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A SINGULAR NONLINEAR BOUNDARY VALUE PROBLEM

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We consider the singular non-linear boundary value problem

$$\ddot{y} + \frac{\gamma}{t}\dot{y} - y + f(y^2)y = 0, \quad t \in (0, \infty) \tag{1.1}$$

$$\lim_{t \to 0+} y(t) > 0, \lim_{t \to \infty} y(t) = 0, \lim_{t \to 0+} \dot{y}(t) = 0, \tag{1.2}$$

where $1 \leqq \gamma \leqq 2$. It is shown that for certain functions f, positive in $(0, \infty)$ and continuous in $[0, \infty)$, the equation (1.1) has solutions $y_n(t)$, $n = 0, 1, 2, \ldots$, which satisfy (1.2) and vanish at n distinct points in $(0, \infty)$.

The problem is motivated by a model for stationary self-focusing of light beams given by Zakharov, Sobolev, and Synakh [12]; and others. After some simplification, their equation becomes

$$\ddot{y} + \frac{1}{t}\dot{y} - y + f(y^2)y = 0. \tag{1.3}$$

Of particular interest is the case $f(s) = s$, in which case (1.3) becomes

$$\ddot{y} + \frac{1}{t}\dot{y} - y + y^3 = 0. \tag{1.4}$$

Ryder [11] and Macki [6] have considered the equation

$$\ddot{x} - x + xF(x^2, t) = 0,$$

which under the substitutions

$$F(x^2, t) = f(x^2/t^2), y(t) = t^{-1}x(t)$$

becomes our equation (1.1) with $\gamma = 2$. The range $1 \leqq \gamma < 2$ is not included, and our condition (III) on the nonlinearity is different from theirs, so that neither result is contained in the other even for $\gamma = 2$. Nehari [10] has considered the equation

$$\ddot{y} + \frac{2}{t}\dot{y} - y + y^3 = 0, \tag{1.5}$$

which is also included in (1.1) for $\gamma = 2$. The thrust of this paper is to

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give a unified treatment which will cover both (1.4) and (1.5). We will use many of the techniques of Nehari [8], but the primary new idea is to write (1.1) in self-adjoint form

$$-t^{-\gamma}(t^{\gamma}y')' + y = f(y^2)y; \quad t \in (0, \infty), \quad 1 \leqq \gamma \leqq 2 \tag{1.6}$$

and introduce "weighted norms".

In all that follows we will assume that the function f satisfies

(I) $f \in C[0, \infty)$,
(II) $f(s) > 0$ if $s > 0$,
(III) $|f(s)| \leqq k|s|$, and
(IV) $\exists \delta > 0$ such that $s^{-\delta}f(s)$ is strictly increasing on $(0, \infty)$.

We define $g(s) = \int_0^s f(\eta)d\eta$, and consider the variational problem

$$\min_A J(y) = \min_A \int_0^\infty \{y^2 + \dot{y}^2 - g(y^2)\}t^{\gamma}dt \tag{1.7}$$

$A = \{y \mid y \in D^1(0, \infty), y(t) \geqq 0, y(t) \not\equiv 0, \|y\|^2 = \int_0^\infty f(y^2)y^2t^{\gamma}dt\}$, where $\|y\|^2 = \int_0^\infty (y^2 + \dot{y}^2)t^{\gamma}dt$, and $D^1(0, \infty)$ denotes the class of functions continuous on $(0, \infty)$ with piecewise continuous derivatives.

LEMMA 1.1. *If $y \in D^1(0, \infty)$ and $\|y\| < \infty$, then*

$$\sup_{0<t<\infty} t^{\gamma}y^2(t) \leqq \|y\|^2 \tag{1.8}$$

and

$$\lim_{t\to 0+} t^{\gamma}y^2(t) = 0. \tag{1.9}$$

PROOF. For the first part we have

$$y^2(t) = -2\int_t^\infty y\dot{y}d\tau \leqq 2t^{-\gamma}\int_t^\infty |y\dot{y}|\tau^{\gamma}d\tau \leqq t^{-\gamma}\|y\|^2.$$

To prove (1.9), for a given $\varepsilon > 0$, we choose $s < T$ such that $0 < t < s$ implies $\int_t^s |y\dot{y}|\tau^{\gamma}d\tau < \varepsilon/6$. Fixing s, we then choose δ with $0 < \delta < s$ so that $\delta^{\gamma}y^2(T) < \varepsilon/3$ and $(\delta/s)^{\gamma}\int_s^T |y\dot{y}|\tau^{\gamma}d\tau < \varepsilon/6$. Then we have

$$y^2(t) \leqq y^2(T) + 2\int_t^T |y\dot{y}|d\tau,$$

and for $0 < t \leqq \delta$,

$$t^{\gamma}y^2(t) \leqq \delta^{\gamma}y^2(T) + 2\int_t^s |y\dot{y}|\tau^{\gamma}d\tau + 2\left(\frac{\delta}{s}\right)^{\gamma}\int_s^T |y\dot{y}|\tau^{\gamma}d\tau$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

LEMMA 1.2. *If* $y \in D^1(0, \infty)$ *and* $\|y\| < \infty$, *then*

$$\int_0^\infty y^2 dt \leqq C\|y\|^2. \tag{1.10}$$

PROOF. Let $\varphi(t) \in C^1(0, \infty)$ be chosen so that $0 \leqq \varphi(t) \leqq 1$ and $\varphi(t) = 1$ for $t \in [0, 1]$, $\varphi(t) = 0$ for $t \in [2, \infty)$. We set $v = \varphi y$, so $\dot{v} = \dot{\varphi} y + \varphi \dot{y}$ and $|\dot{v}| \leqq |\dot{y}| + C|y|$. Then we have

$$\begin{aligned}\int_0^\infty y^2 dt &= \int_0^1 y^2 dt + \int_1^\infty y^2 dt \\ &\leqq \int_0^1 |v|^2 dt + \int_1^\infty |y|^2 t^\gamma dt.\end{aligned}$$

