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GROWTH AND GAP STRUCTURE OF FUNCTIONS IN THE UNIT DISC

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1. **Introduction and Statement of Results.** Let f be an analytic function in the unit disk $D = \{z \mid |z| < 1\}$. Let $M(r) = M(r, f)$ be the maximum in modulus of $f(z)$ on $|z| = r$. We define ρ^* by

$$\limsup_{r \to 1^-} \frac{\log^+ \log^+ M(r)}{-\log(1 - r)} = \rho^*.$$

If $T(r) = T(r, f)$ is the Nevanlinna characteristic of f (see, for example, Hayman [1]), we define ρ by

$$\limsup_{r \to 1^-} \frac{\log T(r)}{-\log(1 - r)} = \rho.$$

$M(r)$ and $T(r)$ are known to be related by the following inequality

$$T(r) \leqq \log^+ M(r) \leqq \frac{R + r}{R - r} T(R) \quad (0 \leqq r < R). \tag{1.1}$$

It follows from (1.1) that

$$\rho \leqq \rho^* \leqq \rho + 1,$$

but it is known that ρ and ρ^* need not be equal. In fact, when f is defined by

$$f(z) = \exp((1 + z)/(1 - z)),$$

it is easy to see $\rho^* = 1$, but $\rho = 0$.

In [4] we have seen that if f is defined in D by

$$f(x) = \sum_{k=0}^{\infty} c_k z^{n_k} \tag{1.2}$$

where the $\{n_k\}$ are those integers for which $c_k \neq 0$, and if the sequence $\{n_k\}$ satisfies a Hadamard gap condition (i.e., there is a constant q with $q > 1$ such that $(n_{k+1})/(n_k) \geqq q$ for $k = 0, 1, 2, \cdots$), *then* $\rho = \rho^*$. We prove

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THEOREM 1. *Let f be an analytic function in D for which $f(z)$ has the form* (1.2) *where $0 \leqq \rho^* < \infty$. Assume there is a positive integer k_0 such that $\{n_k\}$ satisfies one of the conditions* (i)–(iii) *below for $k \geqq k_0$:*

(i) $n_{k+1} - n_k \geqq n_k/(\log_\ell n_k)$ *where* $\log_1 n_k = \log n_k$ *and* $\log_\ell(n_k) = \log(\log_{\ell-1}(n_k))$ *for* $\ell \geqq 2$;

(ii) *for some γ satisfying $0 < \gamma < 1$ there is a positive number Q such that the number of $\{n_k\}$ in each interval* $((1-\gamma)n_k, (1+\gamma)n_k)$ *is at most Q;*

(iii) $n_{k+1} - n_k \geqq Cn_{k+1}^\gamma$ *where* $(1/2) + (1/(2(1+\rho^*))) < \gamma < 1$ *and C is an absolute constant.*

Then $\rho = \rho^*$.

Theorem 1 is a consequence of Theorems 2 and 3. To introduce these theorems we need some additional notation.

If f is an analytic function in D we shall denote the maximum term on $|z| = r$ of the power series expansion of f about zero by $\mu(r) = \mu(r, f)$ and the rank of the maximum term by $\nu(r) = \nu(r, f)$. Further, if

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{1.3}$$

we define

$$\begin{aligned} M_2(r) &= \left\{ \sum_{k=0}^{\infty} |a_k|^2 r^{2k} \right\}^{1/2} \\ &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right\}^{1/2}. \end{aligned} \tag{1.4}$$

We now state Theorems 2 and 3 which may have some independent interest.

THEOREM 2. *Let f be an analytic function in D which has the form* (1.3). *Suppose r is a number satisfying $0 < r < 1$ for which*

$$|f(re^{1\theta})| \leqq K\mu(r), \tag{1.5}$$

where K is an absolute constant. Then for any δ with $0 < \delta < 1$

$$|f(re^{i\theta})| > K^{-2}\delta M_2(r) \tag{1.6}$$

on a set of θ of measure not less than $2\pi(1 - \delta)^2 K^{-4}$.

