', 0)$ $\quad$ if $\hat{x}(x) = \hat{x}(x')$.

The equivalence class of (x, t) is $[x, t]$ and i_1: $X \to K_C X$: $x \to [x, 1]$ is the natural embedding of X in $K_C X$.

The suspension of X in Top C is defined by

$$\Sigma_C X = \frac{X \times I}{R}$$

where R is the equivalence relation generated by

a) $(\check{x}(c), t) \sim (\check{x}(c), t')$ $\quad$ for $\quad c \in C,\ t, t' \in I$, and

b) $(x, i) \sim (x', i)$ $\quad$ if $\quad \hat{x}(x) = \hat{x}(x'),\ i = 0, 1$.

If f: $X \to Y \in$ Top C, then the suspension of f is Σf: $\Sigma_C X \to \Sigma_C Y$ defined as usual.

If $f: A \to X$ and g: $B \to X$ are two maps in $\mathrm{Top}(C \to D)$, then $\langle f, g\rangle : A +_C B \to X$ denotes the join of the two maps, where $A +_C B$ is the pushout of $\check{a}$: $C \to A$ and $\check{b}$: $C \to B$.

For other constructions and results in this category the reader is referred to Kruse [3] or McClendon [6].

1. The spaces and maps referred to in this section, unless otherwise specified, are those in the following $\mathrm{Top}(C \to D)$ diagram:

$$\begin{tikzcd}
C \arrow[r, "u"] & D & \\
K_C A \arrow[u] \arrow[r, "\bar{f}"] & RK(f) \arrow[u] \arrow[dr, dashed, "h"] & \\
A \arrow[u, "i_1"] \arrow[r, "f"] & B \arrow[u, "i"] \arrow[r, "k"] & Z \\
C \arrow[u, "\check{a}"] \arrow[r] & E \arrow[u, "b"] &
\end{tikzcd}$$

DEFINITION 1.1. Let $A \in$ Top C and f: $A \to B \in \mathrm{Top}(C \to D)$. The pushout of i_1: $A \to K_C A$ and f: $A \to B$ in Top$(C \to D)$ is denoted by $RK(f)$. The map i: $B \to RK(f)$ is called the *relative principal cofibration induced by f*. The map f is the *characteristic map*, and $RK(f)$ is called the *relative mapping cone*.

In terms of spaces and relations, $RK(f) = \dfrac{(A \times I) \bigcup B}{R}$

where R is the equivalence relation generated by

a) $(\check{a}(c), t) \sim (\check{a}(c), t')$ if $c \in C$, $t, t' \in I$,

b) $(a, 0) \sim (a', 0)$ if $\hat{a}(a) = \hat{a}(a')$, $a, a' \in A$, and

c) $(a, 1) \sim f(a)$.

If $u = \mathrm{id}$: $C \to D$, then the relative principal cofibration is just the principal cofibration of f in Top C denoted by $K_C(f)$.

DEFINITION 1.2. Let $\check{\Sigma}_E B$ be the Top B Co-H-Space defined by $\check{\Sigma}_E B = B \times I/R$ where R is the equivalence relation generated by

a) $(b(e), t) \sim (b(e), t')$ $e \in E$, $t, t' \in I$, and

b) $(b, 0) \sim (b, 1)$ $b \in B$.

$\check{\Sigma}_E B$ is called the *free suspension of B*, so called because it is adjoint to the space of free loops.

Let $\delta = \delta(h, f)$: $[\check{\Sigma}_E B, Z]_D^B \to [\Sigma_C A, Z]_D^C$ be the homomorphism defined by $\delta[v] = [h\bar{f} + v\Sigma f - h\bar{f}]$ where $+$ and $-$ are used in the sense of adding paths.

