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THE GENERALIZED SPECTRUM OF SECOND ORDER ELLIPTIC SYSTEMS

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1. **Introduction.** Let $\Omega \subset R^n$ be a bounded domain with smooth boundary. Suppose that $L_{\alpha\beta}$, α, $\beta = 1, 2, \cdots, m$ are second order elliptic operators without zero order terms which act on functions $u : \Omega \to \mathbf{C}^m$. The spectrum of the system

$$\sum_{\beta=1}^{m} L_{\alpha\beta} + \mu \sum_{\beta=1}^{m} c_{\alpha\beta} u^\beta = 0, \quad \alpha = 1, 2, \cdots, m$$

subject to appropriate homogeneous boundary conditions is known to consist of a discrete increasing set of numbers $\mu_1, \mu_2, \cdots, \mu_n, \cdots$.

In the case of a single equation with the Laplace operator as principal part and with homogeneous Dirichlet boundary conditions, a particularly simple method for obtaining a lower bound to the first eigenvalue μ_1 was obtained by Barta [1] who showed that

$$\mu_1 \geqq \inf_{x \in \Omega} \left(- \frac{\Delta \varphi}{\varphi} \right).$$

Here φ is an arbitrary C^2 function defined in Ω. This estimate is useful and of interest since the function φ is required to satisfy only a smoothness condition and not a boundary condition. This inequality was extended and generalized to general second order operators in [10]. There it is shown, for example, that μ_1, the first eigenvalue for the Laplace operator subject to zero boundary conditions satisfies the inequality

$$\mu_1 \geqq \inf_{x \in \Omega} (\operatorname{div} P - |P|^2)$$

where P is a vector field in Ω which is only required to satisfy a mild smoothness condition. The Barta inequality is recovered by setting $P_i = -\varphi_{x_i}/\varphi$ with φ an arbitrary C^2 function. Further extensions of these inequalities were obtained by Hersch [4]. Hooker [5] developed analogous results for second order equations with mixed boundary conditions and he also treated the eigenvalue problem for the biharmonic operator subject to a variety of boundary conditions.

Upper and lower bounds for the eigenvalues of second order operators have been obtained by a variety of methods. We mention the investigations of Fichera [3], Payne and Weinberger [9], Weinberger [12],

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Bazley and Fox [2], and Weinstein and Stenger [13]. Further references may be found in these papers, especially in [13]. For second order systems, estimates for eigenvalues are closely connected to comparison theorems and the generalization to partial differential operators of Sturm Liouville theory. In this connection, see Kreith [6] and Swanson [11].

In this paper we extend the results of [10] to second order elliptic systems. This extension is particularly useful in obtaining lower bounds for the first eigenvalue of those higher order elliptic equations which can be reduced to a second order system. In § 3 we illustrate the technique for the biharmonic operator. The process for obtaining lower bounds for the spectrum of a second order system is improved substantially by the introduction of a generalization of the spectrum of an operator. For the operators $L_{\alpha\beta}$ we consider the system

$$\sum_{\beta=1}^{m} L_{\alpha\beta}u + \lambda^\alpha \sum_{\beta=1}^{m} c_{\alpha\beta}u^\beta = 0, \; \alpha = 1, 2, \cdots, m$$

where $\lambda = (\lambda^1, \lambda^2, \cdots, \lambda^m)$ is an element in $\mathbf{C}^m$. The set S of values λ for which the above system has a solution subject to a set of homogeneous boundary conditions is called the *generalized spectrum* of the set $\{L_{\alpha\beta}\}$ with respect to these boundary conditions.

In § 2 we establish certain basic properties of the generalized spectrum for the case of homogeneous Dirichlet boundary conditions. This information implies statements on the ordinary spectrum which consists of the "diagonal" $\lambda^1 = \lambda^2 = \cdots = \lambda^m$ of the generalized spectrum S. In the most general case the generalized spectrum of a system will consist of a continuum in $\mathbf{C}^m$, and the determination of the geometric properties of this set remains as an interesting subject of investigation.

In § 3 we develop concrete techniques for obtaining lower bounds of the generalized spectrum and we apply the method for several types of boundary conditions. It is easily seen that the method is well adapted for actual computations.

2. **Basic Estimates.** Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary. For functions $u : \Omega \to \mathbf{R}^m$ or $\mathbf{C}^m$ we define the operators

$$L_{\alpha\beta}u \equiv \sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}(x) \frac{\partial^2 u^\beta}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^{\alpha\beta}(x) \frac{\partial u^\beta}{\partial x_i},$$

$$\alpha, \beta = 1, 2, \cdots, m,$$

where $u = (u^1, u^2, \cdots, u^m)$ and $x = (x_1, x_2, \cdots, x_n)$. We shall suppose that $a_{ij}^{\alpha\beta} = a_{ji}^{\alpha\beta}$ for all α, β, x. For $\lambda = (\lambda^1, \cdots, \lambda^m) \in \mathbf{C}^m$ we consider solutions of the linear second order uniformly elliptic system

$$\sum_{\beta=1}^{m} L_{\alpha\beta}u + \lambda^{\alpha} \sum_{\beta=1}^{m} c_{\alpha\beta}(x)u^{\beta} = 0, \ \alpha = 1, 2, \cdots, m, \tag{1}$$

subject to the homogeneous boundary conditions

$$\sum_{\beta=1}^{m} B_{\alpha\beta}u^{\beta}(x) = 0 \text{ for } x \in \partial\Omega, \ \alpha = 1, 2, \cdots, m. \tag{2}$$