By Theorem 330 of [**4**],

$$\begin{aligned}\int_0^\infty |v|^2 dt &\leqq 4\int_0^\infty t^2|\dot{v}|^2 dt = 4\int_0^2 t^2|\dot{v}|^2 dt \\ &\leqq 4\cdot 2^{2-\gamma}\int_0^2 |\dot{v}|^2 t^\gamma dt.\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^\infty |y|^2 dt &\leqq 4\cdot 2^{2-\gamma}\int_0^\infty |\dot{v}|^2 t^\gamma dt + \int_0^\infty |y|^2 t^\gamma dt \\ &\leqq 4\cdot 2^{2-\gamma}\int_0^\infty (|\dot{y}| + C|y|)^2 t^\gamma dt + \int_0^\infty |y|^2 t^\gamma dt \\ &\leqq C\|y\|^2.\end{aligned}$$

LEMMA 1.3. *The collection* $\{t^\gamma y(t) | y \in D^1(0, \infty), \|y\| < M\}$ *is equicontinuous on* $(0, \infty)$.

PROOF. If $0 < t_1 < t_2 < \infty$, then

$$\begin{aligned}|t_2^\gamma y(t_2) - t_1^\gamma y(t_1)| &= \left|\int_{t_1}^{t_2} (t^\gamma \dot{y} + \gamma t^{\gamma-1} y) dt\right| \\ &\leqq \left(\int_{t_1}^{t_2} \dot{y}^2 t^\gamma dt\right)^{1/2}\left(\int_{t_1}^{t_2} t^\gamma dt\right)^{1/2} \\ &\quad + \gamma\left(\int_{t_1}^{t_2} y^2 dt\right)^{1/2}\left(\int_{t_1}^{t_2} t^{2\gamma-2} dt\right)^{1/2}.\end{aligned}$$

Using (1.10) gives

$$\begin{aligned}&|t_2^\gamma y(t_2) - t_1^\gamma y(t_1)| \\ &\quad\leqq \|y\|\left\{\left(\frac{t_2^{\gamma+1} - t_1^{\gamma+1}}{\gamma + 1}\right)^{1/2} + C\left(\frac{t_2^{2\gamma-1} - t_1^{2\gamma-1}}{2\gamma - 1}\right)^{1/2}\right\}\end{aligned}$$

which proves the lemma.

LEMMA 1.4. *If* $y \in D^1(0, \infty)$, $y(t) \geqq 0$, $y(t) \not\equiv 0$, *and* $\|y\| < \infty$, *then* $\int_0^\infty f(y^2)y^2 t^\gamma dt$ *and* $\int_0^\infty g(y^2)t^\gamma dt$ *both exist, and* $\exists \alpha > 0$ *such that* $\alpha y \in A$.

PROOF. Condition (IV) implies that f is increasing, so

$$g(y^2) = \int_0^{y^2} f(s)ds \leqq y^2 f(y^2).$$

Using (1.8), (III), and (1.10), we have

$$\begin{aligned}\int_0^\infty g(y^2)t^\gamma dt &\leqq \int_0^\infty y^2 f(y^2)t^\gamma dt \\ &\leqq C\|y\|^2 \int_0^\infty y^2 dt \leqq C\|y\|^4 < \infty.\end{aligned} \tag{1.11}$$

We now define $H(\alpha) = [\int_0^\infty y^2 f(\alpha^2 y^2)t^\gamma dt]/\|y\|^2$. It is easily seen from (I) that $H(\alpha)$ is continuous for $\alpha > 0$. and using (IV) we see that if $0 < \alpha < 1$,

$$H(\alpha) \leqq \frac{\alpha^{2\delta}}{\|y\|^2} \int_0^\infty y^2 f(y^2)t^\gamma dt = \alpha^{2\delta} H(1).$$

Correspondingly, $\alpha \geqq 1$ gives $H(\alpha) \geqq \alpha^{2\delta} H(1)$. Therefore $H(\alpha)$ is strictly increasing, and there exists a unique $\alpha > 0$ such that $H(\alpha) = 1$. A simple computation shows $\alpha y \in A$.

We observe from the above that if $y \in A$, then

$$\|y\|^2 = \int_0^\infty y^2 f(y^2)t^\gamma dt \leqq C\|y\|^4$$

so that

$$\|y\|^2 \geqq 1/C > 0. \tag{1.12}$$

LEMMA 1.5. *If* $\lambda = \inf\{J(y) | y \in A\}$, *then* $\lambda \geqq 0$, $\exists \{y_n\} \in A$ *such that* $J(y_n) \to \lambda$, *and* $\|y_n\|^2 \leqq C$.

PROOF. Suppose $y \in A$. Using (IV) we have

$$\begin{aligned}\int_0^\infty g(y^2)t^\gamma dt &= \int_0^\infty \int_0^{y^2} s^\delta[s^{-\delta} f(s)]t^\gamma ds\, dt \\ &\leqq \int_0^\infty y^{-2\delta} f(y^2)t^\gamma \int_0^{y^2} s^\delta ds = \frac{\|y\|^2}{1+\delta}.\end{aligned}$$

Then

$$J(y) = \|y\|^2 - \int_0^\infty g(y^2)t^\gamma dt \geqq \delta(1+\delta)^{-1}\|y\|^2 > 0, \tag{1.13}$$

hence $\lambda \geqq 0$. Now we choose $\{y_n\} \subseteqq A$ such that $J(y_n) \to \lambda$. Using (1.13) we have

$$\|y_n\|^2 \leqq \delta^{-1}(1+\delta)J(y_n) \leqq C. \tag{1.14}$$

By Lemma 1.1 and Lemma 1.3 $\{t^\gamma y_n(t)\}$ is uniformly bounded and equicontinuous on $(0, \infty)$, and it follows by the Arzela-Ascoli theorem that a subsequence of $\{y_n(t)\}$, call it again $\{y_n(t)\}$, converges uniformly on compact subsets of $(0, \infty)$ to a function $y \in C(0, \infty)$.

