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MULTI-DIMENSIONAL GENERALIZATIONS OF THE PADÉ TABLE

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ABSTRACT. The Padé table is a method of generating rational approximating functions to a given function. Recently, various authors have considered generalisations giving approximants which are algebraic functions or satisfy algebraic differential equations. We show how these schemes fit into an even more general theory of algebraic approximation of functions and list the rather few known examples in which the construction can be given explicitly.

1. **Introduction.** Let $f(z)$ be a function of the complex variable z which is regular at the origin. The Padé table for $f(z)$ is an array of rational functions,

$$a_{mn}(z)/b_{mn}(z) \qquad (m, n = 1, 2, \cdots),$$

defined as follows: $a_{mn}(z)$ and $b_{mn}(z)$ are polynomials of degrees $m-1$ and $n-1$ respectively and not both identically zero such that the function $b_{mn}(z)\, f(z) - a_{mn}(z)$ has a zero of order at least $m+n-1$ at the origin. The definition implies that $b_{mn}(z)$ is not identically zero. (See [13], chapter 10 and [17], chapter 20). The Padé table has proved empirically useful both in providing efficient rational approximations to special functions and as a method of approximate analytic continuation of functions defined locally by power series. (See [2], [3] and [5]).

Various generalisations of the above scheme have been suggested which appear to give better approximating functions than the rational functions of the Padé table. Thus, Padé [12] considered approximation by algebraic functions. If $\rho_0, \cdots, \rho_m$ are non-negative integers, there are polynomials $a_k(z)$ $(0 \leqq k \leqq m)$ of respective degrees at most $\rho_k - 1$ and not all zero such that the function

$$\sum_{k=0}^{m} a_k(z)\, f(z)^k$$

has a zero of order at least $\rho_0 + \cdots + \rho_m - 1$ at the origin. Let $g(z)$ be an appropriately chosen root of the polynomial equation

$$\sum_{k=0}^{m} a_k(z)\, g(z)^k = 0.$$

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The functions $g(z)$ are a multi-dimensional array of algebraic approximants to $f(z)$ indexed by $\rho_0, \cdots, \rho_m$. Recently, Shafer [14] has discussed quadratic approximation (the case $m = 2$ of the above) and his examples show that this algorithm can produce very effective results.

In the same way, we may consider approximating functions which satisfy differential equations. If $\rho_0, \cdots, \rho_m$ are non-negative integers, there are polynomials $a_k(z)$ $(0 \leqq k \leqq m)$ of respective degrees at most $\rho_k - 1$ and not all zero such that the function

$$a_m(z) f^{(m-1)}(z) + \cdots + a_1(z) f(z) + a_0(z)$$

has a zero of order at least $\rho_0 + \cdots + \rho_m - 1$ at the origin. The approximants now are the solutions of the differential equations

$$a_m(z) g^{(m-1)}(z) + \cdots + a_1(z) g(z) + a_0(z) = 0$$

with

$$g^{(k)}(0) = f^{(k)}(0) \quad (0 \leqq k \leqq m-2).$$

In a recent report, Joyce and Guttman [9] give a slightly different version of this algorithm and indicate some applications in series analysis.

Both of the schemes just described are special cases of a general theory of algebraic approximation of functions constructed by Mahler in the thirties. (See [11]). The initial ideas of this theory come from Hermite's basic work [6] and [7] on the arithmetic properties of the exponential function. In particular, Hermite considers the problem of finding polynomials $a_1(z), \cdots, a_m(z)$ such that the function

$$\sum_{k=1}^{m} a_k(z) e^{\omega_k z}$$

vanishes to a high order at the origin and the dual problem of finding polynomials $\mathfrak{a}_1(z), \cdots, \mathfrak{a}_m(z)$ such that all the functions

$$\mathfrak{a}_\ell(z)\, e^{\omega_k z} - \mathfrak{a}_k(z)\, e^{\omega_\ell z} \quad (1 \leqq k, \ell \leqq m)$$

vanish to a high order at the origin.

In this note, we sketch those parts of Mahler's theory which seem immediately applicable to approximation problems. We also give some examples, culled mainly from transcendental number theory, to illustrate the methods. In a subsequent paper, we hope to discuss the question of the convergence of algorithms of this type.

2. **The Latin and German polynomial systems.** Throughout, m is a fixed positive integer and $f_0(z) = 1, f_1(z), \cdots, f_m(z)$ are $m + 1$ functions

of the complex variable z which are regular at the origin.