THEOREM 1.3. a) *The sequence*

$$[A, Z]_D^C \xleftarrow{f^*} [B, Z]_D^E \xleftarrow{i^*} [RK(f), Z]_D^E$$

is an exact sequence of sets.

b) $[\Sigma_C A, Z]_D^C$ *acts on* $[RK(f), Z]_D^E$ *and the action is transitive on the set* $i^{*-1}[k]$ *for any* $[k] \in [B, Z]_D^E$.

c) *The stability subgroup of the action of* $[\Sigma_C A, Z]_D^C$ *on* $[h]$ *is the image of* $\delta(h, f)$.

d) $[\Sigma_C A, Z]_D^C/\mathrm{Im}\ \delta(h, f) \leftrightarrow i^{*-1}(k) = \{$*extensions of k rel E*$\}$.

PROOF. a) Standard.

b) Action is defined as in Puppe sequence and the proof of transitivity is straightforward.

c) Dual to McClendon [5].

d) (Group/stability subgroup) $\leftrightarrow$ Orbit.

The results of this section allow us to classify homotopy extensions of k: $B \to Z$ to $RK(f) \to Z$ relative to E. Since $B \to RK(f)$ is a Top $C \to D$ cofibration, each class of homotopy extensions contains an actual extension; so the elements of $i^{*-1}[k]$ give the number of actual extensions relative to E.

SPECIAL CASES 1.4. a) If $C = D = E = *$, we get the ordinary reduced mapping cone of f and base pointed classes of extensions.

b) If $E = B$, we get classes of extensions relative to B. In this case Theorem 1.3(d) becomes

$$\{\text{extensions of } k \text{ rel } B\} = i^{*-1}[k] = [RK(f), Z]_D^B \leftrightarrow [\Sigma_C A, Z]_D^C.$$

c) If $A = S^n$, $C = D = *$ and $E = B = K$, then δ can be shown to be naturally equivalent to the homomorphism α_R found in Barcus-Barratt **[1]**, and in this case we recover their result.

As an illustration we will use Theorem 1.3(d) to enumerate the H-space structures on S^n. These results are already well known.

EXAMPLE 1.5. Let $A = S^{2n-1}$, $E = B = S^n \vee S^n$ and $f = [\iota_n, \iota_n] =$ Whitehead product, so $RK(f) = K(f) = S^n \times S^n$. Let $k = \langle \mathrm{id}, \mathrm{id}\rangle$: $S^n \vee S^n \to S^n$. Any extension of k would give an H-space structure to S^n. Since $k \circ f = [\iota_n, \iota_n]$: $S^{2n-1} \to S^n$ is null-homotopic if and only if $n = 1, 3$ or 7, we get extensions in these three cases only and the extensions correspond one-to-one to $[\Sigma_C A, Z]_C^C = \pi_{2n}(S^n)$. So if $n = 1$, 3 or 7, we get 1, 12 or 120 H-space structures resp. More generally, let $X \in \mathrm{Top}\ C$. X is a Top C H-space if there is a product map m: $X \times_C X \to X$, which extends the folding map ∇: $X +_C X \to X$. If X is a Top C suspension, then there is a Top C Whitehead map w such that $X \times_C X$ is the Top C mapping cone of w: $X' *_C X' \to X +_C X$, where $\Sigma_C X' = X$ and $*_C$ denotes the Top C join. If a product map m exists for X, then the number of such products is given by Theorem 1.3d;

$$[\text{Top } C \text{ products on } X] \leftrightarrow [\Sigma_C(X' *_C X'), X]_C^C \approx [X \wedge_C X, X]_C^C$$

since $\Sigma_C(X' *_C X') \equiv X \wedge_C X$.

EXAMPLE 1.6. If $X = X' \vee C$ where X' is a Top $*$ H-space, then

$$\begin{aligned}[\text{Top } C \text{ products on } X] \leftrightarrow [X \wedge_C X, X]_C^C &= [X' \wedge_* X', X']_*^* \\ &= [\text{Top}^* \text{ products on } X'].\end{aligned}$$

In this case no new products are obtained.