The quantities $B_{\alpha\beta}$ are boundary operators of order zero or one. The set λ in $\mathbf{C}^m$ for which non-trivial solutions of (1), (2) exist is called the *generalized spectrum of the operator* $L \equiv \{L_{\alpha\beta}\}$ with boundary conditions (2). We denote this set in $\mathbf{C}^m$ by S and we shall be interested in obtaining explicit upper and lower bounds either for S or a portion of S. The subset of S for which $\lambda^1 = \lambda^2 = \cdots = \lambda^m$, i.e., the diagonal of S, coincides with the ordinary spectrum of the operator L; we designate this set by S_0. With $B_{\alpha\beta} = \delta_{\alpha\beta}$ and appropriate conditions on the matrix $C = \{c_{\alpha\beta}\}$, it is known that S_0 is a discrete set. See Morrey [8, p. 251].

We first consider the simpler strongly elliptic system

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}^{\alpha}(x)\frac{\partial u^{\alpha}}{\partial x_i} \right) + \lambda^{\alpha} \sum_{\beta=1}^{m} c_{\alpha\beta}(x)u^{\beta} = 0, \tag{3}$$
$$\alpha = 1, 2, \cdots, m,$$

subject to the homogeneous Dirichlet boundary conditions

$$u^{\alpha}(x) = 0 \quad \text{for } x \in \partial\Omega, \ \alpha = 1, 2, \cdots, m. \tag{4}$$

We shall suppose that the coefficients in (3) are real-valued, bounded, and sufficiently smooth so that the basic existence theory is valid for the Dirichlet problem for (3). Assume that the elements of C satisfy

$$\sum_{\beta=1}^{m} c_{\alpha\beta}(x) \geqq 0, \ \alpha = 1, 2, \cdots, m \tag{5a}$$

and

$$c_{\alpha\beta}(x) \leqq 0 \quad \text{for } \alpha \neq \beta, \ \alpha, \beta = 1, 2, \cdots, m \tag{5b}$$

for all $x \in \Omega$. We write $\lambda \geqq 0$ whenever $\lambda^i \geqq 0$ for all i between 1 and m, and we observe that the system (3) satisfies a maximum principle for $\lambda \leqq 0$ whenever (5) holds [8]. Hence it follows that the real part of the generalized spectrum of (3), (4) can never be negative if (5) holds. This observation yields a simple example of a lower bound on the real part of S.

To obtain useful bounds on S, we let $P^\alpha = (P_1{}^\alpha, \cdots, P_n{}^\alpha)$, $\alpha = 1, 2, \cdots, m$, be C^1 vector fields on $\overline{\Omega}$. For any $u \in C^2(\Omega)$ such that (4) holds, we have

$$\int_\Omega \nabla \cdot [P^\alpha (u^\alpha)^2] = 0, \quad \alpha = 1, 2, \cdots, m. \tag{6}$$

Multiplication of (3) by u^α and an application of Green's Theorem yields

$$\int_\Omega \left[\sum_{i,j=1}^m a_{ij}^\alpha(x) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\alpha}{\partial x_j} - \lambda^\alpha \sum_{\beta=1}^m c_{\alpha\beta} u^\alpha u^\beta \right] = 0, \tag{7}$$

$$\alpha = 1, 2, \cdots, m.$$

Adding (6) and (7) and then summing on α, we obtain

$$\begin{aligned} \int_\Omega \sum_{\alpha=1}^m \Bigg\{ \sum_{i,j=1}^n a_{ij}^\alpha \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\alpha}{\partial x_j} + 2 \sum_{i=1}^n P_i{}^\alpha u^\alpha \frac{\partial u^\alpha}{\partial x_i} \\ + (\operatorname{div} P^\alpha - \lambda^\alpha c_{\alpha\alpha})(u^\alpha)^2 - \lambda^\alpha \sum_{\beta \neq \alpha} c_{\alpha\beta} u^\alpha u^\beta \Bigg\} = 0 \end{aligned} \tag{8}$$

The integrand in (8) is a quadratic form in u and $\partial u / \partial x_i$. The coefficient matrix of this form, denoted by M, may be written

$$M = \begin{pmatrix} A & B^T \\ B & D \end{pmatrix}$$

where the matrices A, B, and D are defined as follows: let A^α, $\alpha = 1, 2, \cdots, m$ denote the $n \times n$ matrix with entries (a_{ij}^α). We define A to be the $mn \times mn$ matrix

$$A = \begin{pmatrix} A^1 & 0 & \cdots & 0 \\ 0 & A^2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A^m \end{pmatrix}$$

The matrix B is $m \times mn$ with entries

$$B = \begin{pmatrix} P^1 & 0 & \cdots & 0 \\ 0 & P^2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & P^m \end{pmatrix}.$$