If $a(z)$ is a polynomial, we denote its degree by $\deg a(z)$, with the convention that the zero polynomial has degree -1. Also, if $f(z)$ is a function regular at the origin, we denote by $\operatorname{ord} f(z)$ the order of the zero of $f(z)$ at the origin.

Let $\rho = (\rho_0, \rho_1, \cdots, \rho_m)$ be an $(m+1)$-tuple of non-negative integers and set $\sigma = \rho_0 + \cdots + \rho_m$. Following Mahler [11], we introduce two systems of polynomials. A Latin polynomial system at ρ is a system of polynomials $a_k(z)$ $(0 \leqq k \leqq m)$, not all identically zero, which together with the Latin remainder function

$$r(z) = \sum_{k=0}^{m} a_k(z)\, f_k(z)$$

satisfy the inequalities

$$\deg a_k(z) \leqq \rho_k - 1 \quad (0 \leqq k \leqq m)$$

and

$$\operatorname{ord} r(z) \geqq \sigma - 1.$$

Such a system always exists, for the polynomials $a_k(z)$ have in all σ coefficients and the condition on the remainder $r(z)$ gives $\sigma - 1$ linear equations to be satisfied by these coefficients and this system always has a non-trivial solution.

A German polynomial system at ρ is a system of polynomials $\mathfrak{a}_k(z)$ $(0 \leqq k \leqq m)$, not all identically zero, which together with the German remainder functions

$$\mathfrak{r}_{k\ell}(z) = \mathfrak{a}_\ell(z) f_k(z) - \mathfrak{a}_k(z)\, f_\ell(z) \quad (0 \leqq k, \ell \leqq m)$$

satisfy the inequalities

$$\deg \mathfrak{a}_k(z) \leqq \sigma - \rho_k \quad (0 \leqq k \leqq m)$$

and

$$\operatorname{ord} \mathfrak{r}_{k\ell}(z) \geqq \sigma + 1 \quad (0 \leqq k, \ell \leqq m).$$

The polynomials $\mathfrak{a}_k(z)$ have in all $m(\sigma + 1) + 1$ coefficients and, from the identities.

$$\mathfrak{r}_{kk}(z) = 0 \quad (0 \leqq k \leqq m),$$

$$\mathfrak{r}_{k\ell}(z) + \mathfrak{r}_{\ell k}(z) = 0 \quad (0 \leqq k, \ell \leqq m),$$

$$f_j(z)\,\mathfrak{a}_{k\ell}(z) + f_k(z)\,\mathfrak{a}_{\ell j}(z) + f_\ell(z)\mathfrak{a}_{jk}(z) = 0 \quad (0 \leqq j, k, \ell \leqq m),$$

it suffices to choose these coefficients so that ord $\mathfrak{r}_{0k}(z) \geqq \sigma + 1$ $(1 \leqq k \leqq m)$, giving a system of $m(\sigma + 1)$ linear homogeneous equations to be satisfied by these coefficients. Thus a German polynomial system always exists.

By using Newton interpolation series for the functions $f_0(z), \cdots, f_m(z)$ in place of their Taylor series, the above construction can be easily extended to examine the approximation of functions at a finite or countably infinite set of points, instead of just at the origin. The generalization is given at varying levels of abstraction by Mahler [11], Jager [8] and Coates [4]. Formally, the theory is essentially unchanged, but the idea may have useful applications. (See, for example, the remarks in [2]).

3. **The fundamental determinant.** As before, let $\rho = (\rho_0, \rho_1, \cdots, \rho_m)$ be an $(m + 1)$-tuple of non-negative integers and $\sigma = \rho_0 + \cdots + \rho_m$. To describe the properties of the two types of polynomial systems, we introduce the determinant $\Delta(\rho) = \Delta(\rho_0, \rho_1, \cdots, \rho_m)$, which is the determinant of order $\sigma - \rho_0$ with the element

$$f_k^{(\rho_0+i-\ell)}(0)/(\rho_0 + i - \ell)!$$

in the i-th row and j-th column, where $j = \rho_1 + \cdots + \rho_{k-1} + \ell$ $(1 \leqq i \leqq \sigma - \rho_0,\ \ 1 \leqq k \leqq m,\ \ 1 \leqq \ell \leqq \rho_k)$. We also denote by ϵ_k $(0 \leqq k \leqq m)$ the $(m + 1)$-tuple having k-th coordinate 1 and all other coordinates 0.

An examination of the defining systems of linear equations for the coefficients of the Latin and German polynomials yields the following uniqueness theorems.