EXAMPLE 1.7. If $X = X' \times C$ where X' is a Top* H-space, then

$$[\text{Top } C \text{ products on } X] \leftrightarrow [X \wedge_C X, X]_C^C \leftrightarrow [X' \wedge X' \times C, X']^C.$$

For example if $X' = S^3$ and $C = S^1$, then

$$\begin{aligned}[S^3 \wedge S^3 \times S^1, S^3]^{S^1,*} = [S^6 \times S^1, S^3]^{S^1,*} &= [S^6 \times S^1/* \times S^1, S^3]^* \\ &= [S^6 \vee S^7, S^3]^* = [S^6, S^3] \times [S^7, S^3] \\ &= \mathbf{Z}_{12} \times \mathbf{Z}_2.\end{aligned}$$

Therefore we obtain some additional products on $S^3 \times S^1$.

2. As in Top*, homotopically equivalent maps induce homotopically equivalent mapping cones. The proof of this fact in our category is similar to the proof for Top* but care must be taken to keep straight which maps and homotopies are in which category. To do this consider the following diagram:

$$
\begin{tikzcd}
 & & A \arrow[dll, "\hat{a}"'] \arrow[r, "f"] & B \arrow[r, "i"] & RK(f) \\
F & D \arrow[l, "\hat{d}"'] & & & \\
 & & A' \arrow[ul, "\hat{a}'"] \arrow[uu, "u"] \arrow[r, "f'"] & B' \arrow[uu, "v"] \arrow[r, "j"] & RK(f') \arrow[uu, dotted, "w"]
\end{tikzcd}
\tag{2.1}
$$

where A, $A' \in \text{Top}\, D$ and B, B', u, v, f, f', H: $fu \sim vf'$, K: $\hat{a}' \sim \hat{a}u$ are in $\hat{d}$: $\text{Top}\, D \to F$. $RK(f)$ and $RK(f')$ are relative principal cofibrations of f and f' in $\text{Top}(D \to F)$. Define $w = w(u, v, H, K)$: $RK(f') \to RK(f)$ by

$$
w(a', t] = \begin{cases} [\check{a}K(a', 3t), 0] & 0 \leqq t \leqq 1/3 \\ [u(a'), 3t - 1] & 1/3 \leqq t \leqq 2/3 \\ [H(a', 3t - 2)] & 2/3 \leqq t \leqq 1 \end{cases}
$$

$$
w[b'] = [v(b')] \qquad b' \in B'.
$$

$w \in \text{Top}(D \to F)$ and makes diagram 2.1 commute.

THEOREM 2.2. *Under the conditions of* (2.1), *if u and v are homotopy equivalences in* $\text{Top}(D \to F)$, *then so is $w = w(u, v, H, K)$. If in addition, $v = \text{id}$, then w is a homotopy equivalence in* $\text{Top}(B \to F)$.

DEFINITION 2.3. Let $A \in \text{Top}\ C$, $B \in \text{Top}\ D$ and $f \in \text{Top}(C \to F)$ in the following diagram:

$$
\begin{tikzcd}
C \arrow[r, "\check{d}"] \arrow[d, "\check{a}"'] & D \arrow[d, "\check{b}"] & \\
A \arrow[r, "f"] \arrow[d, "\hat{a}"'] & B \arrow[d, "\hat{b}"] & \\
C \arrow[r, "d"'] \arrow[ur, phantom, "H"] & D \arrow[r, "\hat{d}"'] & F
\end{tikzcd}
\tag{2.4}
$$

where the top square commutes and H: $\check{d}\hat{a} \sim \hat{b}f$ in $\text{Top}(C \to F)$.

Since $A \in \text{Top}\ C$ and $B \in \text{Top}\ D$, their suspensions $\Sigma_C A$ and $\Sigma_D B$ are defined. However since the bottom square is only homotopy commutative, the usual definition of Σf is not well defined. If we take the homotopy into account and define $\bar{\Sigma} f$ as follows:

$$
\bar{\Sigma} f[a, t] = \begin{cases} [\check{b}H(a, 3t), 0] & 0 \leqq t \leqq 1/3 \\ [f(a), 3t - 1] & 1/3 \leqq t \leqq 2/3 \\ [\check{b}H(a, 3 - 3t), 0] & 2/3 \leqq t \leqq 1, \end{cases}
$$

