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EMBEDDING NONCOMPACT MANIFOLDS

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0. **Introduction.** Let X and Y denote PL spaces; that is, locally compact, separable, metric spaces each of which possesses a piecewise linear structure. The map $f: X \to Y$ is k-connected provided $\pi_i(f) = \pi_i(M_f, X) = 0$ for $i \leqq k$ where M_f denotes the mapping cylinder of f. In [6] Hudson proves that if f is a map between a compact PL manifold M^m and a PL manifold Q^q, $f \mid \partial M$ is an embedding of ∂M into ∂Q and $q - m \geqq 3$, then f is homotopic rel ∂M to a PL embedding provided $\pi_i(f) = 0$ for $i \leqq 2m - q + 1$ and $\pi_i(Q) = 0$ for $i \leqq 3m - 2q + 3$. Theorem 4.2 extends this theorem to the case where M is noncompact with appropriate additional assumptions. The assumption that Q be $3m - 2q + 3$ connected in Hudson's Theorem was later shown to be unnecessary (see [5, Ch. 12]) using surgery techniques. The techniques of this paper, which are an extension of those of [6] and [12] require this connectivity. Using PL approximation techniques Berkowitz and Dancis [1] were able to prove a theorem similar to Theorem 4.2 in the 3/4 range which does not require connectivity of Q.

The term space shall always mean a locally compact, separable, metric space. A polyhedron is a compact PL space. A PL m-manifold is a PL space locally homeomorphic with euclidean m-space. A map f between spaces X and Y is proper provided $f^{-1}(C)$ is compact for each compact subset C of Y. All maps and homotopies are assumed to be proper unless stated otherwise. The symbol "$\simeq$" is read "is homotopic to". The symbol Λ denotes the halfline $[0, \infty)$ and a subspace of a PL space X which is homemorphic to Λ is called a ray in X. All deformation retractions are assumed to be strong deformation retractions in the sense of [8]. The symbol ∂ denotes boundary and the abbreviation int denotes interior.

Sections 1, 2, and 3 should provide a self-contained treatment of infinite engulfing and its relation to connectivity at infinity (c.f., Lemma 2.1 of [1]).

1. **Proper Collapsing.**

DEFINITION 1.1. There is an elementary collapse from the polyhedron P to the polyhedron Q, denoted $P \searrow eQ$, provided $P = Q \cup D$ where D

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is an n-cell for some n and $D \cap Q$ is a face of ∂D. There is a collapse from P to Q, denoted $P \searrow Q$, provided there are polyhedra $P_0, P_1, \cdots, P_k$ with $P = P_0$, $Q = P_k$ and $P_{i-1} \searrow eP_i$ for $i = 1, 2, \cdots, k$.

The definitions for the companion notion of an elementary simplicial collapse and a simplicial collapse are omitted. For those unfamiliar with these definitions, see [3] or [5]. The notation for simplicial collapse will be the same as that for collapse; the context will prevent any confusion.

The following definition is based on the definition of infinite simple homotopy equivalence contained in [2]. The idea for this definition is also mentioned in the remark at the end of § 8 of [3].

Definition 1.2. There is an elementary proper collapse from the *PL* space Y to the subspace X, denoted $Y \searrow ep\, X$, provided $Y = X \cup C_1 \cup C_2 \cup \cdots$ where the possible infinite collection of (compact) polyhedra $\{C_i\}$ satisfy

(i) $(C_i \setminus X) \cap (C_j \setminus X) = \Phi$ for $i \neq j$ and

(ii) $C_i \searrow C_i \cap X$ or each i.

This definition is easier to work with than the combinatorial analogue of A. Scott found in [10].

Definition 1.3. There is a proper collapse from Y to X, denoted $Y \searrow p\, X$, provided there exist a sequence $Y_0, Y_1, \cdots, Y_k$ of *PL* spaces such that $Y = Y_0$, $X = Y_k$ and $Y_{i-1} \searrow ep\, Y_i$ for $i = 1, 2, \cdots, k$.

Lemma 1.4. *If $Y \searrow p\, X$, then there is a proper PL deformation retraction of Y to X.*

Proof. The proof follows easily by induction from the fact that if $Y \searrow ep\, X$, then there is a *PL* deformation retraction of C_i onto $C_i \cap X$ for each i and hence a proper deformation retraction of Y onto X.

Let $f: K \to L$ be a simplicial map between locally finite simplicial complexes of finite dimension. Let M_f denote the simplicial mapping cylinder as defined by Zeeman in [12].

Lemma 1.5. *Let σ be an n-simplex and $f: \sigma \to \tau$ a simplicial map onto τ. Then M_f is a combinatorial $(n + 1)$ ball with σ and τ simplexes in the combinatorial boundary.*

Proof. See Lemma 46 of [12].

Theorem 1.6. *If $f: K \to L$ is a simplicial map between finite dimensional complexes, then $M_f \searrow p\, L$.*

PROOF. Assume inductively the theorem is true for any complex of dimension less than n. Suppose $\dim K = n$. Let σ be an n-simplex of K, $\tau = f(\sigma)$. Then by Lemma 1.5, $M_{f|\sigma} \searrow e\, M_{f|\partial\sigma}$. Since the collapses for each n-simplex of K are disjoint, there is an elementary proper collapse of M_f to $M_{f|K'}$ where K' is the $(n-1)$-skeleton of K. But $M_{f|K'} \searrow p\, L$ by induction. Thus $M_f \searrow p\, L$.