The $m \times m$ matrix D is symmetric with the form

$$D = \begin{pmatrix} \operatorname{div} P^1 - \lambda^1 c_{11} & -\frac{1}{2}(\lambda^1 c_{12} + \lambda^2 c_{21}) & \cdots\cdots & -\frac{1}{2}(\lambda^1 c_{1m} + \lambda^m c_{m1}) \\ -\frac{1}{2}(\lambda^1 c_{12} + \lambda^2 c_{21}) & \operatorname{div} P^2 - \lambda^2 c_{22} & \cdots\cdots & -\frac{1}{2}(\lambda^2 c_{2m} + \lambda^m c_{m2}) \\ \vdots & & \ddots & \vdots \\ -\frac{1}{2}(\lambda^1 c_{1m} + \lambda^m c_{m1}) & \cdots\cdots & \cdots\cdots & \operatorname{div} P^m - \lambda^m{}_{mm} \end{pmatrix}$$

Since (3) is elliptic the matrix A has an inverse and, denoting the determinant of any matrix C by $|C|$, we find

$$\begin{aligned} |M| &= |A| \cdot |D - BA^{-1}B^T| \\ &= |A^1| \cdots |A^m| \cdot |D - BA^{-1} B^T|. \end{aligned}$$

It is a simple fact that the quadratic form (8) is positive definite whenever $D - BA^{-1}B^T$ is a positive definite matrix. See, for example, the Lemma in Kusano and Yoshida [7]. The symmetric matric $D - BA^{-1}B^T$ which we denote by $\overline{D}$ has entries

$$\overline{D} = \begin{pmatrix} \operatorname{div} P^1 - \lambda^1 c_{11} - P^1(A^1)^{-1}P^{1T} & -\frac{1}{2}(\lambda^1 c_{12} + \lambda^2 c_{21}) & \cdots\cdots & -\frac{1}{2}(\lambda^1 c_{1m} + \lambda^m c_{m1}) \\ -\frac{1}{2}(\lambda^1 c_{12} + \lambda^2 c_{21}) & \operatorname{div} P^2 - \lambda^2 c_{22} - P^2(A^2)^{-1}P^{2T} & \cdots & -\frac{1}{2}(\lambda^2 c_{2m} + \lambda^m c_{m2}) \\ \vdots & & \ddots & \vdots \\ -\frac{1}{2}(\lambda^1 c_{1m} + \lambda^m c_{m1}) & \cdots\cdots & \cdots\cdots & \operatorname{div} P^m - \lambda^m c_{mm} - P^m(A^m)^{-1}P^{mT} \end{pmatrix}$$

The unform ellipticity in Ω of the system (3) implies that there are positive constants m_α and M_α such that

$$m_\alpha|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x)\xi_i\xi_j \leqq m_\alpha|\xi|^2, \tag{9}$$
$$\text{for all } \xi \in \mathbf{R}^n,\ x \in \Omega,\ \alpha = 1, 2, \cdots, m.$$

The notation $|\xi|$ indicates the Euclidean norm of the vector $(\xi_1, \cdots, \xi_n)$.

The following result yields a lower bound for the generalized spectrum of problem (3), (4).

THEOREM 1. *Let* (3) *be uniformly elliptic in a bounded domain* Ω. *Then there is a vector* $\lambda_0 = (\lambda_0^1, \cdots, \lambda_0^m)$ *with* $|\lambda_0^\alpha| > 0$, $\alpha = 1, 2, \cdots, m$ *such that no point of the generalized spectrum* S *of* (3), (4) *is contained in the set* $S' = \{\lambda : |\lambda| \leqq |\lambda_0|\}$.

PROOF. let d denote the diameter of Ω, suppose the origin of coordinates is in Ω at distance no greater than d from the boundary, and set

$$P_i^\alpha = \delta \tan \gamma x_i,\ \alpha = 1, \cdots, m,\ \ i = 1, 2, \cdots, n$$

where $\delta > 0$ is a constant and $\gamma > 0$ is a constant less than $\pi/2d$. From (9) we have

$$|P^\alpha (A^\alpha)^{-1} P^{\alpha T}| \leqq m_\alpha^{-1} |P^\alpha|^2 = m_\alpha^{-1}\, \delta^2 \sum_{i=1}^{n} \tan^2 \gamma x_i.$$

We now estimate the diagonal terms of $\bar{D}$. We find

$$\begin{aligned} \operatorname{div} P^\alpha &- \lambda^\alpha c_{\alpha\alpha} - P^\alpha (A^\alpha)^{-1} P^{\alpha T} \\ &\geqq \delta\gamma \sum_{i=1}^{n} \sec^2 \gamma x_i - |\lambda^\alpha c_{\alpha\alpha}| - m_\alpha^{-1} \delta^2 \sum_{i=1}^{n} \tan^2 \gamma x_i \\ &\geqq \delta\gamma - |\lambda^\alpha c_{\alpha\alpha}| + \delta(\gamma - m_\alpha^{-1}\delta) \sum_{i=1}^{n} \tan^2 \gamma x_i. \end{aligned}$$

We now choose $\delta < m_\alpha \gamma$ and select λ_0^α positive and such that $\lambda_0 < \delta\gamma/\bar{c}_{\alpha\alpha}$ where $\bar{c}_{\alpha\alpha} = \sup_{x\in\Omega} |c_{\alpha\alpha}|$. With these choices each diagonal element of $\bar{D}$ is positive for all λ such that $|\lambda| \leqq |\lambda_0|$. We now observe that all the non-diagonal terms of $\bar{D}$ contain as factors two components of λ. Reducing the absolute magnitude of λ_0 increases (does not decrease) the positive lower bound of every diagonal term and decreases every non-diagonal term. Hence for a sufficiently small value of $|\lambda_0|$ the matrix $\bar{D}$ is positive definite. The form (8) is positive definite and (3), (4) has no solution for $|\lambda| \leqq |\lambda_0|$.