We now turn our consideration to the equation

$$-(t^\gamma y')' + t^\gamma y = \alpha_n f(y_n^2) y_n t^\gamma \tag{1.15}$$

where $y_n(t)$ is from above. We define

$$u_n(t) = \alpha_n \int_0^\infty g(t, \tau) f(y_n^2) y_n d\tau \tag{1.16}$$

where

$$g(t, \tau) = \begin{cases} t^{-\nu} K_\nu(t) \tau^{-\nu} I_\nu(\tau) \tau^\gamma; \ 0 < \tau \leqq t \\ t^{-\nu} I_\nu(t) \tau^{-\nu} K_\nu(\tau) \tau^\gamma; \ \tau > t \end{cases}$$

Taking for the moment $\alpha_n = 1$ we have

$$u_n(t) = t^{-\nu} K_\nu(t) \varphi_n(t) + t^{-\nu} I_\nu(t) \psi_n(t)$$

where

$$\varphi_n(t) = \int_0^t \tau^{-\nu} I_\nu(\tau) f(y_n^2) y_n \tau^\gamma d\tau$$

$$\psi_n(t) = \int_0^\infty \tau^{-\nu} K_\nu(\tau) f(y_n^2) y_n \tau^\gamma d\tau.$$

In the above, $\nu = (\gamma - 1)/2$, and I_ν, K_ν are the modified Bessel functions of order ν.

LEMMA 1.6. *The following estimates hold for $\varphi_n(t)$, $\psi_n(t)$, uniformly with respect to n:*

(a) $\varphi_n(t) = o(e^t)$ *as* $t \to \infty$;
(b) $\psi_n(t) = O(t^{-\nu-1/2} e^{-t})$ *as* $t \to \infty$;
(c) $\varphi_n(t) = O(t^{1/2})$ *as* $t \to 0+$; *and*
(d) $\psi_n(t) = O(t^{-\nu})$ *as* $t \to 0+$.
In addition, $\varphi_n(t) = o(t^{1/2})$ *as* $t \to 0+$ *for each* n.

PROOF. We first note that $I_\nu(t) = O(t^\nu)$ and $K_\nu(t) = O(t^{-\nu})$ as $t \to 0+$, whereas $I_\nu(t) = O(t^{-1/2} e^t)$ and $K_\nu(t) = O(t^{-1/2} e^{-t})$ as $t \to \infty$. Using Hölder's inequality,

$$\begin{aligned} \varphi_n(t) \leqq & \left(\int_0^T f(y_n^2)^2 \tau^\gamma d\tau \right)^{1/2} \left(\int_0^T [\tau^{-\nu} I_\nu]^2 y_n^2 \tau^\gamma d\tau \right)^{1/2} \\ & + C e^t T^{-\nu-1/2} \left(\int_T^t f(y_n^2)^2 \tau^\gamma d\tau \right)^{1/2} \left(\int_T^t y_n^2 \, \tau^\gamma d\tau \right)^{1/2}. \end{aligned} \tag{1.17}$$

By condition (III), (1.8), and (1.10) we have

$$\begin{aligned}\int_0^\infty f(y_n^2)^2\tau^\gamma d\tau &\leqq C\int_0^\infty y_n^4\tau^\gamma d\tau \leqq C\|y_n\|^2\int_0^\infty y_n^2 d\tau \\ &\leqq C\|y_n\|^4 \leqq C\left(\frac{1+\delta}{\delta}\right)^2 J^2(y_n) \leqq C.\end{aligned} \tag{1.18}$$

Combining this with (1.17), and again using (1.8), we have $\varphi_n(t) \leqq \eta(T) + Ce^tT^{-\nu-1/2}$, and (a) follows. Next we have the estimate

$$\begin{aligned}\psi_n(t) &\leqq C\int_t^\infty \tau^{-\nu-1/2}e^{-\tau}f(y_n^2)y_n\tau^\gamma d\tau \\ &\leqq Ct^{-\nu-1/2}e^{-t}\left(\int_t^\infty f(y_n^2)^2\tau^\gamma d\tau\right)^{1/2}\left(\int_t^\infty y_n^2\tau^\gamma d\tau\right)^{1/2} \\ &\leqq Ct^{-\nu-1/2}e^{-t},\end{aligned}$$

by virtue of (1.18). For part (c) we use (1.18) and Lemma 1.1 to get

$$\varphi_n(t) \leqq C\left(\int_0^t f(y_n^2)^2\tau^\gamma d\tau\right)^{1/2}\left(\int_0^t y_n^2\tau^\gamma d\tau\right)^{1/2} = o(t^{1/2})$$

for each n, and $\varphi_n(t) = O(t^{1/2})$ uniformly with respect to n. Finally,

$$\begin{aligned}\psi_n(t) &\leqq t^{-\nu}\left(\int_t^\infty K_\nu^2 y_n^2\tau^\gamma d\tau\right)^{1/2}\left(\int_t^\infty f(y_n^2)^2\tau^\gamma d\tau\right)^{1/2} \\ &\leqq Ct^{-\nu}\left(\int_0^\infty K_\nu^2 d\tau\right)^{1/2},\end{aligned}$$

and it follows that $\psi_n(t) = O(t^{-\nu})$ as long $1 \leqq \gamma < 2$. If $\gamma = 2$, we use (III) and (1.8) to obtain

$$\begin{aligned}\psi_n(t) &\leqq C_1\int_t^1 y_n^3\tau d\tau + C_2\int_1^\infty \tau^{-1/2}e^{-\tau}d\tau \\ &\leqq C_1\left(\int_t^\infty y_n^4\tau^2 d\tau\right)^{1/2}\left(\int_t^\infty y_n^2 d\tau\right)^{1/2} + C_2.\end{aligned}$$