THEOREM 3. *Let f be an analytic function in D for which $f(z)$ has the form* (1.2) *and for which $0 < \rho^* < \infty$. Assume there is a positive in-*

teger k_0 such that $\{n_k\}$ satisfies one of the conditions (i)–(iii) *of Theorem* 1 *for $k \geqq k_0$. Then there exists a set E of values of r containing r values arbitrarily near one and an absolute constant K such that*

$$|f(re^{i\theta})| \leqq K\mu(r) \quad (r \in E). \tag{1.7}$$

2. **Proof of Theorem 2.** We shall use the following lemma which appears in Zygmund [**6**, p. 216].

LEMMA 1. *Suppose g is a non-negative function defined in a set E of positive measure E and that*

(i) $|E|^{-1} \int_E g(x)\, dx \geqq A > 0$,

and

(ii) $|E|^{-1} \int_E (g(x))^2\, dx \leqq B$.

Then for any δ with $0 < \delta < 1$ the subset of E in which $g(x) \geqq \delta A$ is of measure not less than $|E|\ (1 - \delta)^2(A^2/B)$.

PROOF OF THEOREM 2. If $\lambda \geqq 1$, we define

$$M_\lambda(r) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\lambda\, d\theta \right\}^{1/\lambda}.$$

We shall obtain a lower bound for $M_1(r)$ using a line of reasoning similar to that in Zygmund [**6**, p. 213].

It is well known that $M_\lambda(r)$ is a logarithmically convex function. Consequently,

$$(M_2(r))^2 \leqq (M_1(r))^{2/3}(M_4(r))^{4/3}.$$

Using (1.5) we see

$$2 \log M_2(r) \leqq \left(\frac{2}{3}\right) \log M_1(r) + \left(\frac{4}{3}\right) \log K + \left(\frac{4}{3}\right) \log \mu(r),$$

so

$$\left(\frac{2}{3}\right) \log M_2(r) \leqq \left(\frac{2}{3}\right) \log M_1(r) + \left(\frac{4}{3}\right) \log K.$$

Thus

$$K^{-2}M_2(r) \leqq M_1(r).$$

We now apply Lemma 1 with $g(\theta) = |f(re^{i\theta})|$ on $[0, 2\pi]$, $A = K^{-2}M_2(r)$, and $B = M_2(r)^2$.

3. **Proof of Theorem 3.** We use the following lemma formulated by Saxer [**3**].

LEMMA 2. *Let* $\{P_k\}$ *be a strictly increasing sequence of positive integers such that*

$$\lim_{k\to\infty} P_k = 1.$$

Then the function h defined by

$$h(z) = \sum_{k=0}^{\infty} b_k z^k = 1 + \sum_{k=1}^{\infty} \frac{z^k}{P_1 \cdots P_k}$$

converges in D.

Furthermore, if f is an analytic function in D for which f(z) has the form (1.3) *and if* ϕ *defined by*

$$\phi(z) = \sum_{k=0}^{\infty} (a_k/b_k) z^k$$

is an analytic function in D for which $\nu(r, \phi) \to \infty$ *as* $r \to 1$, *then there exists a strictly increasing sequence of non-negative integers* $\{\lambda_k\}$ *and a sequence of positive numbers* $\{r_{\lambda_k}, r'_{\lambda_k}\}$ *such that*

$$0 = r_{\lambda_1} < r'_{\lambda_1} < r_{\lambda_2} < r'_{\lambda_2} < \cdots < r_{\lambda_k} < r'_{\lambda_k} < \cdots < 1,$$

and

$$\sum_{k=1}^{\ell-1} \log ((r_{\lambda_{k+1}})/(r'_{\lambda_k})) = \log P_{\lambda_\ell} - \log P_{\lambda_1},$$

which satisfies for every r in $(r_{\lambda_k}, r'_{\lambda_k})$ *the following properties*:

$$\text{(3.1)} \qquad \text{(i)} \quad \frac{|a_{\lambda-m-1}|\, r^{\lambda-m-1}}{|a_\lambda|\, r^\lambda} \leqq \frac{P_{\lambda-m} \cdots P_\lambda}{P_\lambda^{\,m+1}} \ (m = 0, 1, 2, \cdots, \lambda - 1),$$

$$\text{(3.2)} \qquad \frac{|a_{\lambda+n+1}|\, r^{\lambda+n+1}}{|a_\lambda|\, r^\lambda} \leqq \frac{P_\lambda^{\,n+1}}{P_{\lambda+1} \cdots P_{\lambda+n+1}} \ (n = 0, 1, 2 \cdots),$$

where $\lambda = \lambda_k$;

$$\text{(ii)} \qquad \nu(r, f) = \nu(P_{\lambda_k}, h);$$

$$\text{(3.3)} \qquad \text{(iii)} \qquad \mu(r, f) = \mu(P_{\lambda_k}, h) \cdot \mu(r/P_{\lambda_k}, \phi).$$

PROOF OF THEOREM 3. MacLane [2, p. 38] has shown that

$$\limsup_{k\to\infty} \frac{\log^+ \log^+ |a_k|}{\log k - \log^+ \log^+ |a_k|} = \rho^*.$$

Thus if ϵ satisfies $0 < 3\epsilon < (\rho^*)/(1 + \rho^*)$ and $\alpha = ((\rho^*)/(1 + \rho^*)) - \epsilon$ we see

$$\limsup_{k\to\infty} \frac{\log |a_k|}{k^\alpha} > 0.$$

We now let h and ϕ be as in Lemma 2 where

$$P_j = \left(\prod_{k=j+1}^{\infty} (1 + 1/k^\beta) \right)^{-1} \tag{3.4}$$

and $\beta = 1 + (1/(1 + \rho^*)) + 2\epsilon$. Note that $1 < \beta < 2$ and $2 - \beta < \alpha$.

We perform calculations like those in Wiman [5].

Since

$$P_j^{-1} < \exp\left\{ \sum_{j+1}^{\infty} (1/k^\beta) \right\}$$

and

$$\sum_{j+1}^{\infty} (1/k^\beta) < \int_j^\infty x^{-\beta}\, dx = (\beta - 1)^{-1} j^{1-\beta},$$

we see

$$b_j < \exp\left\{ (\beta - 1)^{-1} \left(\sum_{k=1}^{j} (1/k^{\beta-1}) \right) \right\}.$$

From

$$\sum_{k=2}^{j} (1/k^{\beta-1}) < \int_1^j x^{1-\beta}\, dx = (2 - \beta)^{-1} j^{2-\beta},$$

we then get

$$b_j < K_1 \exp\{(2 - \beta)^{-1}(\beta - 1)^{-1} j^{2-\beta}\}$$

where K_1 is an absolute constant. Similar estimates show

$$b_j > \exp\{K_2((j + 2)^{2-\beta} - 2^{2-\beta})\},$$

where K_2 is an absolute constant. Therefore ϕ has radius of convergence one, and since

$$\limsup_{j\to\infty} |a_j|\, b_j^{-1} = \infty,$$

we have $\nu(r, \phi) \to \infty$ as r approaches one.

Using Lemma 2 we see we can get estimates for terms of the power series expansion for f on $|z| = r$ in terms of multiples of $\mu(r, f)$ by estimating the quotients on the right-hand side of (3.1) and (3.2).