The next result shows that the generalized spectrum is always bounded away from zero and, in fact is confined to a region in $\mathbf{C}^m$ outside an arbitrarily large ball if Ω has a sufficiently small diameter.

THEOREM 2. *Let* (3) *be uniformly elliptic in a domain* Ω_0 *and suppose* λ *is fixed. If* Ω *is contained in a ball* $B \subset \Omega_0$ *of sufficiently small radi-*

us, its size depending only on the coefficients in (3), *then there are no nontrivial solutions of* (3), (4).

PROOF. Without loss of generality, choose the origin in Ω_0 and define

$$P_i^\alpha = \delta \tan \gamma x_i.$$

As in (10) the diagonal terms of $\overline{D}$ are estimated by

$$\begin{aligned}\operatorname{div} P^\alpha - \lambda^\alpha c_{\alpha\alpha} - P^\alpha (A^\alpha)^{-1} P^{\alpha T} \\ \geqq \delta\gamma - |\lambda^\alpha c_{\alpha\alpha}| + \delta(\gamma - m_\alpha^{-1}\delta) \sum_{i=1}^{n} \tan^2 \gamma x_i.\end{aligned}$$

We choose $\delta = \gamma \inf_\alpha m_\alpha$. Let $c_0 = \max_\alpha \sup_{x \in \Omega_0} |c_{\alpha\alpha}(x)|$. If γ is chosen so large that

$$\gamma > \max_\alpha \left(\frac{\lambda^\alpha c_0}{m_\alpha} \right)^{1/2}$$

then all the diagonal elements of $\overline{D}$ are positive. Now if γ is increased, the diagonal elements increase, but the off-diagonal terms are unchanged. Hence we can choose γ so large that $\overline{D}$ is a positive definite matrix. Hence the quadratic form in (8) is positive definite for this value of γ. Furthermore, if the ball B has radius r so small that $r < \pi/2\gamma$, then all the quantities P_i^α remain bounded. For any Ω in B, there are no non-trivial solutions of (3), (4).

REMARKS. (i) It is clear that the generalized spectrum has a lower bound which tends to infinity as diam $\Omega \to 0$. (ii) If Ω is bounded and contained in a slab of width less than $\pi/2\gamma$ (instead of a ball), the same result as that in Theorem 2 holds. (iii) For sufficiently small domains Theorem 1 yields uniqueness for solutions of the Dirichlet problem even in cases where (5a) and (5b) hold. (iv) From the proof it is clear that the functions P_i^α are required to be C^1 only in the i-th variable. The functions need only be continuous in the remaining variables.

If the system (3) is replaced by the general non-selfadjoint system

$$\begin{aligned}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{ij}^\alpha(x) \frac{\partial u^\alpha}{\partial x_i} \right) + \sum_{i=1}^{n} \sum_{\beta=1}^{m} b_i^{\alpha\beta}(x) \frac{\partial u^\beta}{\partial x_i} \\ + \lambda^\alpha \sum_{\beta=1}^{m} c_{\alpha\beta}(x) u^\beta = 0, \quad \alpha = 1, 2, \cdots, m,\end{aligned} \tag{11}$$

then Theorem 2 holds so long as the funtions $b_i^{\alpha\beta}$ are bounded in Ω. The proof is virtually unchanged. On the other hand Theorem 1 does not hold unless additional restrictions are placed on the $b_i^{\alpha\beta}$.

Theorem 1 has an analogue if the boundary condition (4) is replaced by the more general condition

$$h_\alpha(x)\frac{\partial u^\alpha}{\partial \nu_\alpha} + k_\alpha(x)u^\alpha = 0, \quad \alpha = 1, 2, \cdots, m \tag{12}$$

where $\partial/\partial\nu_\alpha = \sum_{i,j=1}^n a_{ij}^\alpha n_i\, \partial/\partial x_j$ and n_i is the i-th component of the unit normal to $\partial\Omega$. If there is a positive constant C_0 such that $k_\alpha(x) \geqq C_0$ for all $x \in \partial\Omega$, $\alpha = 1, \cdots, m$, then the boundary term

$$\int_{\partial\Omega} \sum_{\alpha=1}^{m} \left(u^\alpha \frac{\partial u^\alpha}{\partial \nu_\alpha} + (u^\alpha)^2 \sum_{i=1}^{n} P_i{}^\alpha n_i \right) dS$$

which will occur in the proof of Theorem 1 if (12) is used instead of (3) will be nonpositive whenever

$$\begin{gathered} h_\alpha(x)P_i{}^\alpha(x)n_i \leqq k_\alpha(x),\ x \in \partial\Omega, \\ i = 1, 2, \cdots, n,\ \alpha = 1, 2, \cdots, m. \end{gathered} \tag{13}$$

If, in the proof of Theorem 1, the constant δ is reduced sufficiently, then (13) will hold throughout. The remainder of the proof is unaffected. On the other hand if $k_\alpha(x)$ is not bounded away from zero, then it may happen that $\lambda^\alpha = 0$ is in the generalized spectrum. For example, if $k_\alpha(x) \equiv 0$ for all α, then choosing for each α, $u^\alpha \equiv$ const, $\lambda^\alpha = 0$ yields a solution of (3).