Using (1.18) we get

$$\psi_n(t) \leqq C_1\left(\int_t^\infty y_n^2 d\tau\right)^{1/2} + C_2. \tag{1.19}$$

Because of Lemma 1.1 we know $\lim_{t\to\infty} y_n(t) = 0$, so we can write $y_n^2(\tau) = -2\int_\tau^\infty y_n(s)\dot{y}_n(s)ds$, so that

$$\begin{aligned}\int_t^\infty y_n^2(\tau)d\tau &= -2\int_t^\infty\int_\tau^\infty y_n(s)\dot{y}_n(s)ds\, d\tau \\ &\leqq 2\int_t^\infty |y_n(s)\dot{y}_n(s)|\int_t^s d\tau\, ds \\ &\leqq 2t^{-1}\int_t^\infty |y_n(s)\dot{y}_n(s)|s^2 ds \\ &\leqq t^{-1}\|y_n\|^2 \leqq Ct^{-1}.\end{aligned}$$

Using this estimate and (1.19), we have $\psi_n(t) = O(t^{-1/2})$ as $t \to 0+$.

Applying the estimates of Lemma 1.6 to

$$u_n(t) = t^{-\nu}K_\nu(t)\varphi_n(t) + t^{-\nu}I_\nu(t)\psi_n(t)$$

and

$$\dot{u}_n(t) = -t^{-\nu}K_{\nu+1}(t)\varphi_n(t) + t^{-\nu}I_{\nu+1}(t)\psi_n(t)$$

gives $u_n(t) = o(t^{-\gamma/2})$ and $\dot{u}_n(t) = o(t^{-\gamma/2})$ as $t \to \infty$, so that $t^\gamma u_n(t)\dot{u}_n(t) = o(1)$ as $t \to \infty$, uniformly with respect to n. As $t \to 0+$, we have $u_n(t) = O(t^{1/2-\gamma/2})$ and $\dot{u}_n(t) = o(t^{1/2-\gamma})$, so that $t^\gamma u_n(t)\dot{u}_n(t) = o(t^{1-\gamma/2}) = o(1)$. This last estimate is uniform with respect to n except for the case $\gamma = 2$.

LEMMA 1.7. *The function $u_n(t)$, as defined in* (1.16), *is a solution of* (1.15), *and for an appropriate choice of α_n we have $u_n \in A$, $J(u_n) \leqq J(y_n)$, and $J(u_n) = J(y_n)$ if and only if $u_n(t) \equiv y_n(t)$ in* $(0, \infty)$.

PROOF. Clearly our function $u_n \in C^2(0, \infty)$, $u_n(t) \geqq 0$, $u_n(t) \not\equiv 0$, and u_n satisfies (1.15). Setting $y(t) = u_n(t)$ in (1.15), multiplying both sides by $u_n(t)$, and integrating by parts gives

$$\begin{aligned} H_n^2(T) &= \int_0^T (u_n^2 + \dot{u}_n^2)t^\gamma dt \\ &= \alpha_n \int_0^T f(y_n^2)y_n u_n t^\gamma dt + T^\gamma u_n(T)\dot{u}_n(T). \end{aligned}$$

We note that

$$\begin{aligned} \int_0^1 f(y_n^2)y_n u_n t^\gamma dt &\leqq \left(\int_0^1 f(y_n^2)^2 t^\gamma dt\right)^{1/2}\left(\int_0^1 y_n^2 u_n^2 t^\gamma dt\right)^{1/2} \\ &\leqq C\left(\int_0^1 y_n^2 t dt\right)^{1/2} \end{aligned}$$

since $u_n^2(t) = O(t^{1-\gamma})$, and by (1.18). But

$$\int_0^1 y_n^2 t dt \leqq \left(\int_0^1 y_n^2 t^\gamma dt\right)^{1/2}\left(\int_0^1 t^{2-\gamma} dt\right)^{1/2} < \infty,$$

and it follows that both $\int_0^T f(y_n^2)y_n u_n t^\gamma dt$ and $\int_0^T (u_n^2 + \dot{u}_n^2)t^\gamma dt$ are convergent. Now we use (III) and (1.18) to obtain

$$\begin{aligned} \int_0^T f(y_n^2)y_n u_n t^\gamma dt &\leqq C\int_0^T y_n^3 u_n t^\gamma dt \\ &\leqq C\left(\int_0^T y_n^4 t^\gamma dt\right)^{3/4}\left(\int_0^T u_n^4 t^\gamma dt\right)^{1/4} \\ &\leqq C\left(\int_0^T u_n^4 t^\gamma dt\right)^{1/4}. \end{aligned}$$

An inequality of Adams ([1], page 129) gives

$$\int_0^T u_n^4 t^\gamma dt \leqq \left(\frac{\gamma+3}{T}\int_0^T u_n^2 t^\gamma dt + 4\int_0^T |u_n \dot{u}_n| t^\gamma dt\right)\int_0^T u_n^2 dt$$
$$\leqq C\left(\frac{\gamma+3}{T} + 2\right)H_n^4(T) = \eta^4(T)H_n^4(T).$$

Thus we have

$$H_n^2(T) \leqq \alpha_n H_n(T)\eta(T) + T^\gamma u_n(T)\dot{u}_n(T),$$

or after completing the square,

$$\left(H_n(T) - \frac{\alpha_n \eta(T)}{2}\right)^2 \leqq \frac{\alpha_n^2\eta^2(T)}{4} + Tu_n(T)\dot{u}_n(T).$$