LEMMA 1.7. *If $Y \searrow p\, X$, $X \subset Z$, and $Y \cup Z$ is a PL subspace of the PL manifold M, then there exists a PL ambient isotopy H of M which is fixed on Y such that $Y \cup H_1(Z) \searrow p\, H_1(Z)$. (The special notation for proper collapse is omitted from proofs but maintained in the statement of results.)*

PROOF. The proof is by induction on the number of elementary proper collapses. Suppose first that $Y \searrow ep\, X$. Then $Y = X \cup C_1 \cup C_2 \cup \cdots$ where each C_i is a polyhedron, $C_i \searrow C_i \cap X$ and $C_i \backslash X \cap C_j \backslash X = \emptyset$ if $i \neq j$. For each i, apply Lemma 42 of [**12**] to $C_i \searrow C_i \cap X$ and choose an ambient isotopy H^i of M such that $H_t^i \mid C_i \cap X = 1_X$, and $C_i \cup H_1^i(Z) \searrow H_1^i(Z)$. The proof of Lemma 42 reveals that H^i can be constructed so that the support of H^i is the open star of $C_i \backslash X$. Thus by triangulating carefully, one may choose each H^i so that $\sup H^i \cap \sup H^j = \emptyset$ if $i \neq j$. Thus one may define an isotopy H of M by

$$H_t(x) = \begin{cases} H_t^i(x) & \text{if } x \in \sup H^i \\ x & \text{otherwise.} \end{cases}$$

Suppose the theorem is true for all proper collapses consisting of k or less elementary proper collapses. Further suppose that $Y = Y_0 \searrow ep\, Y_1 \searrow ep \cdots \searrow ep\, Y_k \searrow ep\, Y_{k+1} = X$. By induction one may choose an ambient isotopy H of M such that $H|_X = 1_X$ and $Y_1 \cup H_1(Z) \searrow H_1(Z)$. Let $Z' = Y_1 \cap H_1(Z)$. Then $Y \searrow Y \cap Z'$. Let K be an isotopy of M such that $K|_{y_1} = I_{Y_1}$ and $Y \cup K_1(Z') \searrow K_1(Z')$. Since $Z' \searrow H_1(Z)$, $K_1(Z') \searrow K_1 H_1(Z)$.

LEMMA 1.8. *Suppose $Y \searrow pX$. Let W be a subspace of Y. Then there is a subspace Z of Y containing W such that $Y \searrow pZ \cup X \searrow pX$,* $\dim Z \leqq \dim W + 1$, *and* $\dim (Z \cap X) \leqq \dim W$.

PROOF. The proof is by induction on the number of elementary proper collapses in $Y \searrow X$. Suppose $Y \searrow ep\, X$. Then $Y = X \cup C_1 \cup C_2 \cup C_3 \cup \cdots$ with the $\{C_i\}$ satisfying the conditions of Definition 1.2. For each i, let $W_i = W \cap C_i$. Applying Lemmas 44 and 45 of [**12**], one has the existence of a polyhedron Z_i of C_i such that $W_i \subset Z_i$, $X \cup C_i \searrow Z_i \cup X \searrow X$ and the dimension condition on Z_i is satisfied.

In the terminology of [12], Z_i is the trail of W_i in a simplicial collapse of C_i to $C_i \cap X$. Let $Z = \cup Z_i$ and one has the desired subspace.

Suppose the theorem holds whenever $Y \searrow X$ by k or less elementary proper collapses. Suppose $Y = Y_0 \searrow ep\, Y_1 \searrow ep \cdots \searrow ep\, Y_k \searrow ep\, Y_{k+1} = X$. By induction there is a PL subspace $Z' \subset Y$ such that $Y \searrow Z' \cup Y_k \searrow Y_k$, $W \subset Z'$, $\dim Z' \leqq \dim W + 1$, and $\dim (Z' \cap Y_k) \leqq \dim W$. Let $W' = (W \cap Y_k) \cup (Z' \cap Y_k)$. There exists a subspace $Z^2 \subset Y_k$ with $W' \subset Z^2$, $Y_k \searrow Z^2 \cup Y_{k+1} \searrow Y_{k+1}$, $\dim Z^2 \leqq \dim W' + 1$, and $\dim Z^2 \cap Y_{k+1} \leqq \dim W'$. But since $Z' \cap Y_k \subset Z^2$ and $Z^2 \subset Y_k$, $Z' \cup Y_k \searrow Z' \cup Z^2 \cup Y_{k+1} \searrow Z^2 \cup Y_{k+1} \searrow Y_{k+1}$. This is an application of the notion of excision as described in [12]. Let $Z = Z' \cup Z^2$. Then $W \subset Z$, $Y \searrow Z \cup X \searrow X$, $\dim Z \leqq \dim W + 1$, and $\dim Z \cap X \leqq \dim W$.

LEMMA 1.9. *Suppose $f: Y \to Z$ is a proper PL map between PL spaces and $S(f)$ denotes the singular set of f. If $S(f) \subset X$ and $Y \searrow pX$, then $f(Y) \searrow p\,(f(X))$.*

PROOF. The proof is by induction on the number of elementary proper collapses. Suppose $Y \searrow ep\, X$. Then $Y = Y_1 \cup C_1 \cup C_2 \cup C_3 \cup \cdots$. For each i, let $f_i = f \mid C_i \cup X$; then $S(f_i) \subset X$. By Lemma 38 of [12], $f_i(C_i \cup X \searrow f_i(X)$. Since $(C_i \setminus X) \cap (C_j \setminus X) = \Phi$ and $S(f) \subset X$, $f(C_i \setminus X) \cap f(C_j \setminus x) = \Phi$. Hence

$$\begin{aligned} f(Y) &= f(X) \cup f(C_1) \cup f(C_2) \cup f(C_3) \cup \cdots \\ &= f(X) \cup f_1(C_1) \cup f_2(C_2) \cup f_3(C_3) \cdots \searrow f(X). \end{aligned}$$

Now, suppose that $Y = Y_0 \searrow ep\, Y_1 \searrow \cdots \searrow ep\, Y_k \searrow ep\, Y_{k+1} = X$. Let $f_1 = f \mid Y_1$. Then $S(f_1) \subset X$. So inductively, $f_1(Y_1) \searrow f_1(X)$. But $S(f) \subset Y_1$, so $f(Y) \searrow f(Y_1) = f_1(Y_1) \searrow f_1(X) = f(X)$.