THEOREM 1. *Let $0 \leqq k \leqq m$. The determinant $\Delta(\rho - \epsilon_k)$ is non-zero if and only if the Latin polynomial system $a_0(z), \cdots, a_m(z)$ at ρ is unique up to a scalar multiple and $a_k(z)$ has exact degree $\rho_k - 1$.*

THEOREM 2. *The following three statements are equivalent:*

(i) *the determinant $\Delta(\rho)$ is non-zero;*

(ii) *the Latin polynomial system at ρ is unique up to a scalar multiple and the Latin remainder function $r(z)$ has exact order $\sigma - 1$;*

(iii) *the German polynomial system $\mathfrak{a}_0(z), \cdots, \mathfrak{a}_m(z)$ at ρ is unique up to a scalar multiple and $\mathfrak{a}_0(0) \neq 0$.*

THEOREM 3. *Let $0 \leqq k \leqq m$. The determinant $\Delta(\rho + \epsilon_k)$ is non-zero if and only if the German polynomial system $\mathfrak{a}_0(z), \cdots, \mathfrak{a}_m(z)$ at ρ is unique up to a scalar multiple and $\mathfrak{a}_k(z)$ has exact degree $\sigma - \rho_k$.*

We shall verify these assertions by explicitly describing the linear equations which determine the Latin and German polynomial systems. Plainly, the Latin polynomials $a_1(z), \cdots, a_m(z)$ at ρ can be specified by their coefficient vector $a(\rho)$, which is the $(\sigma - \rho_0)$-tuple consisting of the coefficients of these polynomials written in the order of increasing powers of z, beginning with the coefficients of $a_1(z)$ and ending with those of $a_m(z)$. We also define the coefficient vector $\mathfrak{a}(\rho)$ of the German polynomial $\mathfrak{a}_0(z)$ *at* ρ to be the $(\sigma - \rho_0 + 1)$-tuple consisting of the coefficients of $\mathfrak{a}_0(z)$ written in the order of decreasing powers of z. Then, because $f_0(z) = 1$, the complete Latin and German polynomial systems at ρ are determined uniquely when their respective coefficient vectors $a(\rho)$ and $\mathfrak{a}(\rho)$ are known.

Let $M(\rho)$ be the square matrix of order $\sigma - \rho_0$ with the same entries as the determinant $\Delta(\rho)$, and let $M_{ij}(\rho)$ be the matrix obtained by deleting the i-th row and j-th column of $M(\rho)$. The defining equations of the Latin coefficient vector $a(\rho)$ are the first $\sigma - \rho_0 - 1$ equations of the linear system (of $\sigma - \rho_0$ equations and $\sigma - \rho_0$ unknowns)

$$M(\rho)x = 0. \tag{1}$$

To see this, recall that the Latin polynomials $a_1(z), \cdots, a_m(z)$ at ρ are defined by the condition $\operatorname{ord} r(z) \geqq \sigma - 1$ on the Latin remainder function. Because $\deg a_0(z) \leqq \rho_0 - 1$, this condition becomes the $\sigma - \rho_0 - 1$ conditions

$$r^{(\rho_0+i-1)}(0)/(\rho_0 + i - 1)! = \sum_{k=1}^{m} \sum_{\ell=1}^{\rho_k} a_{k\ell} f_k^{(\rho_0+i-\ell)}(0)/(\rho_0 + i - \ell)! = 0$$

$$(1 \leqq i \leqq \sigma - \rho_0 - 1),$$

where we have written

$$a_k(z) = \sum_{\ell=1}^{\rho_k} a_{k\ell} z^{\ell-1} \quad (1 \leqq k \leqq m).$$

It follows that the Latin polynomial system at ρ is unique up to a scalar multiple if and only if some matrix

$$M_{\sigma-\rho_0, j}(\rho)$$

is non-singular. Further, the coefficient vector $a(\rho)$ satisfies the complete set of equations (1) if and only if the Latin remainder function $r(z)$ has $\operatorname{ord} r(z) \geqq \sigma$. Thus (i) and (ii) of Theorem 2 are equivalent, and Theorem 1 follows because if $\deg a_k(z) < \rho_k - 1$ then some Latin remainder functions at $\rho - \epsilon_k$ and ρ coincide, where by Theorem 2,

$\Delta(\rho - \epsilon_k) = 0$. Conversely if $\Delta(\rho - \epsilon_k) \neq 0$ then $a_k(z)$ has exact degree $\rho_k - 1$ and indeed some matrix $M_{\sigma-\rho_0, j}(\rho)$ is non-singular (namely, for $j = \rho_1 + \cdots + \rho_k$), so we obtain the uniqueness of the Latin polynomial system as asserted by Theorem 1.