The right-hand side of (3.1) is majorized by

$$\begin{aligned}\prod_{\ell=0}^{m} (1 + (\lambda - \ell)^{-\beta})^{-(m-\ell)} &\leqq (1 + \lambda^{-\beta})^{-m(m+1)/2} \\ &\leqq \exp\left\{ - \frac{m(m+1)}{4\lambda^{\beta}} \right\}.\end{aligned} \tag{3.5}$$

The right-hand side of (3.2) is majorized by

$$\begin{aligned}\prod_{\ell=0}^{n} (1 + (\lambda + \ell + 1)^{-\beta})^{-(n-\ell)} &\leqq (1 + (\lambda + n + 1)^{-\beta})^{-n(n+1)/2} \\ &< \exp\left\{ - \frac{n^2}{4(\lambda + n)^{\beta}} (1 + \epsilon') \right\}\end{aligned} \tag{3.6}$$

where ϵ' is arbitrarily small if λ (and n) are sufficiently large. From (3.1) and (3.5) we see

$$\begin{aligned}&(|a_\lambda| r^\lambda)^{-1} \sum_{m=q-1}^{\lambda} (|a_{\lambda-m-1}| r^{\lambda-m-1}) \\ &\leqq \int_{q-2}^{\lambda-1} \exp\{(-K_3 x^2)/\lambda^{\beta}\}\, dx \\ &\leqq \int_{q-2}^{\infty} \exp\{(-K_3 x^2)/\lambda^{\beta}\}\, dx \\ &\leqq K_3 \lambda^{\beta} (2(q-2))^{-1} \exp\{(-K_3 (q-2)^2 \lambda^{-\beta})\},\end{aligned} \tag{3.7}$$

where K_3 is an absolute constant. From (3.2) and (3.6) we have

$$\begin{aligned}&(|a_\lambda| r^\lambda)^{-1} \sum_{n=q-1}^{\infty} (|a_{\lambda+n+1}| r^{\lambda+n+1}) \\ &< \int_{q-2}^{\infty} \exp\left\{ - \frac{x^2}{(\lambda + x)^{\beta}} \left(\frac{1+\epsilon'}{4} \right) \right\} dx.\end{aligned} \tag{3.8}$$

We split the integral on the right-hand side of (3.8) into two parts. First we note

$$\int_{\lambda}^{\infty} \exp\left\{ - \left(\frac{1+\epsilon'}{4} \right) \frac{x^2}{(\lambda + x)^{\beta}} \right\} dx$$

$$\begin{aligned}(3.9)\qquad &\leqq \int_{\lambda}^{\infty} x^{\beta-1}(x^{-(\beta-1)}) \exp\left\{-\left(\frac{1+\epsilon'}{2^{2+\beta}}\right) x^{2-\beta}\right\} dx \\ &\leqq K_4 \lambda^{\beta-1} \exp\left\{-\left(\frac{1+\epsilon'}{2^{2+\beta}}\right) \lambda^{2-\beta}\right\},\end{aligned}$$

where K_4 is an absolute constant. The remaining part of the integral in (3.8) is estimated as in (3.7) giving

$$\begin{aligned}&\int_{q-2}^{\lambda} \exp\left\{-\left(\frac{1+\epsilon'}{4}\right) \frac{x^2}{(\lambda+x)^\beta}\right\} dx \\ (3.10)\qquad &\leqq K_5 \lambda^\beta (q-2)^{-1} \exp\left\{-\left(\frac{1+\epsilon'}{2^{1+\beta}}\right) \frac{(q-2)^2}{\lambda^\beta}\right\},\end{aligned}$$

where K_5 is an absolute constant.

To see (1.7) holds if we assume (iii) of Theorem 3, we set $q = \lambda^\eta$ where $\eta = (\beta + \epsilon)/2$ in (3.7), (3.9), and (3.10) and note that the terms on the right-hand side of these inequalities get arbitrarily small as λ goes to infinity. Condition (iii) of Theorem 3 implies there is only one index ℓ such that

$$\lambda - \lambda^\eta \leqq \ell \leqq \lambda + \lambda^\eta$$

and $a_\ell \neq 0$. So (1.7) is true. In a similar manner condition (i) of Theorem 3 implies (1.7).