A space X is collapsible provided $X \searrow p\,\Lambda$. Let H^n denote n-dimensional half space; i.e., $\{(x_1, \cdots, x_n) \mid x_n \geqq 0\}$.

THEOREM 1.10. *Suppose M^n is a PL manifold and X is a collapsible PL subspace of M^n. Then a regular neighborhood of X is homeomorphic to H^n.*

PROOF. Let N be a regular neighborhood o X. Then N is a regular neighborhood of Λ where $X \searrow \Lambda$. Assume without loss of generality that N is a derived neighborhood of Λ in some triangulation of M. Let A_1 be an arc in Λ from the initial point of Λ and let v_1 be the other endpoint of A_1. Let N_1 be the simplicial neighborhood of A_1 rel v_1. N_1 is a PL ball. Let A_2 be an arc in Λ starting with v_1 and sufficiently long that if v_2 is the terminal point of A_2, $\mathrm{st}(v_2, N) \cap N_1 = \Phi$. Let n_2

be the simplical neighborhood of A_2 in $\mathrm{cl}(N - N_1)$ rel v_2. Then $N_1 \cap N_2 = \mathrm{st}(v_1, \partial N_1) = \mathrm{st}(v_1, \partial N_2)$ is a face of each of ∂N_1 and ∂N_2. Furthermore, $\mathrm{st}(v_2, \partial N_2) \cap \mathrm{st}(v_1, \partial N_2) = \Phi$, hence $\mathrm{cl}(\partial N_2 -)\mathrm{st}(v_1, \partial n_2) \cup \mathrm{st}(v_2, \partial N_2))$ is homeomorphic to $\mathrm{lk}(v_1, \partial N_2) \times I$. By repeating this process one can write N as UN_1, where N_i is an n-ball for each i, $N_i \cap N_{i+1}$ is a face of each of ∂N_i and ∂N_{i+1}, and $\mathrm{cl}[\partial N_i - (\mathrm{st}(v_{i-1}, \partial N_i) \cup \mathrm{st}(v_i, \partial N_i))]$ is homeomorphic to $\mathrm{lk}(v_{i-1}, \partial N_i) \times I$. Using the N_i's one can now easily build a homeomorphism between N and H^n.

2. Inessential Maps and Subspaces.

Definition 2.1. A proper map $f : X \to Y$ is inessential provided there are proper maps $\alpha : X \to \Lambda$ and $g : \Lambda \to Y$ such that $f \simeq (g \circ \alpha)$.

Remark. Note that if $f' \simeq f$, then f' is inessential.

Lemma 2.2. *Suppose that in Definition 2.1 X and Y are PL spaces and f is a proper PL map. Then we may require α, g and the homotopy to be PL.*

Proof. Follows directly from the relative simplicial approximation theorem.

Theorem 2.3. *A proper (PL) map f between (PL) spaces X and Y is inessential if and only if there are proper (PL) maps $\alpha : X \to \Lambda$ and $\bar{g} : M_\alpha \to Y$ such that $\bar{g} \mid X = f$.*

Proof. The proof of the *PL* version of the theorem is provided. The proof of the theorem with *PL* omitted is basically the same.

Suppose first that one has *PL* maps α and $\bar{g}$ as above. Let r_t, $0 \leqq t \leqq 1$, denote a proper *PL* deformation retraction of M_α onto Λ. The existence of such a *PL* map follows from the fact that $M_\alpha \searrow \Lambda$ (Theorem 1.6). Define $H_t : X \to Y$ by $H_t(x) = \bar{g}r_1(x)$. But $r_1 \mid X$ is properly homotopic to α so $\bar{g} \circ r_1 \simeq \bar{g} \circ \alpha$ and $f \simeq \bar{g} \circ \alpha$.

Suppose f is inessential. By Lemma 2.2 one has *PL* maps $\alpha : X \to \Lambda$ and $g : \Lambda \to Y$ such that $g \circ \alpha \simeq f$. Let $H : X \times I \to Y$ be a *PL* homotopy with $H_0 = f$ and $H_1 = g \circ \alpha$. Consider the following diagram:

$$\begin{tikzcd} X \times I \cup \Lambda \arrow[rr, "{(h, g)}"] \arrow[d, "p"'] & & Y \\ M_\alpha \arrow[urr, "g'"'] & & \end{tikzcd}$$

where p denotes projection onto the mapping cylinder. (See [**12**, Lemma 47].) Pick $\langle x, t\rangle$ in M_α. Then since (H, g) is constant on $p^{-1}(\langle x, t\rangle)$, g' is defined and continuous and makes the diagram commute. But $g'|X = g'|p(X \times 0) = H_0 = f$. Since p is not PL, g' is not PL. But g' is homotopic, relative to $p(X \times 0) = X$, to a PL map $\bar{g}$ using relative simplicial approximation.

DEFINITION 2.4. A subspace X of Y is inessential (in Y) provided the inclusion map $j : X \to Y$ is inessential.

REMARK. If K is a compact subspace of Y then this definition is easily seen to agree with the usual definition as in [**6**] or [**12**].

LEMMA 2.5. *Suppose Y is a PL subspace of the PL space M' and $Y \searrow p\, X$ with X inessential in M. Then Y is inessential in M.*

PROOF. Since $Y \searrow p\, X$, there is a proper deformation retraction of Y onto X. Let $r : Y \to X$ denote the resulting retraction. Now choose $\alpha : X \to \Lambda$ and $g : \Lambda \to M$ such that $g \circ \alpha \cong i_x$ where i_x denotes the inclusion of X into M. Since $i_x \circ r \simeq i_y$, one has $(g \circ \alpha) \circ r \simeq i_x \circ r \simeq i_y$. Hence Y is inessential in M.