we get a well defined map such that in the following diagram, the top square commutes and $H'\colon \check{d}\hat{a} \sim \hat{b}\bar{\Sigma} f$ in Top$(C \to F)$:

$$\begin{tikzcd}
C \arrow[r, "\check{d}"] \arrow[d, "\check{a}"'] & D \arrow[d, "\check{b}"] & \\
\Sigma A \arrow[r, "\bar{\Sigma} f"] \arrow[d, "\hat{a}"'] \arrow[dr, phantom, "H'"] & \Sigma_D B \arrow[d, "\hat{b}"] & \\
C \arrow[r, "\check{d}"'] & D \arrow[r, "\hat{d}"'] & F
\end{tikzcd}$$

If $H =$ constant in (2.2), then the regular definition of suspension Σf, $(\Sigma f[a, t] = [f(a), t])$ works and $\bar{\Sigma} f$ is homotopic in Top$(C \to D)$ to Σf. $\bar{\Sigma} f$ is called the *one-sided suspension* of f since if $X \in$ Top$(C \to F)$ and $B = D + {}_C X$, then

$$\bar{\Sigma} f\colon \Sigma_C A \to \Sigma_D(D + {}_C X) = D + {}_C \Sigma_C X.$$

In the special case $A = \Sigma_* A'$ and $C = * = F$, the one sided suspension may be naturally defined in terms of exact sequences of the pairs $(K_D B, B)$ and $(\Sigma_D B, D)$. To do this consider the folloiwng diagram:

$$\begin{tikzcd}
 & & \partial^{-1} \arrow[d] & \ker \hat{b}_* & \\
{[\Sigma^2 A', B]} \arrow[r] \arrow[d, "\hat{b}_*"'] & {[\Sigma^2 A', K_D B]} \arrow[r] \arrow[d, "h_*"] & {[\Sigma(KA', A'), (K_D B, B)]} \arrow[r, "\partial"] \arrow[d, "h_*"] & {[\Sigma A', B]} \arrow[r] \arrow[d, "\hat{b}_*"] \arrow[u, phantom, "\cap"] & {[\Sigma A', K_D B]} \\
{[\Sigma^2 A', D]} \arrow[r, shift right, "(\check{\Sigma} b)_*"'] & {[\Sigma^2 A', \Sigma_D B]} \arrow[l, shift right, "(\hat{\Sigma} b)_*"'] \arrow[r, "j_*"'] \arrow[d, phantom, "\cup"] & {[\Sigma K(A', A'), (\Sigma_D B, D)]} \arrow[r] \arrow[dl, "\phi"] & {[\Sigma A', D]} & \\
 & \ker (\hat{\Sigma} b)_* & & &
\end{tikzcd}$$

The top row is part of the exact sequence of the pair $(K_D B, B)$ and the bottom row is part of the exact sequence of the pair $(\Sigma_D B, D)$. If f is as in (2.4), then $f \in \ker \hat{b}^*$ and we may define $\tilde{\Sigma} f = \phi h_* \partial^{-1}(f)$. ϕ is the unique splitting map of j_* whose image is $\ker(\Sigma b)_*$, and ∂^{-1} is well defined since ∂ is an isomorphism onto $\ker \hat{b}^*$.

THEOREM 2.5. *If in diagram* (2.4), $C = F = *$ *and* $A = \Sigma_* A'$, *then* $\bar{\Sigma} f \sim \tilde{\Sigma} f$ *in* Top $*$.