Similarly, a theorem analogous to Theorem 2 holds if (3) is replaced by (12) and the constant C_0 exists. To see this we choose δ so small that

$$k_\alpha \delta n_1 \leqq C_0$$

for all $x \in \partial\Omega$, $i = 1, 2, \cdots, n$. We choose γ as before. Then we select r so small that $r < \pi/4\gamma$(rather than $\pi/2\gamma$) and the result follows.

The boundary condition (2) may be put in the form

$$\sum_{\beta=1}^{m} \left(f_{\alpha\beta}(x) \frac{\partial u^\beta}{\partial \nu_\beta} + g_{\alpha\beta}(x)u^\beta \right) = 0, \quad \alpha = 1, 2, \cdots, m. \tag{14}$$

If $f = \{f_{\alpha\beta}\}$ is an invertible matrix and $g = \{g_{\alpha\beta}\}$ is such that $f^{-1}g$ is positive definite, then boundary condition (14) is essentially the same as (12) with Theorems 1 and 2 applicable to (3), (14). On the other hand, there are important cases in which f and g are singular. Then special techniques are required and we shall give an example illustrating the method in such cases. However, we first give an example yielding a lower bound for (3), (4) which applies to the biharmonic operator.

EXAMPLE. Let $u = (u^1, u^2)$ be a solution in a bounded domain Ω of the system

$$\begin{aligned} \Delta u^1 - \lambda^1 u^2 &= 0 \\ \Delta u^2 - \lambda^2 u^1 &= 0 \end{aligned} \tag{15}$$

with the boundary condition

$$u^1 = u^2 = 0 \text{ on } d\Omega. \tag{16}$$

The generalized spectrum $\lambda = (\lambda^1, \lambda^2)$ is real in this case and the matrix M has the form

$$M = \begin{pmatrix} I & 0 & P^{1T} & 0 \\ 0 & I & 0 & p^{2T} \\ P^1 & 0 & \operatorname{div} P^1 & \frac{1}{2}(\lambda^1 + \lambda^2) \\ 0 & P^2 & \frac{1}{2}(\lambda^1 + \lambda^2) & \operatorname{div} P^2 \end{pmatrix}.$$

Then M is positive definite provided that

$$(\operatorname{div} P^1 - |P^1|^2)(\operatorname{div} P^2 - |P^2|^2) > \tfrac{1}{4}(\lambda^1 + \lambda^2)^2.$$

That is, for arbitrary vector fields P^1, P^2, such that P_i^α is C^1 in x_i and C^0 in x_j, $j \neq i$, $\alpha = 1, 2$, we have

$$(\lambda^1 + \lambda^2)^2 \geqq 4 \inf_{x \in \Omega} (\operatorname{div} P^1 - |P^1|^2)(\operatorname{div} P^2 - |P^2|^2). \tag{17}$$

We show that it is frequently possible to choose P^1 and P^2 so that the bound (17) is actually achieved at a point $\bar\lambda = (\bar\lambda^1, \bar\lambda^2)$ in the spectrum S such that $(\lambda^1)^2 + (\lambda^2)^2$ is a minimum. Thus all of S is outside the disk of radius $(\bar\lambda^1)^2 + (\bar\lambda^2)^2$. To see this let $u^1 = \varphi$, $u^2 = \psi$ be the eigenfunctions which yield the solution of (15), (16) with $|\bar\lambda| \in$ S. It is clear that for any α, β, the functions $\alpha\varphi$, $\beta\psi$ satisfy (15), (16) with the eigenvalues $\beta\bar\lambda^1/\alpha$, $\alpha\bar\lambda^2/\beta$. Then minimizing the quantity $(\beta\bar\lambda^1/\alpha)^2 + (\alpha\bar\lambda^2/\beta)^2$ for all α/β, we find that the minimum value is $2\bar\lambda^1\bar\lambda^2$ which occurs for $\alpha/\beta = \bar\lambda^2/\bar\lambda^1$. If φ, ψ are never zero in Ω, then we choose

$$P_i^1 = -\varphi_{x_i}/\varphi,\ P_i^2 = -\psi_{x_i}/\psi.$$

Then (17) yields the inequality

$$(\lambda^1 + \lambda^2)^2 \geqq 4\bar\lambda^1\bar\lambda^2$$

and the lower bound of $2\bar\lambda^1\bar\lambda^2$ is achieved for $\lambda^1 = \bar\lambda^1$, $\lambda^2 = \bar\lambda^2$.