THOEREM 2.6. *Let M^m denote a PL manifold. If the PL subspace X^x is inessential in* $\operatorname{int} M$, *then there exist subspaces Y^y, Z^z in* $\operatorname{int} M$ *such that $X \subset Y \searrow p\, Z$, $y \leqq x + 1$, and $z \leqq 2x - m + 2$.*

PROOF. First, a weaker result; namely, $X \subset Y \searrow Z$, $y \leqq x + 1$ and $z \leqq 2x - m + 3$. Since X is inessential in M there is a PL map $\alpha : X \to \Lambda$ and a PL map $\bar{g} : M_\alpha \to M$ such that $\bar{g}|_X = i$, where i denotes the inclusion map of X into M. Using the relative general position Theorem 4 of [**5**], one may assume that $\dim S_2(\bar{g}) \leqq 2x - m + 2$. $M_\alpha \searrow \Lambda$. Let $W = S_2(g)$. Lemma 1.8 gives one a subspace Z' of M_α containing W with $M_\alpha \searrow Z' \cup \Lambda$ and $\dim Z' \leqq 2x - m + 3$. The results now follow from Lemma 1.9 if $Y = \bar{g}(M_\alpha)$ and $Z = \bar{g}(Z')$. For the stronger result $\dim Z \leqq 2x - m + 2$ the proof proceeds as above until one is ready to find the subspace Z' containing $S_2(g)$. In order to choose Z' with $\dim Z' \leqq 2x - m + 2$, one needs a proper version of Zeeman's piping lemma [**12**, Lemma 48]. Stated below is the proper version needed.

LEMMA 2.7. *Let M^m be a manifold and let X^x, $J_0^x \subset J^{x+1}$ be cyclinderlike, $x \leqq m - 3$. Let $f : J \to M$ be a proper map in general position for the pair J, $X \cup J_0$ and such that $f(J - J_0) \subset$* $\operatorname{int} M$. *Then there exists a proper map $f_1 : J \to M$, properly homotopic to f keeping $X \cup J_0$ fixed, and a subspace $J_1 \subset J$ such that $f_1(J - J_0) \subset$* $\operatorname{int} M$, $S(f_1) \subset J_1$, $\dim J_1 \leqq 2x - m + 2$, $\dim (J_0 \cap J_1) \leqq 2x - m + 1$ *and* $J \searrow J_0 \cup J_1 \searrow J_0$.

REMARK. The long and detailed proof by Zeeman extends without alteration. The key to being able to handle the case where J is noncompact is in the nature of the piping technique. Essentially the piping takes place through a sequence of independent alterations on cylinder-like subcomplexes of J. A proof is omitted.

Returning to the proof of Theorem 2.6, choose a PL map $\alpha : X \to \Lambda$ and a PL map $\bar{g} : M_\alpha \to M \ni \bar{g} \mid X = i$. Assume $\bar{g}$ is a general position map. Then if J_0 denotes the submapping cylinder of M_α determined by restricting α to the $(x-1)$-skeleton of a triangulation, X^x, $J_0 \subset M_\alpha$ is cylinderlike. Hence one has a proper homotopy of $\bar{g}$ to a map $\bar{g}_1 : M_\alpha \to M$ and a subspace J_1 such that $S_2(\bar{g}_1) \subset J_1$, $\dim J_1 \leqq 2x - m + 2$, $\dim J_0 \cap J_1 \leqq 2x - m + 1$, and $J \searrow J_0 \cup J_1 \searrow J_0$. By Lemma 1.8, there exists a subspace $Z' \subset J_0$ such that $Z' \supset [(J_1 \cap J_0)]$, $\dim Z' \leqq 2x - m + 2$, and $J_0 \searrow Z' \cup \bar{\Lambda} \searrow \Lambda$. Hence $\bar{A} \searrow A$. Let $v J_1 \cup Z'$. Then $J \searrow Z \cup A \searrow A$. But $S_2(\bar{g}_1) \subset Z$. Thus $\bar{g}_1(J) \searrow \bar{g}_1(Z)$ and dimension $g_1(Z) \leqq 2x - m + 2$. The subspace Y of the theorem is $\bar{g}_1(J)$.

3. Connectivity at ∞.

DEFINITION 3.1. Let X be a closed subspace of Y. The statement that (Y, X) is locally n-connected at ∞ means given any cofinal family $\{C_j\}$ of compact subsets of Y there exist a cofinal family $\{D_j\}$ of compact subsets of Y such that

(1) $D_j \supset C_j$ for each j and

(2) the inclusion induced map

$$i* : \pi_k(Y - D_j, X - D_j) \to \pi_k(Y - C_j, X - C_j)$$

is the zero map for each $j \geqq 1$ and each $k \leqq n$.

REMARK. It is straightforward to check that local n-connectivity at ∞ for the pair (Y, X) is equivalent to the existence of a cofinal monotone sequence $\{D_i\}$ of compact subsets of Y such that the inclusion induced map

$$i* : \pi_k(Y - D_j, X - D_j) \to \pi_k(Y - D_{j-1}, X - D_{j-1})$$

is the zero map for each $j \geqq 1$ and each $k \leqq n$. For each k, the inverse sequence $G_k = \{\pi_k(Y - D_j, X - D_j), i_*\}$ is said to be the essentially constant and $\varprojlim G_k \simeq \operatorname{im} i* = 0$. For a detailed treatment of this topic see [9].

DEFINITION 3.2. The pair (Y, X) is (m, n)-connected if $\pi_k(Y, X) = 0$ for each $k \leqq m$ and (Y, X) is locally n-connected at ∞.