PROOF. Let $f \in \ker \hat{b}^*$, $f\colon \Sigma A' \to B$, a specific inverse for ∂ is defined by

$$\partial^{-1}[f]\,[a, t] = \begin{cases} [\check{b}H(a, 2t), t] & 0 \leqq t \leqq 1/2 \\ [f(a), 2t - 1] & 1/2 \leqq t \leqq 1 \end{cases}$$

and for $g \in [\Sigma(KA', A'), (\Sigma_D B, D)]$, ϕ may be defined by

$$\phi[g]\,[a, t] = \begin{cases} g[a, 2t] & 0 \leqq t \leqq 1/2 \\ \check{\Sigma} b\; \hat{\Sigma} b\; g[a, 2 - 2t] & 1/2 \leqq t \leqq 1. \end{cases}$$

Therefore

$$\tilde{\Sigma} f(a, t] = \begin{cases} [\check{b}H(a, 4t), 0] & 0 \leqq t \leqq 1/4 \\ [f(a), 4t - 1] & 1/4 \leqq t \leqq 1/2 \\ [\check{b}\hat{b}f(a), 0] & 1/2 \leqq t \leqq 3/4 \\ [\check{b}H(a, 4 - 4t), 0] & 3/4 \leqq t \leqq 1 \end{cases}$$

$$\sim \begin{cases} [\check{b}H(a, 3t), 0] & 0 \leqq t \leqq 1/3 \\ [f(a), 3t - 1] & 1/3 \leqq t \leqq 2/3 \\ [\check{b}H(a, 3 - 3t), 0] & 2/3 \leqq t \leqq 1 \end{cases}$$

$$= \bar{\Sigma} f[a, t].$$

LEMMA 2.6. *Let A, $B \in \mathrm{Top}\, D$, $f\colon A \to B \in \mathrm{Top}(D \to F)$ be such that the following diagram is of the same type as* (2.1):

$$\begin{tikzcd} & B \arrow[dl] \arrow[r, "\hat{b}"] & D \arrow[r, "i"] & RK(\hat{b}) \\ D \arrow[r, phantom, "H"] & \arrow[r, phantom, "-H"] & & \\ & A \arrow[ul] \arrow[uu, "f"] \arrow[r, "\hat{a}"'] & D \arrow[uu, "\mathrm{id}"] \arrow[r, "j"'] & RK(\hat{a}) \arrow[uu, "w"] \end{tikzcd}$$

where $w = w(f,\ \mathrm{id},\ -H,\ H)$. *Then* $RK(\hat{b}) = \Sigma_D B$, $RK(\hat{a}) = \Sigma_D A$, *and* $w = \bar{\Sigma} f$.

THEOREM 2.7. *Let f be as in diagram* (2.4), *then $\Sigma_D(RK(f))$ is homotopically equivalent to $RK(\bar{\Sigma} f)$ in* $\mathrm{Top}(D \to F)$.

PROOF. The basis for this proof is to consider $D + {}_C A$ as an element of Top D in two different ways, $(D + {}_C A,\ \iota,\ \langle \mathrm{id},\ \hat{b}f \rangle)$ and $(D + {}_C A,\ \iota,\ \langle \mathrm{id},\ \check{d}\hat{a} \rangle)$. With these structure maps, $\langle \check{b}, f \rangle\colon D + {}_C A \to B$ is a map in Top D in the first case but is only in $\mathrm{Top}(D \to F)$ in the second case.
Now consider

$$\begin{tikzcd} & (D + {}_C A, \iota, \langle \mathrm{id}, \hat{b}f \rangle) \arrow[dl] \arrow[r, "\langle \check{b}, f \rangle"] & B \arrow[r, "i"] & K\langle \check{b}, f \rangle \arrow[r, "\mu"] & D \arrow[r] & \Sigma_D K\langle \check{b}, f \rangle \\ D \arrow[r, phantom, "\langle \mathrm{id}, H \rangle"] & \arrow[r, phantom, "u = \mathrm{id}\ \text{constant}"] & & & & \\ & (D + {}_C A, \iota, \langle \mathrm{id}, \check{d}\hat{a} \rangle) \arrow[ul] \arrow[uu] \arrow[r, "\langle \check{b}, f \rangle"'] & B \arrow[uu] \arrow[r, "j"'] & RK\langle \check{b}, f \rangle \arrow[uu, "w"] \arrow[r, "\nu"'] & D \arrow[uu, equal] \arrow[r] & \Sigma_D RK\langle \check{b}, f \rangle \arrow[uu, "\bar{\Sigma} w"] \end{tikzcd}$$