A simple method for obtaining a lower bound results from the choice

$$P_1^{\,1} = P_i^{\,2} = -\varphi_{x_i}/\varphi$$

where φ is any positive C^2 function in Ω. We obtain from (17)

$$\lambda^1 + \lambda^2 \geqq 2 \inf_{x\in\Omega} \frac{\Delta\varphi}{\varphi}.$$

In particular if φ is the first eigenfunction of the membrane equation with zero Dirichlet boundary conditions and μ the corresponding eigenvalue, then

$$\lambda^1 + \lambda^2 \geqq 2\mu. \tag{18}$$

The system (15) is related to the biharmonic operator in that

$$\Delta^2 u^1 - \lambda^1\lambda^2 u^1 = 0$$

with the boundary conditions $u^1 = \Delta u^1 = 0$ on $\partial\Omega$. Setting $\nu = \lambda^1\lambda^2$, we see that (18) implies the well-know inequality

$$\nu \geqq \mu^2.$$

3. **Additional comparison functions.** Let Q be an $m \times m$ matrix with entries $Q^{\alpha\beta}$ which are $C^1(\overline{\Omega})$. An application of Green's theorem yields the identity

$$\begin{aligned}
&-\int \Big[Q^{\alpha\gamma}u^\gamma \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\Big(a_{ij}^\alpha \frac{\partial u^\alpha}{\partial x_i}\Big) \\
&\quad + \lambda^\alpha Q^{\alpha\gamma}u^\gamma \sum_{\beta=1}^m c_{\alpha\beta}u^\beta \Big] \\
&= \int_\Omega \Big[Q^{\alpha\gamma} \sum_{i,j=1}^n a_{ij}^\alpha \frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\gamma}{\partial x_j} \\
&\quad + u^\gamma \sum_{i,j=1}^n a_{ij}^\alpha \frac{\partial u^\alpha}{\partial x_i}\frac{\partial Q^{\alpha\gamma}}{\partial x_j} + \lambda^\alpha Q^{\alpha\gamma}u^\gamma \sum_{\beta=1}^m c_{\alpha\beta}u^\beta \Big] \\
&\quad - \int_{\partial\Omega} Q^{\alpha\gamma}u^\gamma \frac{\partial u^\alpha}{\partial \nu_\alpha}.
\end{aligned} \tag{19}$$

The identities for $\alpha = \gamma$ allow an additional useful transformation if the $Q^{\alpha\alpha}$ are $C^2(\overline{\Omega})$. We have

$$-\int \left[Q^{\alpha\alpha}u^\alpha \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}^\alpha \frac{\partial u^\alpha}{\partial x_i} \right) + \lambda^\alpha Q^{\alpha\alpha}u^\alpha \cdot \sum_{\beta=1}^{m} c_{\alpha\beta}u^\beta \right]$$

(20)

$$= \int_\Omega \left[Q^{\alpha\alpha} \sum_{i,j=1}^{n} a_{ij}^\alpha \frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\alpha}{\partial x_j} - \frac{1}{2}(u^\alpha)^2 \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}^\alpha \frac{\partial Q^{\alpha\alpha}}{\partial x_i} \right)\right.$$

$$\left. - \lambda^\alpha Q^{\alpha\alpha}u^\alpha \sum_{\beta=1}^{m} c_{\alpha\beta}u^\beta \right] - \int_{\partial\Omega} u^\alpha \left(Q^{\alpha\alpha}\frac{\partial u^\alpha}{\partial \nu_\alpha} - \frac{1}{2}u^\gamma \frac{\partial Q^\alpha}{\partial \nu_\alpha} \right).$$

Suppose that each $Q^{\alpha\alpha}$ is positive in Ω and in addition that

$$\sup_{x\in\Omega} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}^\alpha \frac{\partial Q^{\alpha\alpha}}{\partial x_i} \right) < 0, \quad \alpha = 1, 2, \cdots, m. \tag{21}$$

With boundary conditions (4), we may combine (20) with (8) and obtain a lower bound which may be an improvement over that determined by the matrix M. The usefulness of such scalar fields depends on the possibility of finding positive functions $Q^{\alpha\alpha}$ which satisfy (21). If Ω is bounded such functions always exist. For example, setting

$$Q^{\alpha\alpha} = 1 - \gamma r^p \tag{22}$$

where $\gamma > 0$, $p > 0$ are constants, we find

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}^\alpha \frac{\partial Q^{\alpha\alpha}}{\partial x_i} \right)$$

$$= -\gamma p r^{p-4} \left[(p-2) \sum_{i,j=1}^{n} (a_{ij}^\alpha x_i x_j) \right.$$

$$\left. + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} (a_{ij}^\alpha) x_i r^2 + r^2 \sum_{i=1}^{n} a_{ii}^\alpha \right].$$

If the coefficients a_{ij}^α are in $C^1(\overline{\Omega})$, then the ellipticity of the operators (a_{ij}^α) yields a positive lower bound for the terms in the bracket on the right if p is sufficiently large. Then choosing γ sufficiently small guarantees that $Q^{\alpha\alpha}$ is positive in Ω.

For boundary conditions (12), the functions $Q^{\alpha\alpha}$ are required to satisfy on $\partial\Omega$ the inequality

$$h_\alpha(x)\frac{\partial Q^{\alpha\alpha}}{\partial \nu_\alpha} - 2k_\alpha(x)Q^{\alpha\alpha} \geqq 0, \quad \alpha = 1, 2, \cdots, m. \tag{23}$$

If there is a constant $m > 0$ such that $k_\alpha(x) \leqq -m$ for $x \in \partial\Omega$, then the functions $Q^{\alpha\alpha}$ defined in (22) will satisfy (23) provided γ is sufficiently small. Thus if k_α is negative and bounded away from zero in Ω, functions $Q^{\alpha\alpha}$ exist which satisfy boundary conditions (12). At the end of this Section we give an example to show how lower bounds of the generalized spectrum can be obtained by use of the functions $Q^{\alpha\gamma}$ with $\alpha \neq \gamma$.