Since the right hand side is bounded as $T \to \infty$, we see that $\int_0^\infty (u_n^2 + \dot{u}_n^2)t^\gamma dt < \infty$. By Lemma 1.4 we may now choose α_n so that $u_n \in A$. Multiplying both sides of (1.15) by y_n (again $y(t) = u_n(t)$) and integrating by parts gives

$$\int_0^\infty (u_n y_n + \dot{u}_n \dot{y}_n)t^\gamma dt = \alpha_n \int_0^\infty f(y_n^2)y_n^2 t^\gamma dt = \alpha_n \|y_n\|^2.$$

Using Hölder's inequality we obtain

$$\left[\alpha_n \int_0^\infty f(y_n^2)y_n^2 t^\gamma dt\right]^2 \leqq \|u_n\|^2 \|y_n\|^2$$
$$= \int_0^\infty f(y_n^2)y_n^2 t^\gamma dt \int_0^\infty f(u_n^2)u_n^2 t^\gamma dt$$

and it follows that

$$\alpha_n^2 \int_0^\infty f(y_n^2)y_n^2 t^\gamma dt \leqq \int_0^\infty f(u_n^2)u_n^2 t^\gamma dt. \tag{1.20}$$

But

$$\left(\int_0^\infty f(u_n^2)u_n^2 t^\gamma dt\right)^2 = \|u_n\|^4 = \alpha_n^2\left(\int_0^\infty f(y_n^2)y_n u_n t^\gamma dt\right)^2$$
$$\leqq \alpha_n^2 \int_0^\infty f(y_n^2)y_n^2 t^\gamma dt \int_0^\infty f(y_n^2)u_n^2 t^\gamma dt. \tag{2.21}$$

Combining (1.20) and (1.21) yields

$$\int_0^\infty f(u_n^2)u_n^2 t^\gamma dt \leqq \int_0^\infty f(y_n^2)u_n^2 t^\gamma dt. \tag{1.22}$$

Since f is strictly increasing by (IV), g is strictly convex, so

$$\int_0^T g(u_n^2)t^\gamma dt \geqq \int_0^T g(y_n^2)t^\gamma dt + \int_0^T (u_n^2 - y_n^2)f(y_n^2)t^\gamma dt, \tag{1.23}$$

with equality if and only if $u_n(t) \equiv y_n(t)$ on $(0, T)$. Combining (1.22) and (1.24) gives

$$\int_0^\infty \{f(u_n^2)u_n^2 - g(u_n^2)\}t^\gamma dt \leqq \int_0^\infty \{f(y_n^2)y_n^2 - g(y_n^2)\}t^\gamma dt,$$

i.e., $J(u_n) \leqq J(y_n)$, with $J(u_n) = J(y_n)$ if and only if $u_n(t) \equiv y_n(t)$ on $(0, \infty)$.

We also observe at this point that because of (1.13) we have $\|u_n\|^2 \leqq \delta^{-1}(1 + \delta)J(u_n) \leqq \delta^{-1}(1 + \delta)J(y_n) \leqq C$. As before with $\{y_n(t)\}$, we can use Lemma 1.1 and Lemma 1.3 to obtain a subsequence of $\{u_n(t)\}$, call it again $\{u_n(t)\}$, such that $u_n(t)$ converges uniformly on compact subsets of $(0, \infty)$ to a function $u_0 \in C(0, \infty)$.

LEMMA 1.8. *The sequence α_n is bounded.*

PROOF. From (1.20) we have $\alpha_n^2 \leqq \|u_n\|^2/\|y_n\|^2$. As in the proof of Lemma 1.5 we have $\|u_n\|^2 \leqq \delta^{-1}(1 + \delta)J(u_n) \leqq C$, and form (1.12) we have $\|y_n\|^2 \geqq C_2$, hence $\alpha_n^2 \leqq C_1/C_2$.

LEMMA 1.9. $\lim_{n\to\infty} J(u_n) = J(u_0) = \lambda > 0$.

PROOF. We first observe that $\{\varphi_n(t)\}$ and $\{\psi_n(t)\}$ are uniformly convergent on compact subsets of $(0, \infty)$, since the same is true of $\{y_n(t)\}$, and because of the estimates

$$\begin{aligned}
&\left|\int_0^\eta t^{-\nu}I_\nu\{f(y_n^2)y_n - f(y_m^2)y_m\}t^\gamma dt\right| \\
&\quad \leqq \left(\int_0^\eta f(y_n^2)^2 t^\gamma dt\right)^{1/2}\left(\int_0^\eta [t^{-\nu}I_\nu]^2 y_n^2 t^\gamma dt\right)^{1/2} \\
&\qquad + \left(\int_0^\eta f(y_m^2)^2 t^\gamma dt\right)^{1/2}\left(\int_0^\eta [t^{-\nu}I_\nu]^2 y_m^2 t^\gamma dt\right)^{1/2} \\
&\quad \leqq C\left(\int_0^\eta [t^{-\nu}I_\nu]^2 dt\right)^{1/2} = o(\eta^{1/2})
\end{aligned}$$

as $\eta \to 0+$ and

$$\begin{aligned}
&\left|\int_T^\infty t^{-\nu}K_\nu\{f(y_n^2)y_n - f(y_m^2)y_m\}t^\gamma dt\right| \\
&\quad \leqq C\left(\int_T^\infty [t^{-\nu}K_\nu]^2 dt\right)^{1/2} = o(1)
\end{aligned}$$

as $T \to \infty$. It now follows that $\dot{u}_n(t) \to \dot{u}_0(t)$ uniformly on compact subsets of $(0, \infty)$.