where $w = w(\mathrm{id},\ \mathrm{id},\ \text{constant},\ \langle \mathrm{id},\ H \rangle)$ is a homotopy equivalence in $\mathrm{Top}(B \to F)$. $\bar{H}$ is a homotopy in $\mathrm{Top}(D \to F)$ making that square homotopy commutative so $\bar{\Sigma} w = w(w,\ \mathrm{id},\ \bar{H},\ \text{constant})$ is a homotopy equivalence in $\mathrm{Top}(D \to F)$. μ and ν are the natural structure maps that make $K\langle \check{b}, f \rangle$ and $RK\langle \check{b}, f \rangle$ in Top D. Next consider

$$\begin{tikzcd}[column sep=small]
& (D+{}_CA,\iota,\langle\mathrm{id},bf\rangle) \arrow[r,"\langle\mathrm{id},\hat{b}f\rangle"] \arrow[dl] & D \arrow[rr] & & (\Sigma_D(D+{}_CA),\iota,\langle\mathrm{id},\hat{b}f\rangle \arrow[r,"\Sigma\langle\check{b},f\rangle"] \arrow[dl] & \Sigma_D B \arrow[r] & K\Sigma\langle\check{b}\ f\rangle \\
D \arrow[r,phantom,"\langle\mathrm{id},H\rangle"] & & & D \arrow[r,phantom,"H'"] & & & \\
& (D+{}_CA,\iota,\langle\mathrm{id},d\hat{a}\rangle) \arrow[r,"\langle\mathrm{id},\hat{b}f\rangle"'] \arrow[ul] \arrow[uu,"\mathrm{id}\,\langle\mathrm{id},-H\rangle"'] & D \arrow[rr] \arrow[uu,equal] & & (\Sigma_D(D+{}_CA,\iota,\langle\mathrm{id},\check{d}\hat{a}\rangle)) \arrow[r] \arrow[ul] \arrow[uu,"\bar{w}"'] & \Sigma_D B \arrow[r,"\bar{\Sigma}\langle\check{b},f\rangle"'] \arrow[uu,equal] & RK(\bar{\Sigma}\langle\check{b},f\rangle)) \arrow[uu,"\hat{w}"']
\end{tikzcd}$$

where $\bar{w} = w(\mathrm{id}, \mathrm{id}, \langle\mathrm{id}, -H\rangle, \langle\mathrm{id}, H\rangle) = \bar{\Sigma}(\mathrm{id})$ and $\hat{w} = w(\bar{w}, \mathrm{id}, \text{constant}, H')$ are homotopy equivalences in $\mathrm{Top}(D \to F)$. The map $\bar{\Sigma}\langle\check{b}, f\rangle$ is the one-sided suspension of $\langle\check{b}, f\rangle$.

Combining the results of the two diagrams we get

$$\Sigma_D RK\langle\check{b}, f\rangle \xrightarrow{\bar{\Sigma}w} \Sigma_D K\langle\check{b}, f\rangle \xrightarrow{\theta} K(\Sigma\langle\check{b}, f\rangle) \xleftarrow{\hat{w}} RK(\Sigma\langle\check{b}, f\rangle)$$

where $\bar{\Sigma}w$ and $\hat{w}$ are *h.e.* in $\mathrm{Top}(D \to F)$ from the diagrams and θ is a homeomorphism obtained from interchanging parameters. Therefore $\Sigma_D RK\langle\check{b}, f\rangle$ is *h.e.* to $RK(\bar{\Sigma}\langle\check{b}, f\rangle)$ in $\mathrm{Top}(D \to F)$.

LEMMA 2.8. *If $F = C$ in diagram* (2.4) *and $D \in \mathrm{Top}\ C$, then $RK\langle\check{b}, f\rangle$ and $RK(f)$ are homeomorphic in* $\mathrm{Top}\ D$. *In addition $RK(f) = K_C(f)$.*

COROLLARY 2.9. *If $F = C$ and $D \in \mathrm{Top}\ C$ in diagrams*(2.4), *then $\Sigma_D K_C(f)$ is h.e. in* $\mathrm{Top}(D \to C)$ *to $K_C(\bar{\Sigma}f)$.*

PROOF. 2.7 and 2.8.