DEFINITION 3.3. A space Y is said to be locally n-connected at ∞ provided there is a closed subspace Λ homeomorphic to $[0, \infty)$ such that (Y, Λ) is locally n-connected at ∞.

LEMMA 3.4. *Suppose there is a proper deformation retraction from the pair (Y, X) to the pair (B, A). Then (Y, X) is (m, n)-connected iff (B, A) is (m, n)-connected.*

PROOF. Let $r : (Y, X) \longrightarrow (B, A)$ denote the retract of pairs. Then $r*$ induces an isomorphism $r* \ : \pi_i(Y, X) \longrightarrow \pi_i(B, A)$ so (Y, X) is m-connected iff (B, A) is m-connected.

Now suppose (B, A) is locally n-connected at ∞. Following the remark after Definition 3.1, choose a cofinal sequence $\{D_j\}$ of compact subsets of B such that the inclusion induced map $i* : \pi_k(B - D_j, A - D_j) \longrightarrow \pi_k(B - D_{j-1}, A - D_{j-1})$ is the zero map for each $j \geqq 1$ and $K \leqq n$. Let $D_j' = r^{-1}(D_j)$. Consider the following diagram:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\pi_k(Y - D'_{j+1}, X - D'_{j+1}) \arrow[r, "r_*"] \arrow[d, "i'_*"] & \pi_k(B - D_{j+1}, A - D_{j+1}) \arrow[dl, "l_*"'] \arrow[d, "i_*"] \\
\pi_k(Y - D'_j, X - D'_j) \arrow[r, "r_*"] \arrow[d, "i'_*"] & \pi_k(B - D_j, A - D_j) \arrow[dl, "l_*"'] \arrow[d, "i_*"] \\
\pi_k(Y - D'_{j-1}, X - D'_{j-1}) \arrow[r, "r_*"] \arrow[d] & \pi_k(B - D_{j-1}, A - D_{j-1}) \arrow[dl] \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

i, i', and l are all inclusion maps. $l \circ r \simeq i'$ provided one includes far enough up the inverse sequence. This means refining each sequence to a subsequence. We assume this has already been done so that $l_* \circ r_* = i_*'$ at each level. Furthermore, $r \circ l = i$ at each level so $r_* \circ l_* = i_*$. Hence $i_* r_* = r_* i_*'$ at each level.

Now suppose each i_*' is the zero map. Since r_* is onto, image $i_* =$ im $i_* \circ r_* =$ im $r_* \circ i_*' =$ zero. Thus i_* is the zero map at each level. Conversely, suppose i_* is the zero map at each level. Then $i_*' \circ i_*') = r_*) =$ zero. So the subsequence of even terms has the desired property.

THEOREM 3.5. *Suppose X is a closed subspace of Y, K is a closed PL subspace of the PL space L, and* $\dim L - K \leqq n$. *Suppose $h : K \times I \cup L \times \{1\} \to Y$ is a proper map such that $h(K \times \{0\}) \subset X$. Then if (Y, X) is (n, n)-connected, h extends to a proper map $H : L \times I \to Y$ such that $H(L \times \{0\}) \subset X$.*

PROOF. If L is a finite complex, the assumption that (Y, X) is locally n-connected at ∞ is unnecessary and the proof is the standard extension argument using induction on the skeleta of a cylindrical triangulation of $Y \times I$. What follows is an outline of how the local connectivity at ∞ is used to extend this type of argument to the case where L is not finite. Let L^{-1} denote the (-1)-skeleton of L; i.e., $L^{-1} = \Phi$. Define $H_{-1} : (K \times I) \cup (L \times \{1\}) \cup (L^{-1} \times I) \to Y$ by $H_{-1} = h$. Then inductively assume one has $H_{k-1} : (K \times I) \cup (L \times \{1\}) \cup (L^{k-1} \times I) \to Y$ defined with the desired properties and one can proceed to construct H_k extending H_{k-1} as follows.

Suppose $\{C_i\}$ and $\{D_i\}$ are cofinal families of compact subsets of Y with the properties provided in Definition 3.1. Define $C_0 = \Phi = D_0$. Let σ denote a k-simplex of $L - K$ and let $J_\sigma = (\partial\sigma \times [0, 1]) \cup (\sigma \times \{1\})$. Let n_σ denote the smallest positive integer j such that $\partial \div \mathrm{p}_k(Y - C_{n_s}, \quad H_{k-1} \quad {}_{(J\sigma)}) \subset Y - D_{n_\sigma}$. Since $i_* : \pi_k(Y - D_{n_\sigma}, X - D_{n_s}) \to \pi_k(Y - C_{n_s}, X - C_{n_s})$ is the zero map, there is a map $\alpha : \sigma \times I \to Y - C_{n_\sigma}$ such that $\alpha | J_\sigma = H_{k-1} | J_\sigma$ and $\alpha(\sigma \times \{0\}) \subset X - C_n$. Thus H_{k-1} together with α defines a map of $(K \cup L^{k-1} \cup \sigma) \times I \cup (L \times \{1\})$ into Y extending H_{k-1}. Repeating this for each k-simplex of L^k defines a map $H_k : (K \cup L^k) \times I \cup (L \times \{1\}) \to Y$ such that $H_k[(K \cup L^k) \times \{0\}] \subset X$ which extends $H_{k=1}$ and consequently h. All that remains is to show that H_k is a proper extension of h. Let E be a compact subset of Y. Pick a positive integer n_0 so that $E \subset C_{n_0}$. Since H_{k-1} is proper, we have $H_{k-1}^{-1}(D_{n_0})$ is a compact subset of $(K \cup L^{k-1}) \times I \cup (L \times 1)$. Thus

$$\mathscr{C} = \{\sigma \in L^k - L^{k-1} \mid H_{k-1}(J) \cap D_{n_0} \neq \Phi\}$$

is finite. If $\sigma \notin \mathscr{C}$, then H_k was defined so that $H_k(\sigma \times i) \cap \mathscr{C}_{n_0} = \Phi$. Thus $H_k{}^{-1}(E) \subset \cup\{\sigma \times I \mid \sigma \in \mathscr{C}\}$ which is compact. $H_k{}^{-1}(E)$ is closed hence compact.