It is frequently the case that boundary conditions (2) imply that the first derivatives of one or more of the functions u^α vanish on $\partial\Omega$. Under these circumstances additional vector fields can be used for comparison. Let $R^\alpha = (R_1{}^\alpha, \cdots, R_n{}^\alpha)$, $\alpha = 1, 2, \cdots, m$ be vector fields of class $C^1(\Omega) \cap C(\overline{\Omega})$. We have the identity

$$
\begin{aligned}
(24)\quad & \int_\Omega \Bigg[\sum_{i,j,k=1}^{n} R_k{}^\alpha \frac{\partial u^\alpha}{\partial x_k} \frac{\partial}{\partial x_j} \left(a_{ij}^\alpha \frac{\partial u^\alpha}{\partial x_i} \right) \\
& \qquad + \lambda^\alpha \sum_{k=1}^{n} \sum_{\beta=1}^{m} R_k{}^\alpha \frac{\partial u^\alpha}{\partial x_k} c_{\alpha\beta} u^\beta \Bigg] \\
& = \frac{1}{2} \int_\Omega (\operatorname{div} R^\alpha) \sum_{i,j=1}^{n} a_{ij}^\alpha \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\alpha}{\partial x_j} \\
& \quad - \int_\Omega \sum_{i,j,k=1}^{n} \frac{\partial R^\alpha{}_k}{\partial x_j} a_{ij}^\alpha \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\alpha}{\partial x_k} \\
& \quad + \frac{1}{2} \int_\Omega \sum_{i,j,k=1}^{n} R_k{}^\alpha \frac{\partial}{\partial x_k} (a_{ij}^\alpha) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\alpha}{\partial x_j} \\
& \quad + \int_{\partial\Omega} R_k{}^\alpha a_{ij}^\alpha \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\alpha}{\partial x_k} n_j \\
& \quad - \frac{1}{2} \int_{\partial\Omega} \sum_{i,j,k=1}^{n} R_k{}^\alpha a_{ij}^\alpha \frac{\partial u^\alpha}{\partial x_j} \frac{\partial u^\alpha}{\partial x_j} n_k \\
& \quad + \lambda^\alpha \int_\Omega \sum_{k=1}^{n} \sum_{\beta=1}^{m} R_k{}^\alpha \frac{\partial u^\alpha}{\partial x_k} c_{\alpha\beta} u^\beta .
\end{aligned}
$$

If the first derivatives of u^α vanish on $\partial\Omega$, then the right side of (24) is the integral of a quadratic form in u^α and its first derivatives. If R^α can be chosen so that this form combined with the quadratic forms considered earlier is positive definite for a set of values λ, then we get further bounds on the generalized spectrum. For example, if $R_k{}^\alpha$ is a function of x_k alone, if the coefficients a^α_{ij} are constant, and if u^α is a solution of (3) with the appropriate boundary conditions, then (24) yields

$$\int_\Omega \frac{1}{2} \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial R_k{}^\alpha}{\partial x_k} - 2\frac{\partial R_j{}^\alpha}{\partial x_j} \right) a^\alpha_{ij} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\alpha}{\partial x_j}$$

$$+ \lambda^\alpha \sum_{k=1}^{n} \sum_{\beta=1}^{m} R_k{}^\alpha c_{\alpha\beta} u^\beta \frac{\partial u^\alpha}{\partial x_k} = 0.$$

If we make the particularly simple choice $R_k{}^\alpha = x_k$, we find

$$\int_\Omega \Bigg[\frac{1}{2} (n-2) \sum_{i,j=1}^{n} a^\alpha_{ij} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\alpha}{\partial x_j}$$

$$+ \lambda^\alpha \sum_{k=1}^{n} \sum_{\beta=1}^{m} x_k c_{\alpha\beta} u^\beta \frac{\partial u^\alpha}{\partial x_k} \Bigg] = 0.$$

When $n \geqq 3$, the above relation provides a positive definite form in the first derivatives, usually of considerable help in counterbalancing the remaining terms, those containing $(u^\alpha)^2$ and the ones involving products of u^α with its first derivatives.

As an example, we consider the problem of determining a bound on the first eigenvalue for the vibration of a clamped plate. The motion is described by the equation

$$\Delta^2\varphi - \nu\varphi = 0 \text{ in } \Omega \tag{25}$$

subject to the boundary conditions

$$\varphi = 0, \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial\Omega. \tag{26}$$

We transform (25) into a second order system by setting $u^2 = \mu\Delta\varphi$ and identifying u^1 with φ. The result is the second order system

$$\Delta u^1 - \lambda^1 u^2 = 0, \quad \Delta u^2 - \lambda^2 u^1 = 0 \tag{27}$$

where $\lambda^1 = \mu^{-1}$ and $\lambda^2 = \mu\nu$. The boundary conditions (26) become

$$u^1 = \frac{\partial u^1}{\partial x_i} = 0, \; i = 1, 2, \cdots, n \text{ on } \partial\Omega. \tag{28}$$

It is important to note that no boundary conditions on u^2 are prescribed. Thus, although (27) is the same system as that considered in the previous example, the conditions (28) require the methods of this section for the determination of a lower bound on the generalized spectrum of (27) and consequently a bound on the first eigenvalue of (25), (26).