COROLLARY 2.10. *Let $C = F, f, D \in \mathrm{Top}\ C$ in diagram* (2.4), *then*

$$\begin{aligned}(2.11)\qquad & [A, Z]^C_C \xleftarrow{f^*} [B, Z]^D_C \xleftarrow{i^*} [K_C(f), Z]^D_C \xleftarrow{k^*} [\Sigma_C A, Z]^C_C \\ & \xleftarrow{(\bar{\Sigma}f)^*} [\Sigma_D B, Z]^D_C \xleftarrow{i^*} [\Sigma_D K(f), Z]^D_C \longleftarrow\end{aligned}$$

is exact for $Z \in \mathrm{Top}(D \to C)$. From $[\Sigma_C A, Z]^C_C$ right we have groups and homomorphisms, from $[\Sigma^2_C A, Z]$ right, abelian groups and homomorphisms. The map k^* can be identified by the action map of $[\Sigma_C A, Z]^C_C$ on $[K_C(f), Z]^D_C$.

PROOF. From the proof of 2.7 we get

$$\begin{tikzcd}[column sep=small]
(D+{}_CA,\iota,\langle\mathrm{id},\hat{b}f\rangle) \arrow[r,"\langle\check{b},f\rangle"] & B \arrow[r] & K\langle\check{b},f\rangle \arrow[r,"k"] & \Sigma_D(D+{}_CA) \arrow[r,"\Sigma\langle\check{b},f\rangle"] \arrow[d,shift left,"\bar{w}"] & \Sigma_D B \arrow[r,"\Sigma i"] & \Sigma_D K\langle\check{b},f\rangle \\
(D+{}_CA,\iota,\langle\mathrm{id},\check{d}\hat{a}\rangle) \arrow[r,"\langle\check{b},f\rangle"'] \arrow[u,equal] & B \arrow[r] \arrow[u,equal] & RK\langle\check{b},f\rangle \arrow[r,"\bar{k}"'] \arrow[u,"w"'] & \Sigma_D(D+{}_CA) \arrow[r,"\bar{\Sigma}\langle\check{b},f\rangle"'] \arrow[u,shift left,"\bar{w}"] & \Sigma_D B \arrow[r,"\Sigma j"'] \arrow[u,equal] & \Sigma_D RK\langle\check{b},f\rangle \arrow[u,"\hat{w}"']
\end{tikzcd}$$

The top row induces exact sequences of classes of maps in $\mathrm{Top}\ D$, see McClendon [**6**]. Since all vertical maps are *h.e.* in $\mathrm{Top}(D \to C)$ and $[X, Z \times_C D]^D_D \approx [X, Z]^D_C$ for $Z \in \mathrm{Top}\ C$, we get

$$\begin{aligned}& [D + {}_CA, Z]^D_C \xleftarrow{\langle\check{b},f\rangle} [B, Z]^D_C \xrightarrow{i^*} [K\langle\check{b}, f\rangle, Z]^D_C \\ & \qquad \xleftarrow{\bar{k}^*} [\Sigma_D(D + {}_CA), Z]^D_C \xleftarrow{\bar{\Sigma}\langle\check{b},f\rangle^*} [\Sigma_D B, Z] \xleftarrow{(\Sigma i)^*} [\Sigma_D K\langle\check{b}, f\rangle, Z]^D_C\end{aligned}$$

is exact. Since $[D + {}_CA, Z]^D_C \approx [A, Z]^C_C$, $K\langle\check{b}, f\rangle \equiv K(f)$, $\Sigma_D(D + {}_CA) \equiv$

$D + {}_C\Sigma_C A$ and $\bar{\Sigma}\langle\check{b}, f\rangle = \bar{\Sigma}f$, the above sequence becomes sequence 2.11. Also $\Sigma\langle\check{b}, f\rangle$ induces homomorphisms, so $(\bar{\Sigma}\langle\check{b}, f\rangle)^*$ and hence $(\bar{\Sigma}f)^*$ are homomorphisms. Similarly, maps to the right of $(\bar{\Sigma}f)^*$ are also homomorphisms.