The map H_n is the desired extension of h to $L \times I$.

THEOREM 3.6. *If Q is (k, k) connected then any closed PL subspace of Q of dimension less than or equal to k is inessential in Q.*

PROOF. Let Λ be a ray in Q with (Q, Λ) (k, k) connected. Let L be a subspace of Q with $\dim L \leqq k$. Let i denote the inclusion of L into Q and j the inclusion of Λ into Q. Applying Theorem 2.1 with $Q = Y$, $\Lambda = X$, $K = \Phi$, $L = L$ and $h = i$ one can conclude that there exist a homotopy $H : L \times I \to Q$ such that $H_1 = i$ and H_0 is a proper map of L into Λ. If $\alpha = H_0$ then one has $j \circ \alpha \simeq i$. Thus L is inessential in Q.

DEFINITION 3.7. The proper map $f : X \to Y$ is said to be (m, n)-connected provided the pair (M_f, X) is (m, n)-connected where M_f denotes the mapping cylinder of f and X is identified with $X \times \{0\}$.

THEOREM 3.8. *If the proper map $f : X \to Y$ is (m, n)-connected and g is properly homotopic to f, then g is (m, n)-connected.*

PROOF. Let H be a proper homotopy between $f = H_0$ and $g = H_1$. Then there exists a proper strong deformation retraction of the pair $(M_H, X \times I)$ to $(M_{h_0}, X \times 0)$ and also to $(M_{H_1}, X \times 1)$. By Lemma 2.4, $(M_{H_0}, X \times 0)$ (m, n)-connected implies $(M_H, X \times I)$ (m, n)-connected which in turn implies $(M_{h_1}, X \times 1)$ is (m, n)-connected.

REMARK. A proof of Theorem 3.8 appears in [7]. The proof provided is considerably shorter than the one in [7] and is provided for completeness.

LEMMA 3.9. *Suppose $f : X \to Y$ is an (n, n)-connected proper map between the spaces X and Y. Suppose K is a PL subspace of the PL space L and $\dim (L - K) \leqq n$. Then given proper maps $\beta : K \to X$ and $\alpha : L \to Y$ with $\alpha \circ i \simeq f \circ \beta$($i$ denotes inclusion of K into L), there exists a proper map $\overline{\beta} : L \to X$ such that $\overline{\beta} \mid K = \beta$ and $f \circ \overline{\beta} \simeq \alpha$.*

PROOF. Let M_f denote the mapping cylinder of f and consider the following diagram where i_1 and i_2 are inclusion maps.

$$\begin{tikzcd} X \arrow[r, "i_2"] & M_f \\ K \arrow[u, "\beta"] \arrow[r, "i_1"'] & L \arrow[u, "\alpha"'] \end{tikzcd}$$

Then by hypothesis one has $\alpha \circ i_1 \simeq i_2 \circ \beta$. Let $h' : K \times I \to M_f$ denote the homotopy with $h_0 = i_2 \circ \beta$ and $h_1 = \alpha \circ i_1$. Then one can use α to extend h' to a map $h:(K \times I) \cup)L \times \{1\}) \rightarrow M_f$ by $h \mid L \times \{1\} = \alpha$. By Theorem 3.5, there exists an extension $H : L \times I \rightarrow M_f$ such that $H(L \times \{0\}) \subset X$. Define $\overline{\beta} = H \mid L \times \{0\}$. Then by construction, $i_2 \circ \overline{\beta} \simeq \alpha$; but $f \simeq i_2$ so $f \circ \overline{\beta} \simeq \alpha$.

THEOREM 3.10. *Suppose X and Y are PL spaces, $f : X \rightarrow Y$ is a proper PL map which is (n, n)-connected. Suppose K is a PL subspace of X with* $\dim X \leqq n - 1$, *i denotes the inclusion of K into X, and $f \circ i$ is inessential. Then there exist proper PL maps $\phi : K \rightarrow \Lambda$, $g_1 : M_\phi \rightarrow Y$ and $g_2 : M_\phi \rightarrow X$ such that $g_1 \mid K = f \circ i$, $g_2 \mid K = i$, and $f \circ g_2$ is inessential.*

PROOF. Since $f \circ i$ is inessential, Theorem 2.3 gives one *PL* maps $\phi : K \rightarrow \Lambda$ and $g_1 : \mathrm{M}_\phi \rightarrow Y$ such that $g_1 \mid K = f \circ i$. Applying Lemma 3.9 with $L = M_\phi$, $\alpha = g_1$ and $\beta = i$, one has a map $\overline{\beta} : M_\phi \rightarrow X$ extending i such that $f \circ \overline{\beta} \simeq g_1$. $\overline{\beta}$ is homotopic, relative to K, to a *PL* map. Let g_2 denote such a map. Then $f \circ g_2 \simeq g_1$ and all one needs is that $f \circ g_2$ is inessential. For this it suffices to know that $j \circ g_2$ is inessential where j is the inclusion of X into M_f. But $f \circ g_2 \simeq g_1$ implies $j \circ g_2 \simeq g_1$. Let $H : M_\phi \times i \rightarrow M_f$ be the homotopy with $H_0 = j \circ g_2$ and $H_1 = g_1$. Let $r_t : M_\phi \rightarrow M_\phi$ denote a deformation retraction of M_ϕ to Λ with $r_0 =$ identity. Let M_{r_1} denote the mapping cylinder of r_1 and define $K : M_{r_1} \rightarrow M_f$ by $K(p(y, t) = H(r_t(y), t)$ for $p(y, t) \in M_{r_1}$. But $r_1 : M_\phi \rightarrow \Lambda$ and $K \mid M_\phi = H_0 = j \circ g_2$ so Theorem 2.3 gives one that $j \circ g_2$ is inessential.