In the same way, if $1 \leqq k \leqq m$, the defining equations of the German coefficient vector $\mathfrak{a}(\rho)$ are the equations of a linear system

$$(2) \qquad M(\rho + \epsilon_k)^t x = 0$$

with the $(\rho_1 + \cdots + \rho_k + 1)$-th equation omitted. To see this, recall that the German polynomials $\mathfrak{a}_0(z), \cdots, \mathfrak{a}_m(z)$ at ρ are defined by the conditions ord $\mathfrak{r}_{k0}(z) \geqq \sigma + 1$, $(1 \leqq k \leqq m)$, on the German remainder functions. Because deg $\mathfrak{a}_k(z) \leqq \sigma - \rho_k (1 \leqq k \leqq m)$, these conditions become the $\sigma - \rho_0$ conditions

$$\mathfrak{r}_{k0}^{(\sigma+1-\ell)}(0)/(\sigma + 1 - \ell)!$$

$$= \sum_{i=1}^{\sigma-\rho_0} \mathfrak{a}_{\sigma+1-\rho_0-i} f_k^{(\rho_0+i-\ell)}(0)/(\rho_0 + i - \ell)! \qquad (1 \leqq k \leqq m;\ 1 \leqq \ell \leqq \rho_k),$$

where we have written

$$\mathfrak{a}_0(z) = \sum_{i=1}^{\sigma-\rho_0} \mathfrak{a}_{\sigma+1-\rho_0-i} z^{\sigma+1-\rho_0-i}.$$

It follows that the German polynomial system at ρ is unique up to a scalar multiple if and only if some matrix

$$M_{j,\rho_1+\cdots\rho_{k+1}}(\rho + \epsilon_k)$$

is non-singular. Further, the coefficient vector $\mathfrak{a}(\rho)$ satisfies the complete set of equations (2) if and only if the German polynomial $\mathfrak{a}_k(z)$ has degree less than $\sigma - \rho_k$. Theorem 3 and the rest of Theorem 2 now follow.

An alternative proof of Theorems 2 and 3 can be derived from Theorem 5 without any appeal to the systems (1) and (2); for details see [**16**], pp. 279–281.

4. **Normal systems.** We say that the functions $f_1(z), \cdots, f_m(z)$ are normal at the point ρ if $\Delta(\rho) \neq 0$. After Theorem 2, this definition is essentially the same as those given by Jager [8], page 202, and Coates [4], pages 433 and 441. We note here two consequences of normality.

THEOREM 4. *Let $\rho_1, \cdots, \rho_m$ be given non-negative integers. The functions $f_1(z), \cdots, f_m(z)$ are normal at the point $\rho = (\rho_0, \rho_1, \cdots, \rho_m)$ for* *infinitely many choices of the non-negative integer* ρ_0 *if and only if they satisfy no equation of the shape*

$$b_0(z) + b_1(z) f_1(z) + \cdots + b_m(z) f_m(z) = 0 \tag{3}$$

where all the $b_k(z)$ *are polynomials and* $\deg b_k(z) \leqq \rho_k - 1$ *for* $1 \leqq k \leqq m$.

PROOF. It suffices to note that an equation of the shape (3) is equivalent to the condition $\Delta(\rho_0, \rho_1, \cdots, \rho_m) = 0$ for all suffciently large integers ρ_0.

THEOREM 5. *Let* $a_0(z), \cdots, a_m(z)$ *and* $\mathfrak{a}_0(z), \cdots, \mathfrak{a}_m(z)$ *be the two polynomial systems at the point* ρ. *Then*

$$\sum_{k=0}^{m} a_k(z)\,\mathfrak{a}_k(z) = cz^{\sigma-1} \tag{4}$$

where c *is a constant. Moreover,* $c \neq 0$ *if and only if* ρ *is a normal point.*

PROOF. Let $r(z)$ and $\mathfrak{r}_{k\ell}(z)$ be the remainder functions corresponding to the given systems of polynomials. From the definitions, we have the identity

$$\sum_{k=0}^{m} a_k(z)\,\mathfrak{a}_k(z) = \mathfrak{a}_0(z)\, r(z) - \sum_{k=0}^{m} a_k(z)\,\mathfrak{r}_{k0}(z). \tag{5}$$

Now, the left-side of (5) is a polynomial of degree at most $\sigma - 1$ and the two terms on the right have orders at least $\sigma - 1$ and $\sigma + 1$ respectively, so we have (4) and the last statement of the theorem follows from Theorem 2.