Next we use Fatou's Lemma, (1.14) and (1.11) to get $\|u_0\|^2 \leqq \lim_{n\to\infty} \inf\|u_n\|^2 < \infty$ and

$$\int_0^\infty g(u_0^2)t^\gamma dt \leqq \liminf_{n\to\infty} \int_0^\infty g(u_n^2)t^\gamma dt < \infty.$$

Then

$$\left|\int_0^T (u_0^2 + \dot{u}_0^2)t^\gamma dt - \int_0^\infty (u_n^2 + \dot{u}_n^2)t^\gamma dt\right|$$
$$\leqq \left|\int_0^T (u_0^2 + \dot{u}_0^2)t^\gamma dt - \int_0^T (u_n^2 + \dot{u}_n^2)t^\gamma dt\right| + \left|\int_T^\infty (u_n^2 + \dot{u}_n^2)t^\gamma dt\right|.$$

Multiplying (1.15) by u_n (set $y(t) = u_n(t)$) and integrating by parts gives

$$\int_T^\infty (u_n^2 + \dot{u}_n^2)t^\gamma dt = \alpha_n \int_T^\infty f(y_n^2)y_n u_n t^\gamma dt + T^\gamma u_n(T)\dot{u}_n(T),$$

and

$$\int_T^\infty f(y_n^2)y_n u_n t^\gamma dt \leqq \left(\int_T^\infty f(y_n^2)^2 t^\gamma dt\right)^{1/2} \left(\int_T^\infty y_n^2 u_n^2 t^\gamma dt\right)^{1/2}.$$

Using (1.17), (1.8), and (1.11) yields

$$\int_T^\infty f(y_n^2)y_n u_n t^\gamma dt \leqq CT^{-\gamma/2}\left(\int_T^\infty (y_n^2 u_n^2 t^{2\gamma} dt\right)^{1/2} \leqq CT^{-\gamma/2}.$$

Therefore

$$\left|\int_0^T (u_0^2 + \dot{u}_0^2)t^\gamma dt - \int_0^\infty (u_n^2 + \dot{u}_n^2)t^\gamma dt\right|$$
$$\leqq C\alpha_n T^{-\gamma/2} + T^\gamma u_n(T)\dot{u}_n(T).$$

If follows that

$$\limsup_{n\to\infty}\left|\int_0^\infty (u_0^2 + \dot{u}_0^2)t^\gamma dt - \int_0^\infty (u_n^2 + \dot{u}_n^2)t^\gamma dt\right|$$
$$\leqq \int_T^\infty (u_0^2 + \dot{u}_0^2)t^\gamma dt + CT^{-\gamma/2} + T^\gamma u_0(T)\dot{u}_0(T).$$

Letting $T \to \infty$ we have $\|u_0\|^2 = \lim_{n\to\infty}\|u_n\|^2$. Next we use (III) and (1.11) to see that

$$\int_0^\eta g(u_n^2)t^\gamma dt \leqq C\int_0^\eta u_n^4 t^\gamma dt \leqq C\int_0^\eta t^{2-\gamma} dt = O(\eta^{3-\gamma})$$

as $\eta \to 0+$ since $u_n(t) = O(t^{1/2-\gamma/2})$ uniformly with respect to n. Also, using (1.8), (1.14) and Lemma 1.7,

$$\int_T^\infty g(u_n^2)t^\gamma dt \leqq CT^{-\gamma}\int_T^\infty u_n^4 t^{2\gamma} dt \leqq CT^{-\gamma}\int_T^\infty u_n^2 t^\gamma dt \leqq CT^{-\gamma}.$$

It follows that

$$\lim_{n\to\infty}\int_0^\infty g(u_n^2)t^\gamma dt = \int_0^\infty g(u_0^2)t^\gamma dt,$$

and hence $J(u_n) \to J(u_0) = \lambda$. Finally,

$$J(u_n) \geqq \delta(1 + \delta)^{-1}\|u_n\|^2 \geqq C$$

by (1.12), so that $J(u_0) = \lambda > 0$, and $u_0(t) \not\equiv 0$. It follows in similar fashion that

$$\lim_{n\to\infty} \int_0^\infty f(u_n^2)u_n^2 t^\gamma dt = \int_0^\infty f(u_0^2)u_0^2 t^\gamma dt,$$

so that $u_0 \in A$.

Next we consider the equation

$$-(t^\gamma u')' + t^\gamma u = \alpha_0 u_0 f(u_0) t^\gamma. \tag{1.23}$$

Then, proceeding as before, $u(t) = \alpha_0 \int_0^\infty g(t, \tau) f(u_0^2) u_0 d\tau$ is a solution, where α_0 has been chosen so that $u \in A$. By Lemma 1.7 we must have $J(u) \leqq J(u_0)$, and hence $J(u) = J(u_0)$, and $u(t) \equiv u_0(t)$. Then $u_0(t) = \alpha_0 \int_0^\infty g(t, \tau) f(u_0^2) u_0 d\tau$ satisfies (1.6). Since $\|u_0\| < \infty$, $u_0(t) = O(t^{-\gamma/2})$ by (1.8), and $\lim_{t\to\infty} u_0(t) = 0$. Also, it is clear that $u_0 \in C^2(0, \infty)$. Next we observe that

$$\int_0^T (u_0^2 + \dot{u}_0^2) t^\gamma dt = \alpha_0 \int_0^T f(u_0^2) u_0^2 t^\gamma dt + T^\gamma u_0(T) \dot{u}_0(T)$$

so that $\|u_0\|^2 = \alpha_0 \int_0^\infty f(u_0^2) u_0^2 t^\gamma dt$, hence $\alpha_0 = 1$ since $u_0 \in A$. It follows from the representation $u_0(t) = \int_0^\infty g(t, \tau) f(u_0^2) u_0 d\tau$ and condition (II) that $u_0(t) > 0$ for $t \in (0, \infty)$.