THEOREM 3.11. *Suppose X^x is inessential in* int W^n, $n - x \geqq 3$. *Then if W is $(2x - n + 2, 2x - n + 2)$-connected, there is a collapsible subspace Z containing X such that* $\dim (Z - X) \leqq x + 1$.

PROOF. Since X is inessential in W^n, one can apply Theorem 2.6 and find subspaces Y^y and $Z_1{}^z$ in int W such that $X \subset Y$, $Y \searrow Z_1$, $Y \leqq x + 1$, and $z \leqq 2x - n + 2$. But Z is inessential in int W by Theorem 3.6. Since $z \leqq x - 1$, by induction one can find a collapsible subspace Z_2 containing Z_1. Furthermore, by Lemma 1.7, one can assume without loss of generality that $Y \cup Z_2 \searrow Z_2 \searrow \Lambda \nabla$ Let $Z = Y \cup Z_2$ and one is done.

THOEREM 3.12. *Suppose the PL manifold W^n is (k, k)-connected, $k \leqq n - 3$, and Y^y, X^x, and P are subspaces of* int W *with $Y \searrow pX$, $x \leqq k$. Then there is a collapsible subspace $Z^z \supset X$ such that $Y \cup Z \searrow pZ$, $z \leqq x + 1$ and $Z \setminus X$ is in general position with respect to P.*

PROOF. By Theorem 3.6, X is inessential in int W so by Theorem 3.11, there is a collapsible subspace $Z_1{}^z$ containing X with $z \leqq x + 1$.

By Lemma 1.7, one can assume without loss of generality that $Y \cup Z_1 \searrow Z_1$. Since $Z_1 \setminus X \cap Y = \Phi$, one can choose an isotopy H of W fixed on X *such that* $H_1(Z_1 \setminus X)$ is in general position with respect to P and, by choosing H small enough, so that $H_1(Z_1 \setminus X) \cap Y = \Phi$. Let $Z = H_1(Z_1)$. Then $Y \cup Z \searrow Z$.

4. The Embedding Theorem.

THEOREM 4.1. *Let $f : M^n \to Q^q$ be a proper PL map with $f^{-1}(\partial Q) = \partial M$, $f \mid \partial M$ a proper PL embedding, and f (k, k)-connected for some $k \leqq q - n - 2$. Then $f \simeq g$ rel ∂M for some proper PL general position map g such that either g is an embedding or $g(S_2(g)) \searrow pY$, where Y is a complex of dimension not exceeding $2n - q - k$. In particular, when $2q \geqq 3(n + 1)$ (the metastable range), g is an embedding if f is $(2n - q + 1, 2n - q + 1)$-connected.*

The proof is an extension of the proof of Proposition 1 of [6] and is found in [7].

THEOREM 4.2. *Suppose $f : W^n \to W^q$ is a proper PL map, $f \mid \partial W$ is an embedding of ∂W into ∂Q, f is $(2n - q + 1,\ 2n - q + 1)$-connected, and Q is $(3n - 2q + 3,\ 3n - 2q + 3)$-connected. Then f is properly homotopic to an embedding provided $n \leqq q - 3$.*

PROOF. Let $k = 2n - q$ and let $l = \min\{k + 1,\ q - n - 2\}$. If $l = k + 1$ then f is homotopic to an embedding by Theorem 4.1 since $k + 1 \leqq q - n - 2$. So suppose $q - n - 2 \leqq k + 1$. Then f is homotopic to a map $g : W^n \to Q^q$ such that $g(S_2 g) \searrow pY$ and $\dim Y \leqq 2n - q - l = 3n - 2q + 2$. The proof proceeds in two steps. The first step consists of finding collapsible polyhedra C and D in W and Q, respectively, such that $\dim C \leqq k + 1$, $\dim D \leqq k + 2$ and $g(C) \subset D$. The second step is to alter C and D to find collapsible polyhedra C' and D' in W and Q respectively such that $g^{-1}(D') = C'$. The proof will then follow by an application of Theorem 1.10.

Step 1. Before starting, the following notation is adopted. Given a *PL* space X let $M(X)$ denote the mapping cylinder of a simplicial map of X to Λ. For the purpose at hand, the particular map chosen is not important and no specific reference need be made to it.

Suppose $g(S_2 g) \searrow Y$. Since $\dim Y \leqq 3n - 2q + 2$ and Q is $(3n - 2q + 3,\ 3n - 2q + 3)$-connected, Y is inessential in Q by Theorem 3.6; hence $g(S_2 g)$ is inessential by Lemma 2.5. Applying Theorem 3.10, choose general position map $h_1 : M(S_2 g) \to W$ such that $h_1 \mid S_2 g = i$, and gh_1 is inessential. By Lemma 2.7, h_1 is homotopic rel $S_2 g$ to a map h_2 such that if $E = \operatorname{im} H_2$, $\dim E \leqq k + 1$ $E \searrow p E_1$