It can be shown that the generalized spectrum of (27), (28) is real, and to find a lower bound for it, we apply (19). We set $Q^{22} = 0$ and choose $Q^{12} = Q^{21}$. The boundary conditions (28) show that Q^{11} and Q^{12} may be chosen arbitrarily in Ω except for the usual smoothness requirements. Writing Q^1 for Q^{11} and Q^2 for Q^{12}, we find from (19) that

$$\int_\Omega \left[Q^1|\nabla u^1|^2 + u^1 \sum_{i=1}^n \frac{\partial u^1}{\partial x_i}\frac{\partial Q^1}{\partial x_i} + Q^1\lambda^1 u^1 u^2 \right] = 0$$

$$\int_\Omega \left[Q^2(\nabla u^1 \cdot \nabla u^2) + u^2 \sum_{i=1}^n \frac{\partial u^1}{\partial x_i}\frac{\partial Q^2}{\partial x_i} + Q^2\lambda^1 (u^2)^2 \right] = 0$$

$$\int_\Omega \left[Q^2(\nabla u^1 \cdot \nabla u^2) + u^1 \sum_{i=1}^n \frac{\partial u^2}{\partial x_i}\frac{\partial Q^2}{\partial x_i} + \lambda^2 (u^1)^2 \right] = 0.$$

We apply Green's theorem in the first equation above, subtract the second equation from the third and then apply Green's theorem again to obtain

$$\text{(29)}\qquad \int_\Omega \left[Q^1|\nabla u^1|^2 - \frac{1}{2}(u^1)^2 \Delta Q^1 + \lambda^1 Q^1 u^1 u^2 \right] = 0,$$

$$\text{(30)}\qquad \int_\Omega \left[2u^2 \sum_{i=1}^n \frac{\partial u^1}{\partial x_i}\frac{\partial Q^2}{\partial x_i} - (\Delta Q^1)(u^1 u^2) + Q^2\lambda^1(u^2)^2 - Q^2\lambda^2(u^1)^2 \right] = 0.$$

We combine (29) and (30) with the form for $P^1 = (P_1^{\,1}, P_2^{\,1}, \cdots, P_n^{\,1})$ which was derived in Section 2. The resulting matrix is

$$M_1 = \begin{pmatrix} Q^1 I & P^T & (\operatorname{grad} Q^2)^T \\ P & \operatorname{div} P - \frac{1}{2}\Delta Q^1 - \lambda^2 Q^1 & \frac{1}{2}(\lambda^1 Q^1 - \Delta Q^2) \\ \operatorname{grad} Q^2 & \frac{1}{2}(\lambda^1 Q^1 - \Delta Q^2) & \lambda^1 Q^2 \end{pmatrix}.$$

When the matrix M_1 is positive definite the corresponding values of $\lambda = (\lambda^1, \lambda^2)$ are excluded from the generalized spectrum. We make the simple choice $Q^1 = Q^2 \equiv 1$ in Ω and find that M_1 is positive definite provided that

$$(\operatorname{div} P - |P|^2 - \lambda^2)\lambda^1 > \tfrac{1}{4}(\lambda^1)^2.$$

Thus for arbitrary vector fields P, the value of λ must satisfy

$$\tfrac{1}{4}\lambda^1 + \lambda^2 \geqq \inf_{x \in \Omega} (\operatorname{div} P - |P|^2).$$

Let τ be the first eigenvalue of $(\Delta + \tau)\varphi = 0$ in Ω with $\varphi = 0$ on $\partial\Omega$. Then clearly

$$\tfrac{1}{4}\lambda^1 + \lambda^2 \geqq \tau$$

or

$$\tfrac{1}{4}\mu^{-1} + \mu\nu \geqq \tau.$$

Now minimizing this inequality with respect to μ, we get $\mu = \sqrt{\nu}/2$. Hence we obtain for the first eigenvalue ν_1 of the clamped plate problem the classical inequality

$$\nu_1 \geqq \tau^2. \tag{31}$$

This inequality corresponds to $\lambda^1 = 2\sqrt{\nu}_1$ and $\lambda^2 = 1/2\sqrt{\nu}_1$, and we observe that it is essential to develop the framework of the generalized spectrum to obtain this result. If we had confined ourselves to (27), (28) with $\lambda^1 = \lambda^2$, i.e., if we had restricted the consideration to the ordinary spectrum of a second order system, the lower bound would not have been as strong as (31). Of course, improved bounds, both theoretical and numerical can be obtained by other choices of Q^1, Q^2 and by selecting Q^{22} different from zero. In this last case, Q^{22} would have to satisfy a boundary condition. Moreover, added information about the generalized spectrum can be found by means of (24) which can be applied with $\alpha = 1$. Condition (28) shows that the boundary term in (24) vanishes and hence $R^1 = (R_1{}^1, R_2{}^1, \cdots, R_n{}^1)$ may be chosen arbitrarily except for smoothness hypotheses. To illustrate this point, suppose that Ω is contained in the domain $D = \{r : 0 < r_0 \leqq r \leqq r_1 < 1\}$ where $r^2 = \sum_{i=1}^n x_i{}^2$. We choose $Q^1 = 1 - r^2$, $Q^2 = 1$ and find for ν_1 the bound

$$\nu_1 \geqq \frac{(\tau + n)^2}{(1 - r_0{}^2)^2(1 - r_1{}^2)}$$

which is an improvement over (31).

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