SPECIAL CASES 2.12. a) Let $f = \langle f_i\rangle\colon\ S^{r(i)} \vee \cdots \vee S^{r(w)} \to D \vee S^n$, $\pi = \langle \mathrm{id}, *\rangle\colon D \vee S^n \to D$ and $\pi \circ f_i \sim 0$ for each i. $\bar{\Sigma}^k f = \langle \bar{\Sigma}^k f_i\rangle$ where $\bar{\Sigma}^k f_i$ is the k-fold one-sided suspension of f and $C = F = *$, then (2.11) becomes

$$\cdots \leftarrow \bigoplus_i \pi_{r(i)+k}(Z) \xleftarrow{(\bar{\Sigma}^k f)^*} \pi_{n+k}(Z) \xleftarrow{i^*} [K(\bar{\Sigma}^k f), Z)^D$$
$$\xleftarrow{k^*} \bigoplus \pi_{r(i)+k+1}(Z) \xleftarrow{(\bar{\Sigma}^{k+1} f)^*} \pi_{n+k+1}(Z) \leftarrow \cdots$$

b) I apply the sequence of corollary 2.10 to compute $[P^n, Z]^{p^{n-2}, f} = G$ where $f\colon P^{n-2} \to Z$ is given with a fixed extension $g\colon p^n \to Z$. We have

$$\begin{tikzcd}
 & P^n \arrow[r] & K_n \\
S^{n-1} \arrow[r, "f_{n-1}"'] \arrow[rr, bend right, "h"'] & P^{n-1} \arrow[u, phantom, "\cup"] \arrow[r, "p"'] & P^n \vee S^{n-1} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

where f_{n-1} is the attaching map and p is the projection for the pushout of $P^{n-2} \subset P^{n-1}$ and $P^{n-2} \subset P^n$. Thus $G = [K_n, Z]^{P^n, g}$. It is not hard to check that

$$h = \begin{cases} \iota_{n-1} + t \circ \iota_{n-1}, & (n-1) \text{ odd} \\ \iota_{n-1} - t \circ \iota_{n-1}, & (n-1) \text{ even} \end{cases}$$

where $0 \neq t \in \pi_1(P^n \vee S^{n-1})$. For $n \geqq 3$, $[n \geqq 4]$ h is a one-sided [double] suspension so $[K, Z]^P$ is a [abelian] group. Corollary 2.10 applied to h, $(C = *,\ D = P^n,\ B = P^n \vee S^{n-1})$ becomes

$$0 \longleftarrow \ker \alpha_{n-1} \longleftarrow G \longleftarrow \operatorname{coker} \alpha_n \longleftarrow 0$$
$$\text{where } \alpha_j\colon \pi_j z \to \pi_j z,\ \alpha_j z = z + tz \quad \text{all } j,\ n \text{ even}$$
$$\alpha_j z = z - tz \quad \text{all } j,\ n \text{ odd}.$$

Thus we have the following results:

b1) If $f_* t$ acts trivially on $\pi_{n-1}Z$, $\pi_n Z$ (e.g., Z is homotopically simple), then the cardinality of G is given by

$$|G| = \begin{cases} |\{z \in \pi_{n-1}Z \mid 2z = 0\}| \times |\pi_n Z/2\pi_n Z| & n \text{ even} \\ |\pi_{n-1}Z| \times |\pi_n Z| & n \text{ odd}; \end{cases}$$

b2) If Z is homotopically simple, G is independent of f;

b3) If $Z = P^n, f = \mathrm{id}$, then $G = Z$ for n even or odd; and if $Z = P^n, f = *$, then

$$G = \begin{cases} Z_2 & n \text{ even} \\ Z & n \text{ odd.} \end{cases}$$

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