and $\dim E_1 \leqq \min\{2k-n+2,\ k-1\}$. Also gh_2 is inessential. So there is a map $k_1 : M(M)S_2g)) \to Q$ such that $k_1 \mid M(S_2g) = gh_2$. As was the case for h_1, k_1 is homotopic, rel $M(S_2(g))$, to a map $k_2 : M(M(S_2g)) \to Q$ such that if $P = \operatorname{im} k_2$, $\dim P \leqq k+2$, $P \searrow p\, P_1$ and $\dim P_1 \leqq \min\{2(k+1)-q+2,\ k-2\}$. But $2(k+1)-q+2 \leqq 3n-2q+1$. Theorem 3.12 there exists a collapsible subspace C_0 of M such that $E_1 \subset C_0$ and $E \cup C_0 \searrow C_o$, $\dim C_0 \leqq \min\{3n-2q+3, k\}$ and $C_0 \searrow E_1$ is in general position with respect to $g^{-1}(P)$. So $g(E \cup C_0) \subset P \cup g(C_0)$; $g(C_0) \cap P = g(E_1) \cup g(C_0 \searrow E_1 \cap g^{-1}(P))$; $\dim g(E_1) \leqq \min\{3n-2q+2, k-1\}$; and $\dim g(C_o \searrow E_1) \cap g^{-1}(P)) \leqq 3n-2q-1$. So $\dim[g(C_0) \cap P] \leqq \min\{3n-2q+2,\ k-1\}$. Applying Lemma 1.8, one has $P \cup g(C_0) \searrow P_1 \cup g(C_0) \cup X$, $\dim X \leqq \min\{3n-2a+3, k\}$, an $\dim(P_1 \cup g(C_0) \cup X) \leqq \min\{3n-2q+3, k\}$. Since Q is $(3n-2q+3,\ 3n-2q+3)$-connected, Theorem 3.12 gives one the existence of a collapsible subspace D_0 containing $(P_1 \cup g(C_0) \cup X)$ such that $\dim D_0 \leqq \min\{3n-2q+4, k+1\}$ and $P \cup D_0 \searrow D_0$. Let $C = E \cup C_0$ and $D = P \cup D_0$. Step 1 is completed.

Step 2. Assume that $D - g(C)$ is in general position with respect to $g(W)$. Then $g^{-1}(D) = C \cup X_1$, and $\dim X_1 \leqq \min\{k-1, 3n-2q+2\}$. The proof proceeds exactly as in the proof of Lemma 5 of [6] so that one has the collapsible subspaces C' and D' of W and Q, respectively, with $g^{-1}(D') = C'$ and $S_2g \subset C'$. To complete the proof let N_1 denote a 2nd derived neighborhood of D' in Q. Then $N_0 = g^{-1}(N_1)$ is a neighborhood of C' in W and $g \mid \partial N_0$ is an embedding. By Theorem 1.10, $N_0 \simeq \partial N_0 c\,[0,\infty)$ and likewise $N_1 \simeq \partial N_c\,[0,\infty)$. Extend $g \mid_{\partial N_0}$ using the product structure to an embedding of N_0 into N_1 and the desired embedding is achieved.

5. **Unknotting.** Let I denote the interval $[0, 1]$. If X is a *PL* space and Q is a manifold, a concordance of X in Q is a proper embedding

$$F : X \times I \to Q \times I$$

such that $F(X \times \{i\}) \subseteq Q \times \{i\}$ for $i = 0, 1$. F is fixed on Y if $Y \subseteq X$ and $F(y, t) = (F_0(y), t)$ for all y in Y and t in I. F is x_0-allowable, or simple allowable, if $F^{-1}(Q \times \{i\}) = X \times \{i\}$ for $i = 0, 1$ and $F^{-1}(\partial Q \times I) = X_0 \times I$ where X_0 is a closed subspace of X. Two embeddings $f, g : X \to Q$ are allowably concordant keeping Y fixed if there is an allowable concordance F of X into Q, fixed on Y, such that $F_0 = f$ and $F_1 = g$.

An isotopy of X in Q is a concordance F of X in Q which is level preserving; that is, $F(X \times \{t\}) \subseteq Q \times \{t\}$ for each t in I. An ambient isotopy of Q is an isotopy H of Q onto Q with $H_0 = 1_Q$. Two embeddings $f, g : X \to Q$ are ambient isotopic if there is an ambient isotopy H of Q such that $H_1 f = g$. X is said to unknot in Q if every pair of homotopic embeddings of X into Q is ambient isotopic.

The following theorem is taken directly from [11].

THEOREM 5.1. *Let $F : X \times I \to Q \times I$ be an allowable concordance fixed on the closed subspace Y, and Q a manifold. Let $F^{-1}(\partial Q \times I) = X_0 \times I$ with $X_0 \subset Y$. If* $\dim Q - \dim X \geqq 3$, *then there is an allowable ambient isotopy H of $Q \times I$, fixed on $(Q \times \{0\}) \cup (\partial Q \times I) \cup F(Y \times I)$ such that $H_1 F = F_0 \times 1 : X \times I \to Q \times I$.*

THEOREM 5.2. *Suppose $f : W^n \to Q^q$ is a proper embedding, f is $(2n - q + 2,\ 2n - q + 2)$-connected and Q is $(3n - 2q + 4,\ 3n - 2q + 4)$-connected. Suppose $g : W^n \to Q^q$ is a proper embedding such that f is properly homotopic to g relative to ∂W. Then f is ambient isotopic to g.*

PROOF. Let $F : W^n \times I \to Q^q \times I$ be defined by $F(x, t) = (h(x, t), t)$ where h denotes the homotopy between f and g. Applying Theorem 4.2, one has an embedding $F' : W^n \times I \to Q^q \times I$ such that $F'(x, 0) = (f(x), 0)$, $F'(x, 1) = (g(x), 1)$, and $F'(x, t) = (h(x, t), t) = (f(x), t)$ for each x in ∂W. By Theorem 5.1, there is an ambient isotopy H of $Q \times I$, fixed on $(Q \times \{0\}) \cup (\partial Q \times I)$ such that $H_1 F' = F_0' \times 1 : W \times I \to Q \times I$. An ambient isotopy of g to f is defined by $L(x, t) = H_t F'(x, 1)$.

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