COROLLARY. *Suppose* ρ *is a normal point. Then the system of German polynomials at* ρ *is relatively prime and the only possible common factor of the system of Latin polynomials at* ρ *is a power of* z.

PROOF. From (4), the only common factor of either system of polynomials is a power of z. Moreover, at a normal point, $\mathfrak{a}_0(0) \neq 0$ by Theorem 2, so the German polynomials have no common factor.

5. **Some explicit constructions.** Only a few isolated instances of normal systems are known. We give three examples.

(i) Let $\omega_0 = 0, \omega_1, \cdots, \omega_m$ be distinct complex numbers. Then the functions

$$e^{\omega_1 z}, \cdots, e^{\omega_m z}$$

are normal at every point $\rho = (\rho_0, \rho_1, \cdots, \rho_m)$. To prove this assertion, we observe that if C is a closed contour containing $\omega_0, \omega_1, \cdots, \omega_m$, then the function

$$r(z) = \sum_{k=0}^{m} \left\{ \sum_{j=1}^{\rho_k} a_{kj} z^{j-1} \right\} e^{\omega_k z}$$

$$= \frac{1}{2\pi i} \int_C \prod_{k=0}^{m} (\zeta - \omega_k)^{-\rho_k} e^{\zeta z} \, d\zeta$$

has a zero of order exactly $\rho_0 + \cdots + \rho_m - 1$ at the origin. This identity, due to Hermite [7], gives explicit expressions for the Latin polynomials $a_k(z)$ and remainder $r(z)$ at ρ. Hermite [6] also obtained formulae for the German polynomials at ρ.

(ii) Let $\omega_0 = 0, \omega_1, \cdots, \omega_m$ be complex numbers, no two of which differ by an integer. Then the functions

$$(1 - z)^{\omega_1}, \cdots, (1 - z)^{\omega_m}$$

are normal at every point $\rho = (\rho_0, \rho_1, \cdots, \rho_m)$ and explicit formulae for the polynomial systems at ρ have been given by Mahler [10] and Jager [8]. The Latin polynomials $a_k(z)$ and remainder $r(z)$ at ρ may be obtained from the identity.

$$r(z) = \sum_{k=0}^{m} a_k(z)(1 - z)^{\omega_k}$$

$$= \frac{1}{2\pi i} \int_C \prod_{k=0}^{m} \prod_{\ell=1}^{\rho_k} (\zeta - \omega_k - \ell + 1)^{-1} (1 - z)^{\zeta} \, d\zeta,$$

where C is a closed contour containing all the $\omega_k + \ell - 1$ $(0 \leqq k \leqq m, 1 \leqq \ell \leqq \rho_k)$ and again this proves the normality assertion made above.

(iii) Let $\omega_0 = 0, \omega_1, \cdots, \omega_m$ be complex numbers, no two of which differ by an integer. The doubly-indexed family of functions

$$(1 - z)^{\omega_r} \log(1 - z)^s \ (0 \leqq r \leqq m, \ 0 \leqq s \leqq n_r, \ (r, s) \neq (0, 0))$$

is normal with respect to every set of parameters ρ_{rs} $(0 \leqq r \leqq m, 0 \leqq s \leqq n_r)$ such that

$$\rho_{r,0} \geqq \rho_{r,1} \geqq \cdots \geqq \rho_{r,n_r} (0 \leqq r \leqq m).$$

In this case, the Latin polynomials $a_{rs}(z)$ and remainder $r(z)$ at ρ are obtained from the identity

$$r(z) = \sum_{r=0}^{m} \sum_{s=0}^{n_r} a_{rs}(z)(1-z)^{\omega_r}(\log(1-z))^s$$

$$= \frac{1}{2\pi i} \int_C \prod_{r=0}^{m} \prod_{s=0}^{n_r} \prod_{t=1}^{\rho_{rs}} (\zeta - \omega_r - t + 1)^{-1}(1-z)^{\zeta}\, d\zeta,$$

where C is a closed contour containing all the $\omega_r + t - 1$ $(0 \leqq r \leqq m$, $1 \leqq t \leqq \rho_{r,0})$. (See [1] and [15]).

Some further examples of normal systems in the more general sense mentioned at the end of section 2 are given in [15].

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