If we now investigate the corresponding functions $\varphi_0(t)$ and $\psi_0(t)$ it is easy to show that

$$t^{-\nu} K_\nu(t) \varphi_0(t) = O(t^{7-3\gamma/2}) = o(1)$$

as $t \to 0+$, so that

$$\lim_{t\to 0+} u_0(t) = \frac{1}{2^\nu \Gamma(\nu + 1)} \int_0^\infty \tau^{-\nu} K_\nu(\tau) f(u_0^2) u_0 \tau^\gamma d\tau.$$

Putting this back into $\varphi_0(t)$ we find $\varphi_0(t) = O(t^{\gamma+1})$, so $t^{-\nu} K_{\nu+1}(t) \varphi_0(t) = o(1)$ and $t^{-\nu} I_{\nu+1}(t) \psi_0(t) = o(1)$ as $t \to 0+$, so that $\lim_{t\to 0+} \dot{u}_0(t) = 0$. We have thus proved the following theorem.

THEOREM 1.1. *If f satisfies conditions* (I)–(IV) *and* $1 \leqq \gamma \leqq 2$, *then there exists a solution of* (1.6) *for which* $y \in C^2(0, \infty)$, $y(t) > 0$ *for* $t \in (0,\infty)$, $\lim_{t\to\infty} y(t) = 0$, $\lim_{t\to 0+} y(t) > 0$, *and* $\lim_{t\to 0+} \dot{y}(t) = 0$.

2. In this section it will be shown that equation (1.6) has, in addition to the positive solution guaranteed by Theorem 1.1, an infinite number of other solutions which may be obtained by solving the minimum problem (1.7) under increasingly restrictive constraints. The associated minimal values of (1.7) will be denoted by $\lambda_1, \lambda_2, \ldots$, where $\lambda_1 = \lambda$ is the number defined in Lemma 1.5, and $0 < \lambda_1 < \lambda_2 < \cdots$.

We first consider the problem

$$(2.1)\quad -t^{-\gamma}(t^{\gamma}y')' + y = f(y^2)y\begin{cases} y(a) = y(b) = 0,\ 0 < a < b \leqq \infty \\ \dot{y}(a) = y(b) = 0,\ 0 = a < b \leqq \infty, \end{cases}$$

where as usual $1 \leqq \gamma \leqq 2$. The variational techniques of §1 may be used on (2.1) simply by choosing the appropriate Green's function $g(t, \tau)$ for the corresponding interval and boundary conditions. The result may be stated as the following theorem.

THEOREM 2.1. *Suppose f satisfies* (I)–(IV), $1 \leqq \gamma \leqq 2$, *and* $A = \{y \mid y \in D^1(a, b),\ y(t) \geqq 0,\ y(t) \not\equiv 0,\ y(a) = y(b) = 0\ (\dot{y}(0+) = 0 \text{ if } a = 0),\ \int_a^b (y^2 + \dot{y}^2)t^{\gamma}dt = \int_a^b f(y^2)y^2t^{\gamma}dt\}$. *If* $J(y) = \int_a^b \{y^2 + \dot{y}^2 - g(y^2)\}t^{\gamma}dt$, *then the minimum problem* $\min_A J(y) = \lambda(a, b)$ *is solved by a solution of* (2.1). *Moreover,* $y(t) > 0$ *in* (a, b), $\lambda(a, b) > 0$, *and if* $a = 0$, *then* $\lim_{t\to 0+} y(t) > 0$.

We now proceed to show the existence of a discrete infinity of solutions $\{y_n(t)\}$ of (1.6) such that $\lim_{t\to\infty} y_n(t) = 0$, $\lim_{t\to 0+} y_n(t) > 0$, $\lim_{t\to 0+} \dot{y}_n(t) = 0$, and $y_n(t)$ has exactly n distinct zeros in $(0, \infty)$. The procedure depends on the following lemma.

LEMMA 2.1. *If* $\lambda(a, b)$ *denotes the minimum of* $J(y)$ *for the interval* $[a, b]$, *then*

(a) *if* $a \leqq a' \leqq b' < b$, *then* $\lambda(a, b) \leqq \lambda(a', b')$.

(b) $\lambda(a, b) \to \infty$ *as* $b - a \to 0$ (*as* $a \to \infty$ *if* $b = \infty$, *as* $b \to 0$ if $a = 0$).

(c) $\lambda(a, b)$ *is a continuous function of a and b* (*of b if* $a = 0$, *of a if* $b = \infty$).

PROOF. Parts (a) and (c) follow precisely as in [9], so we consider only (b). If $0 < a < b < \infty$, we first observe that (1.8) can be proved exactly as in the proof of Lemma 1.1. Using (1.8) and (III) we have

$$\|y\|^2 \leqq C\int_a^b y^4t^{\gamma}dt \leqq Ca^{-\gamma}\int_a^b y^2t^{2\gamma}dt \leqq Ca^{-\gamma}(b - a)\|y\|^4.$$

Since $\|y\| \neq 0$, we have $\|y\|^2 \geqq a^{\gamma}/c(b - a)$ and $\|y\|^2 \to \infty$ as $b \to a$. It follows from (1.13) that also $\lambda(a, b) \to \infty$ as $b \to a$.

In the case $0 < a < b = \infty$ we see (III), (1.8) as before to get

$$\|y\|^2 \leqq Ca^{-\gamma}\|y\|^2\int_a^{\infty} y^2t^{\gamma}dt \leqq Ca^{-\gamma}\|y\|^4,$$

so that $\|y\|^2 \to \infty$, and hence also $\lambda(a, \infty) \to \infty$ as $a \to \infty$.