To prove the theorem when condition (ii) is assumed, we again choose q as above to obtain the bound

$$\left| \sum_{S_1} c_k z^{n_k} + \sum_{S_2} c_k z^{n_k} \right| \leqq \mu(r, f), \quad |z| = r,$$

when $\nu(r, f)$ is large where $S_1 = \{k | n_k \leqq \nu - \nu^\eta\}$ and $S_2 = \{k | n_k \geqq \nu + \nu^\eta\}$. Then the gap condition shows

$$\left| \sum_{S_3} c_k z^{n_k} \right| \leqq Q\mu(r, f), \quad |z| = r,$$

when ν is large where $S_3 = \{k \mid \nu - \nu^\eta < n_k < \nu + \nu^\eta\}$.

4. **Proof of Theorem 1.** We assume $\rho < \rho^*$, so $\rho^* > 0$. We adopt the notation of the previous section and proceed to estimate $\mu(P_s, h)$ from below. Using (3.4) we see

$$\log\left(\frac{(P_s)^s}{P_1P_2\cdots P_s}\right) \geqq \sum_{k=2}^{s}\log(1+1/k^\beta)$$

$$+\sum_{k=3}^{s}\log(1+1/k^\beta)+\cdots+\log(1+1/s^\beta)$$

$$\geqq \frac{1}{2}\left\{\sum_{k=2}^{s}(1/k^\beta)+\sum_{k=3}^{\infty}(1/k^\beta)\right.$$

$$\left.+\cdots+1/s^\beta\right\} \tag{4.1}$$

$$\geqq \frac{1}{2}\left(\frac{1}{\beta-1}\right)\left\{\sum_{\ell=2}^{s}(\ell^{1-\beta}-(s+1)^{1-\beta})\right\}$$

$$\geqq \frac{1}{2}\left(\frac{1}{\beta-1}\right)\left\{\frac{(s+1)^{2-\beta}}{2-\beta}\right.$$

$$\left.-\frac{2^{2-\beta}}{2-\beta}-(s-1)(s+1)^{1-\beta}\right\}.$$

Thus when $s \geqq s_0$ we note (4.1) implies

$$\log\left(\frac{(P_s)^s}{P_1P_2\cdots P_s}\right) \geqq K'(s+1)^{2-\beta} \tag{4.2}$$

where K' is an absolute constant. We also have

$$2(1-P_s) \geqq -\log P_s = \sum_{k=s+1}^{\infty}\log(1+1/k^\beta)$$

$$\geqq \frac{1}{2}\sum_{k=s+1}^{\infty}(1/k^\beta) \geqq \frac{(s+1)^{1-\beta}}{2(\beta-1)}. \tag{4.3}$$

Combining (4.2) and (4.3) we find for $s \geqq s_0$ there is an absolute constant K'' such that

$$\log \mu(P_s, h) \geqq K''(1-P_s)^{-(2-\beta)/(\beta-1)}.$$

Now (3.3) shows us for values of r given by Lemma 2 for which $r \geqq t$ we get

$$\log \mu(r, f) \geqq K''(1-r)^{-(2-\beta)/(\beta-1)}. \tag{4.4}$$

Our choice of β indicates that (4.4) means there are values of r given by Lemma 2 arbitrarily near one for which

$$\log \mu(r, f) \geqq K''(1-r)^{-(\rho^*-\epsilon'')} \tag{4.5}$$

where ϵ'' is an arbitrarily small positive number. Since our proof of Theorem 3 shows such values of r are in the set E of Theorem 3 we may apply Theorem 2 to see (1.6) is valid on a set of θ of measure not less than $2\pi(1-\delta)^2K^{-4} = Q'$. Therefore for such values of r we see

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta \geqq Q' \log^+(K^{-2}\delta M_2(r)). \tag{4.6}$$

We then obtain using (1.4), (4.5), and (4.6), values of r arbitrarily near one for which

$$T(r, f) > Q''(1-r)^{-(\rho^*-\epsilon'')} \tag{4.7}$$

whre Q'' is a positive absolute constant. But the definition of ρ shows for $r \geqq t'$ we have

$$T(r, f) < (1-r)^{-(\rho+\epsilon'')}. \tag{4.8}$$

Sine ϵ'' can be chosen arbitrarily small, we observe (4.7) and (4.8) are contradictory when $\rho < \rho^*$.

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