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TWO COUNTEREXAMPLES FOR MEASURABLE RELATIONS

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In [2] various definitions of measurable relations are made and discussed. Some questions are posed there concerning the equivalence of the various definitions under suitable restrictions of the spaces. It is our purpose to resolve two of these questions by constructing counterexamples.

1. **Definitons and notations.** The pair $(T, \mathscr{A})$ will denote a measurable space and a σ-algebra. We say $(T, \mathscr{A})$ is a *(complete) measure space* if $(T, \mathscr{A}, \mu)$ is a (complete) finite measure space for some measure μ.

A *Polish* space is a separable metrizable space which is complete under some metric. A *Souslin* space is a metrizable continuous image of a Polish space.

A relation $F : T \rightarrow X$ is a subset of $T \times X$. In conformity with [2], for $F : T \rightarrow X$, a relation, we denote by F the corresponding function into the set of subsets of X, and when we want to emphasize the properties of F as a subset of $T \times X$, we will refer to its graph $Gr(F)$ rather than F. If domain $F = T$, we call F a *multifunction* from T to X. As usual, we use the notations

$$F(A) = \{x \in X : (t, x) \in \mathrm{Gr}(F) \text{ for some } t \in A\}$$

and

$$F^{-1}(B) = \{t \in T : F(t) \cap B \neq \Phi\}.$$

All relations under considerations are multifunctions in the present paper.

Let $F : T \rightarrow X$ be a relation from a measurable space $(T, \mathscr{A})$ to X, a metric space whose Borel σ-algebra is denoted by $\mathscr{B}$. We define, as in [2], F to be *measurable, (weakly measurable, $\mathscr{B}$-measurable, $\mathscr{C}$-measurable)* if $F^{-1}(B)$ is measurable for each closed (respectively, open, Borel, compact) subset B of X. Finally, $\mathscr{A} \times \mathscr{B}$ will denote the σ-algebra generated by $\mathscr{A}$ and $\mathscr{B}$ on $T \times X$.

The following theorem is proved in [2].

THEOREM. *Let $(T, \mathscr{A})$ be a measurable space, X be a separable metric space, and $F : \mathrm{T} \rightarrow X$ be closed valued. Consider the following statements*:

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A. *F is $\mathscr{B}$-measurable.*
B. *F is measurable.*
C. *F is weakly measurable.*
D. *F is $\mathscr{C}$-measurable.*
E. $\mathrm{Gr}(F) \in \mathscr{A} \times \mathscr{B}$.

We have:
(i) $A \Rightarrow B \Rightarrow C \Rightarrow D$ and $C \Rightarrow E$.
(ii) If X is σ-compact, then $A \Rightarrow B \Leftrightarrow C \Leftrightarrow D \Rightarrow E$.
(iii) If $(T, \mathscr{A})$ is complete and X is Souslin, then $A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow E \Rightarrow D$.

The questions of Himmelberg concern the reversibility of certain implications in the above theorem.

2. **The first example.** The first example deals with $B \Leftrightarrow C$ when $(T, \mathscr{A})$ is just a measurable space and X is Souslin (or even Polish). We show $C \not\Rightarrow B$ with X a Polish space. Our example is similar to one of Kaniewski [4], page 865.

Let X be the set of irrational numbers with the usual topology and $(T, \mathscr{A}) = (X, \mathscr{B})$. There is a closed set $H \subset X \times X$ such that its projection $p(H)$ into the first coordinate space is not a Borel set. Also, there is a homeomorphism $h : X \to X \times X$. Let $\varphi = p \circ h : X \to X$. φ is an open continuous map which maps the closed set $h^{-1}(H)$ onto $p(H)$. Let $F : X \to X$ be given by $F = \varphi^{-1}$. Then F is closed valued and weakly measurable but not measurable. (See [1] pages 59 to 72 for a discussion of Borel and Souslin sets.)

3. **The second example.** The second example deals with $D \not\Rightarrow B$ in (iii) of the above theorem. We first construct $(T, \mathscr{A})$. Let T be the set of irrational numbers and $A \in \mathscr{A}$ when and only when A is residual in T or coresidual in T (i.e., $T \setminus A$ is residual). Let $\mu(A) = 1$ if A is residual and $\mu(A) = 0$ if A is coresidual. $(T, \mathscr{A}, \mu)$ is a complete finite measure space. Let X be the space of irrational numbers and F be the identity map of T to X. Then F is closed valued. Clearly, F is $\mathscr{C}$-measurable since compact subsets of T are coresidual. But F is not measurable since $F^{-1}(B) \notin \mathscr{A}$ when B is a bounded closed nonempty subset of X.

4. **A summary statement.** In this section we give the present status of the reverse implications in the theorem of Himmelberg.

In [2] it is shown that $B \not\Rightarrow A$ for (i) and (ii).

The example of Kaniewski and the example of § 2 above both show $C \not\Rightarrow B$ for (i).

The second example above shows $D \not\Rightarrow C$ for (i) and (iii), even if T is complete and X is Polish.

It is easily seen that $E \not\Rightarrow C$ for (i). For, suppose T is the set of irrational numbers with $\mathscr{A}$ the Borel sets of T and X is the space of irrational numbers. Let $X = X_1 \cup X_2$ be the union of two nonempty disjoint open sets, A be a closed set in $T \times X_1$ with $p_T(A) \notin \mathscr{A}$ and define the closed-valued multifunction F by $\mathrm{Gr}(F) = A \cup (T \times X_2)$. Since $F^{-1}(X_1) = p_T(A)$, our assertion follows.

The only remaining open question is whether $E \Rightarrow B$ for (ii). By a theorem of Kunugui and Novikoff the answer is yes if T is a Borel set in a Polish space with $\mathscr{A}$ being the Borel subsets of T. (See [3] for a discussion of this last fact.)

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