Finally we suppose $0 = a < b < \infty$. Since $y(b) = 0$, we have

$$(2.2)\quad \begin{aligned} y^2(t) &= \left(\int_t^b \dot{y}dt\right)^2 \leqq (b - t)\int_t^b \dot{y}^2dt \\ &\leqq (b - t)t^{-\gamma}\int_t^b \dot{y}^2\tau^{\gamma}d\tau \leqq (b - t)t^{-\gamma}\|y\|^2. \end{aligned}$$

Using (2.2) and (III) we obtain, with $0 < a < b$,

$$\int_a^b (y^2 + \dot{y}^2)t^\gamma dt \leqq C(b-a)\int_a^b (y^2 + \dot{y}^2)t^\gamma dt \int_a^b y^2 dt. \tag{2.3}$$

But

$$\begin{aligned}\int_a^b y^2 dt &\leqq 2\int_a^b |y\dot{y}|(\tau - a)d\tau \leqq 2a^{1-\gamma}\int_a^b |y\dot{y}|\tau^\gamma d\tau\\ &\leqq a^{1-\gamma}\int_a^b (y^2 + \dot{y}^2)\tau^\gamma d\tau,\end{aligned}$$

so (2.3) becomes

$$\int_a^b (y^2 + \dot{y}^2)t^\gamma dt \leqq c(b-a)a^{1-\gamma}\left(\int_a^b (y^2 + \dot{y}^2)t^\gamma dt\right)^2.$$

We then have

$$\|y\|^2 = \int_0^b (y^2 + \dot{y}^2)t^\gamma dt \geqq \frac{a^{\gamma-1}}{c(b-a)}.$$

Setting $a = b/2$ and letting $b \to 0$ shows that $\|y\|^2 \to \infty$ for $1 \leqq \gamma < 2$. When $\gamma = 2$, we have $\|y\|^2 \geqq ca(b-a)$. Setting $b_n = n^{-1}$ and $a_n = n^{-1}(1 - n^{-1})$, we find $a_n/(b_n - a_n) \to \infty$, hence $\|y\|^2 \to \infty$ as $b \to 0$ for $1 \leqq \gamma \leqq 2$.

To formulate the minimum problem defining λ_n we choose $(n + 2)$ points t_k such that $0 = t_0 < t_1 < \cdots < t_{n+1} = \infty$. In the interval $[t_{k-1}, t_k]$ we consider the minimum problem

$$\min_{A_k} J(y) = \min_{A_k}\int_{t_{k-1}}^{t_k} \{y^2 + \dot{y}^2 - g(y^2)\}t^\gamma dt$$

where

$$\begin{aligned}A_k = \{y \mid y \in D^1(t_{k-1}, t_k),\ &y(t) \geqq 0, y(t) \not\equiv 0,\\ &y(t_k) = y(t_{k-1}) = 0 \quad (\text{for } k = 0, \dot{y}(0+) = 0),\\ &\int_{t_{k-1}}^{t_k} (y^2 + \dot{y}^2)t^\gamma dt = \int_{t_{k-1}}^{t_k} f(y^2)y^2 t^\gamma dt\}\end{aligned}$$

for $1 \leqq k \leqq n + 1$. Theorem 2.1 shows that it is sufficient to consider this minimum problem for functions $y_k(t)$ which in the intervals (t_{k-1}, t_k) are the solutions of (2.1), whose existence is guaranteed by Theorem 2.1.

It now follows by Lemma 2.1, as in [9], that the function

$$\Lambda(t_1, \ldots, t_n) = \sum_{i=1}^{n+1} \lambda(t_{i-1}, t_i)$$

attains its minimum for certain values $0 = t_0 < t_1 < \cdots < t_{n+1} = \infty$, and that $\lambda_1 < \lambda_2 < \cdots$. We now define $y(t)$ on $(0, \infty)$ by setting $y(t) =$

$y_k(t)$ in $[t_{k-1}, t_k]$, where if necessary $y_k(t)$ is replaced by $-y_k(t)$ to assure that $y(t)$ changes sign at each t_k $(1 \leqq k \leqq n)$. Then $y(t)$ has precisely n zeros in $(0, \infty)$, and as in [9], it can easily be shown that

$$\lim_{t \to t_k^-} \dot{y}(t) = \lim_{t \to t_k^+} \dot{y}(t) \quad (1 \leqq k \leqq n).$$

Hence $y(t)$ is a solution of (1.6) on $(0, \infty)$.

The results of this section may be summarized in the following theorem.

THEOREM 2.2. *Let Γ_n denote the class of functions $y(t)$ with the following properties*: $y \in D^1(0, \infty)$, $y(t_k) = 0$ $(1 \leqq k \leqq n,\ n \geqq 1)$, *where* $0 = t_0 < t_1 < \cdots < t_{n+1} = \infty$; *for* $1 \leqq k \leqq n+1$

$$\int_{t_{k-1}}^{t_k} (y^2 + \dot{y}^2) t^\gamma dt = \int_{t_{k-1}}^{t_k} f(y^2) y^2 t^\gamma dt,$$

where f satisfies (I)–(IV). *If* $g(s) = \int_0^s f(u)du$, *the variational problem*

$$\min J(y) = \min \int_0^\infty \{y^2 + \dot{y}^2 - g(y^2)\} t^\gamma dt = \lambda_n, \quad y \in \Gamma_n \tag{2.4}$$

has a solution $y_n \in C^1(0, \infty)$, *and the numbers* λ_n *are strictly increasing. The function* $y_n(t)$ *has precisely n zeros in* $(0, \infty)$ *and satisfies the system*

$$\begin{gathered} -t^{-\gamma}(t^\gamma y')' + y = f(y^2)y, \quad t \in (0, \infty) \\ \lim_{t \to \infty} y(t) = 0, \ \lim_{t \to 0+} y(t) > 0, \ \lim_{t \to 0+} \dot{y}(t) = 0. \end{gathered} \tag{2.5}$$

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